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Summer term 2021

Algebraic Geometry II – Retry Exam 09.09.2021

Begin: 09:00 **End:** 12:00

Technical details:

– Upload a scan of your solutions (max. 500kb per page) until 12:00 to the eCampus folder 'Exams/Exam submission'. Solutions uploaded any later than 12:00 will not be considered which results in you failing the exam.

– The exam will be posted at 9:00 on the usual website `https://www.math.uni-bonn.de/people/gmartin/AlgebraicGeometrySS2021.htmpl` and will also be available in the folder 'Exams' in eCampus (where this document can be found already now).

– Please sign the declaration of honor below and scan it together with your solutions. Without it your solutions will not be considered.

– If at all possible, upload your solution in one(!) pdf document with the declaration of honor as the first page. Name the file **firstname.name.pdf**. Please, write your name on each sheet.

– In case of technical problems (but only then), you can also send the scan to gmartin@math.uni-bonn.de (until 12:00 am).

– Allow at least 30-45min for scanning and uploading your solutions.

Guidelines:

– The exam consists of applying your knowledge of algebraic geometry to fill in the missing arguments in the text below. In particular, reading and understanding the arguments that are given is part of the task.

– It is recommended to work through the exercises in the given order.

– The exam will only lead to pass or fail. It will not be graded. A complete solution will be posted on eCampus, so that you can find out yourself how well you did in the exam.

– All arguments have to be justified. Apart from standard material from commutative algebra, you have to deduce everything by only using results explicitly stated or used, either in the lectures, on the exercise sheets or in the text below.

– You may use available resources to solve the exercise. You are allowed to consult the notes of the class, the exercise sheets, online books, etc. **You are not allowed** to contact any other person during the exam (by email, phone, social media, etc.) or to discuss your answers with anyone before successfully uploading them to eCampus.

I hereby swear that I completed the examination detailed above completely on my own and without any impermissible external assistance or through the use of non-permitted aids. I am aware that cheating during the execution of an examination (as detailed in §63 Para 5 of the Higher Education Act NRW) is a violation of the legal regulations for examinations and an administrative offense. The submission of false affirmation in lieu of an oath is a criminal offense.

Signed at: **Date:**

Name: **Student ID:** **Signature:**

1. Spreading

Exercise 1. Consider the following homogeneous polynomial $F = x_0^{10} + x_1^{10} + x_2^{10}$ in three variables.

(i) Define a scheme $\mathcal{X}$ together with a flat morphism $\pi \colon \mathcal{X} \longrightarrow \mathrm{Spec}(\mathbb{Z})$ such that the fibre over the generic point is $V_+(F) \subset \mathbb{P}^2_{\mathbb{Q}}$.

(ii) Find a minimal positive integer N such that the restriction of π to the open subset $\mathrm{Spec}(\mathbb{Z}[1/N]) \subset \mathrm{Spec}(\mathbb{Z})$ is a smooth morphism.

(iii) Are there fibres of π that are non-reduced or reducible?

2. Curves on surfaces

Exercise 2. Let C be a smooth projective irreducible curve over an algebraically closed field k and let $S = C \times_k C$ with the two projections $p_i \colon S \longrightarrow C$, $i = 1, 2$.

(i) Show that $p_1^* \oplus p_2^* \colon \mathrm{Pic}(C) \oplus \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(S)$ is injective.

(ii) Fix a closed point $x \in C$ and consider the invertible sheaves $\mathcal{O}_S(a, b) := p_1^*\mathcal{O}_C(ax) \otimes p_2^*\mathcal{O}_C(bx)$. Decide for which a and b the sheaf $\mathcal{O}_S(a, b)$ is ample.

(iii) Consider the diagonal $C \cong \Delta \subset S$ and determine the self-intersection number $(\Delta.\Delta)$. Is $\mathcal{O}_S(\Delta)$ contained in the image of $p_1^* \oplus p_2^*$?

(iv) Compute the Hilbert polynomial of the curve Δ with respect to an ample invertible sheaf of the form $\mathcal{O}_S(a, b)|_\Delta$. Is it equal to the Hilbert polynomial of the sheaf $\mathcal{O}_S(\Delta)$ with respect to the ample invertible sheaf $\mathcal{O}_S(a, b)$?

3. Higher direct images

Exercise 3. In the situation of Exercise 2, let $\iota \colon \tilde{S} \longrightarrow S$ be the blow-up of the point $(x, x) \in S$ and consider the composition $f \colon \tilde{S} \xrightarrow{\ \iota\ } S \xrightarrow{\ p_1\ } C$

(i) Describe the maximal open subsets in S and C over which ι resp. f are flat. Answer the same question for 'flat' replaced by 'smooth'.

(ii) Determine the direct image sheaves $R^i f_* \mathcal{O}_{\tilde{S}}$. Do they satisfy base change? In other words, for which points $y \in C$ and for which i is the natural map

$$R^i f_* \mathcal{O}_{\tilde{S}} \otimes k(y) \longrightarrow H^i(\tilde{S}_y, \mathcal{O}_{\tilde{S}_y})$$

an isomorphism?

(iii) Describe a coherent sheaf $\mathcal{F}$ on $\tilde{S}$ that is not f-flat and such that $f(\mathrm{Supp}(\mathcal{F})) = C$. Determine and compare the functions $y \longmapsto h^i(\tilde{S}_y, \mathcal{F}_y)$ and $y \longmapsto \dim R^i f_* \mathcal{F} \otimes k(y)$ (say on k-rational points $y \in C$).

4. Étale morphisms

We study morphisms $f_i\colon X_i \longrightarrow Y_i$, $i = 1, 2$, between finite type k-schemes, where k is an algebraically closed field of characteristic 0.

Exercise 4. (i) Decide whether with f_i smooth (flat, unramified, or étale) also the product $f := f_1 \times f_2\colon X_1 \times_k X_2 \longrightarrow Y_1 \times_k Y_2$ of the f_i is smooth (flat, unramified, resp. étale). What about the composition $p_1 \circ f$ with the first projection $p_1\colon Y_1 \times_k Y_2 \longrightarrow Y_1$?

(ii) Assume the f_i are dominant morphisms between smooth integral k-schemes of the same dimension. Describe the ramification divisor R_f in terms of R_{f_i}. Describe $\omega_{X_1 \times_k X_2}$ in terms of $\omega_{Y_1 \times_k Y_2}$ and R_f.

(iii) Assume X and Y are smooth projective curves. Assume further that there exists an isomorphism $F\colon X \times_k \mathbb{P}^1 \longrightarrow Y \times_k \mathbb{P}^1$. Are X and Y isomorphic?

(iv) How many étale morphisms $\mathbb{P}^n \times_k \mathbb{P}^m \longrightarrow \mathbb{P}^N \times_k \mathbb{P}^M$ of degree at least 2 are there? (Hint: First, show that such a morphism can only exist if $\{n, m\} = \{N, M\}$.)