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Summer term 2021

Algebraic Geometry II – Exam 04.08.2021

Begin: 09:00 **End:** 12:00

Technical details:

– Upload a scan of your solutions (max. 500kb per page) until 12:00 to the eCampus folder 'Exams/Exam submission'. Solutions uploaded any later than 12:00 will not be considered which results in you failing the exam.

– The exam will be posted at 9:00 on the usual website `https://www.math.uni-bonn.de/people/gmartin/AlgebraicGeometrySS2021.htmpl` and will also be available in the folder 'Exams' in eCampus (where this document can be found already now).

– Please sign the declaration of honor below and scan it together with your solutions. Without it your solutions will not be considered.

– If at all possible, upload your solution in one(!) pdf document with the declaration of honor as the first page. Name the file **firstname.name.pdf**. Please, write your name on each sheet.

– In case of technical problems (but only then), you can also send the scan to gmartin@math.uni-bonn.de (until 12:00 am).

– Allow at least 30-45min for scanning and uploading your solutions.

Guidelines:

– The exam consists of applying your knowledge of algebraic geometry to fill in the missing arguments in the text below. In particular, reading and understanding the arguments that are given is part of the task.

– It is recommended to work through the exercises in the given order.

– The exam will only lead to pass or fail. It will not be graded. A complete solution will be posted on eCampus, so that you can find out yourself how well you did in the exam.

– All arguments have to be justified. Apart from standard material from commutative algebra, you have to deduce everything by only using results explicitly stated or used, either in the lectures, on the exercise sheets or in the text below.

– You may use available resources to solve the exercise. You are allowed to consult the notes of the class, the exercise sheets, online books, etc. **You are not allowed** to contact any other person during the exam (by email, phone, social media, etc.) or to discuss your answers with anyone before successfully uploading them to eCampus.

I hereby swear that I completed the examination detailed above completely on my own and without any impermissible external assistance or through the use of non-permitted aids. I am aware that cheating during the execution of an examination (as detailed in §63 Para 5 of the Higher Education Act NRW) is a violation of the legal regulations for examinations and an administrative offense. The submission of false affirmation in lieu of an oath is a criminal offense.

Signed at: **Date:**

Name: **Student ID:** **Signature:**

1. Spreading Let X be a projective scheme over a field K.

Exercise 1. (i) Consider a subfield $k \subset K$. Show that there exists an integral finite type k-scheme S and a flat, projective morphism $f\colon \mathcal{X} \longrightarrow S$ of k-schemes such that the function field of S is a sub-extension, i.e. $k \subset K(S) \subset K$, and the base change $(\mathcal{X}_\eta)_K$ of the generic fibre $\mathcal{X}_\eta$ considered as a scheme over $k(\eta) = K(S)$ is isomorphic to X.

(ii) If we drop 'flatness' in (i), prove that one can choose S to be projective.

(iii) Assume X is a smooth K-scheme. Can one choose $\mathcal{X} \longrightarrow S$ to be smooth (and projective)?

(iv) Consider the curve $X = V_+(F) \subset \mathbb{P}^2_K$ with $F = x_0^2x_2 - x_1^3 + tx_1x_2^2$ and $K = \mathbb{Q}(t)$. Try to find an explicit flat family $f\colon \mathcal{X} \longrightarrow S$ as above with S projective. Study one singular fibre.

Solution (i) Pick a closed embedding $X \subset \mathbb{P}^N_K$, so we write X as $V_+(F_1, \ldots, F_n)$ for finitely many homogenous polynomials $F_1, \ldots, F_n \in K[x_0, \ldots, x_N]$. Consider the finite type k-algebra A generated by the finitely many coefficients occurring in the finitely many polynomials F_i. Then we let $S \coloneqq \operatorname{Spec}(A)$ and we can use the same polynomials to define the closed subscheme $\mathcal{X} \coloneqq V_+(F_1, \ldots, F_n) \subset \mathbb{P}^N_A = \mathbb{P}^N_k \times_k \operatorname{Spec}(A)$. Clearly, $K(S)$ is the quotient field of A, a finitely generated field extension of k contained in K and the generic fibre $\mathcal{X}_\eta$ is a projective scheme over $K(S)$ described as $V_+(F_1, \ldots, F_n) \subset \mathbb{P}^N_{K(S)}$. Therefore, its base change to K gives back X.

Flatness is an open condition (Lecture 10), i.e. the set $U \subset \mathcal{X}$ of points $x \in \mathcal{X}$ for which $\mathcal{O}_{\mathcal{X},x}$ is flat over $\mathcal{O}_{S,f(x)}$ is open. The set is not empty, as it contains all the points in the generic fibre. The complement $Z \coloneqq \mathcal{X} \setminus U$ is closed and, since f is projective, its image is closed in S. However, $f(Z)$ does not contain the generic point and, therefore, $S \setminus f(Z)$ is open and non-empty. Thus, $f\colon \mathcal{X} \longrightarrow S$ restricted to $S' \coloneqq S \setminus f(Z)$ yields the desired flat projective family.

(ii) The (flat) family $\mathcal{X} \longrightarrow S$ constructed above, can be projectivized as follows: The S was an open subset of $\operatorname{Spec}(A)$ with A a finite type k-algebra, i.e. there exists a surjection $k[y_1, \ldots, y_m] \twoheadrightarrow A$. Hence, $\operatorname{Spec}(A) \subset \mathbb{A}^m_k \subset \mathbb{P}^m_k$ and we may simply take the closure of S inside $\mathbb{P}^m_k$ and the closure of $\mathcal{X} \subset \mathbb{P}^N_A$ inside $\mathbb{P}^N \times_k \mathbb{P}^m_k$.

(iii) Smoothness is also an open condition and, as in (i), we may restrict the original family $\mathcal{X} \longrightarrow S$ to some open subset $S' \subset S$ to obtain a smooth projective family.

(iv) The obvious choice is $S = \mathbb{P}^1_\mathbb{Q}$ and $\mathcal{X} \subset \mathbb{P}^2_\mathbb{Q} \times_\mathbb{Q} \mathbb{P}^1_\mathbb{Q}$ defined by the polynomial $t_0x_0^2x_2 - t_0x_1^3 + t_1x_1x_2^2$, where t_0, t_1 are the homogenous coordinates on $\mathbb{P}^1_\mathbb{Q}$. The fibre over $[1:0]$ is the curve with a cusp $x_0^2x_2 - x_1^3$ (on an affine chart given by $x^2 - y^3$). Flatness follows from the the fact that $\mathcal{X}$ is irreducible and S is one-dimensional (see Lecture 9). Alternatively, one could argue via the Hilbert polynomial which is constant in the family as all fibres are curves of degree 3.

2. Curves and surfaces For simplicity we assume that k is an algebraically closed field. Add simplifying assumptions on k, e.g. on the characteristic, when needed.

Exercise 2. Consider the surfaces $S_t \coloneqq V_+(x_0^4 + \cdots + x_3^4 + 4t\prod x_i) \subset \mathbb{P}^3_k$ depending on a parameter $t \in k$.

(i) Determine for which value of t the surface S_t is smooth. How would you define S_∞ and what are its properties?

(ii) Compute the Hilbert polynomial of the surfaces S_t and explain how to view the surfaces S_t as the fibres over k-rational points of a flat morphism $\mathcal{S} \longrightarrow \mathbb{P}^1$.

(iii) Consider the projection $\pi\colon S_0 \longrightarrow \mathbb{P}^2$, $[x_0 : x_1 : x_2 : x_3] \longmapsto [x_0 : x_1 : x_2]$ from the point $[0:0:0:1]$ onto the plane $V_+(x_3) \cong \mathbb{P}^2$. Is π flat? Describe the locus of points in $\mathbb{P}^2$ over which π is étale. Is there are a line $L \subset \mathbb{P}^2$ such that its pre-image in S_0 is smooth? If there is, how many such lines are there? Can you find one explicitly?

(iv) In the above situation, determine the higher direct image sheaves $R^i\pi_*\mathcal{O}_{S_0}$ and $\pi^!\mathcal{O}(-3)$. Describe the sheaf $\Omega_{S_0/\mathbb{P}^2}$.

Solution (i) We assume $\mathrm{char}(k) \neq 2$. The partial derivatives are $4x_i^3 + 4t\prod_{j\neq i} x_j$. Hence S_0 is smooth. Moreover, S_t, $t \neq 0$, is singular if and only if there exists a point $[a_0 : a_1 : a_2 : a_3]$ with $\sum a_i^4 + 4t\prod a_i = 0$ and $t\prod a_i = -a_j^4$ for $j = 0, \ldots, 3$. (In fact, by Euler's equation the first equation follows from the latter ones.) The latter equalities imply $a_0^4 = \cdots = a_3^4$ and $t^4 = 1$. Conversely, if $t^4 = 1$, then indeed S_t is singular. More explicitly, for $t = \pm 1$ the point $[\mp 1 : 1 : 1 : 1]$ is a singular point of S_t and for $t = \pm\sqrt{-1}$ the point $[\pm\sqrt{-1} : 1 : 1 : 1]$ is a singular point of S_t. Hence, S_t is singular if and only if $t^4 = 1$. The surface S_∞ is naturally defined as $V_+(\prod x_i)$, which is the union of four hyperplanes and in particular neither irreducible nor smooth.

(ii) The Hilbert polynomial of any quartic $S \subset \mathbb{P}^3$ is computed as follows: $P_S(n) \coloneqq \chi(S, \mathcal{O}_S(n)) = \chi(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) - \chi(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n-4)) = 2n^2 + 2$. Alternatively, one can use the Riemann–Roch formula $\chi(S, \mathcal{O}_S(n)) = (1/2)(\mathcal{O}_S(n), \mathcal{O}_S(n)) + 2 = (1/2)4n^2 + 2$. The universal family $\mathcal{S}_{\mathrm{univ}} \longrightarrow \mathbb{P} \coloneqq |\mathcal{O}_{\mathbb{P}^3}(4)|$ of quartic hypersurfaces in $\mathbb{P}^3$ is flat and the surfaces S_t are the fibres over $t \in \mathbb{P}^1(k)$ of the family $\mathcal{S} \coloneqq \mathcal{S}_{\mathrm{univ}} \times_{\mathbb{P}} \mathbb{P}^1$, where $\mathbb{P}^1 \longrightarrow \mathbb{P}$ is the linear embedding $[t_0 : t_1] \longmapsto t_0 \sum x_i + 4t_1 \prod x_i$. As flatness is preserved under base change, also $\mathcal{S} \longrightarrow \mathbb{P}^1$ is flat.

(iii) Since S_0 and $\mathbb{P}^2$ are both smooth, the morphism π is flat if and only if the fibre dimension is constant (miracle flatness, mentioned in Lecture 11). However, $\pi^*\mathcal{O}(1) = \mathcal{O}_{S_0}(1)$ is ample and trivial on all fibres of π, which therefore have to be all zero-dimensional. This implies the flatness of π. The fibre over a point $[a_0 : a_1 : a_2] \in \mathbb{P}^2$ consists of four reduced points if and only if $a_0^4 + a_1^4 + a_2^4 \neq 0$. Hence, the maximal open subset over which π is étale is $D_+(x_0^4 + x_1^4 + x_2^4)$. By Bertini, the pre-image of a non-empty open subset of lines in $\mathbb{P}^2$, which we can be viewed as zero sets of the linear system $H^0(\mathbb{P}^2, \mathcal{O}(1)) \subset H^0(S_0, \mathcal{O}_S(1))$, is smooth. Here one may want to assume $k = \bar{k}$ and $\mathrm{char}(k) = 0$, although this is not necessary. For example, the pre-image of the line $V_+(x_0)$ is smooth.

(iv) Since π is finite, we clearly have $R^i\pi_*\mathcal{O}_{S_0} = 0$ for $i > 0$. To compute $\pi_*\mathcal{O}_{S_0}$, observe that S_0 can be realized as the closed subscheme of $\mathbb{V}(\mathcal{O}(1))$ defined by $f^*(x_0^4 + x_1^4 + x_2^4) - t^4 \in H^0(\mathbb{V}(\mathcal{O}(1)), f^*\mathcal{O}(4))$. Here, $f\colon \mathbb{V}(\mathcal{O}(1)) \longrightarrow \mathbb{P}^2$ is the natural projection and $t \in H^0(\mathbb{V}(\mathcal{O}(1)), f^*\mathcal{O}(1))$ is the canonical section. Hence, S_0 is the relative spectrum $\mathbb{S}\mathrm{pec}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3))$ and, therefore, $\pi_*\mathcal{O}_{S_0} \cong \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3)$.

To describe $\Omega_{S_0/\mathbb{P}^2}$ we use the exact sequence $\pi^*\Omega_{\mathbb{P}^2} \longrightarrow \Omega_{S_0} \longrightarrow \Omega_{S_0/\mathbb{P}^2} \longrightarrow 0$. In fact, since the first two sheaves are locally free and the morphism is generically étale, this is a short exact sequence. Also recall the natural surjection $\Omega_{S_0}|_C \twoheadrightarrow \Omega_C$ for any curve $C \subset S_0$. Applied to the ramification curve $\pi\colon C \xrightarrow{\sim} V_+(x_0^4 + x_1^4 + x_2^4) \subset \mathbb{P}^2$, this eventually yields $\Omega_{S_0/\mathbb{P}^2} \cong i_*\omega_C \cong i_*\mathcal{O}_C(1)$, where $i\colon C \hookrightarrow S_0$ and $\mathcal{O}_C(1)$ is the restriction of $\mathcal{O}(1)$ on $\mathbb{P}^2$ to $V_+(x_0^4 + x_1^4 + x_2^4)$.

In class we have seen that $\pi^!\omega_{\mathbb{P}^2} \cong \omega_{S_0}$. Since $\omega_{\mathbb{P}^2} \cong \mathcal{O}(-3)$ and $\omega_{S_0} \cong \mathcal{O}_{S_0}$, this implies $\pi^!\mathcal{O}(-3) \cong \mathcal{O}_{S_0}$.

Exercise 3. Let $L \subset \mathbb{P}^2$ and $C \subset \mathbb{P}^2$ be a line and a smooth conic (i.e. a plane curve of degree two), respectively. In the following we shall consider $\mathbb{P}^2$ embedded as a hyperplane in $\mathbb{P}^3$.

(i) Assume L is contained in a smooth quartic $S \subset \mathbb{P}^3$, i.e. in a hypersurface of degree four. Show that for its self intersection number as a curve on S we have $(L.L) = -2$. Similarly, what is $(C.C)$ if the conic C is contained in S?

(ii) Assume L and C are contained in the same hyperplane $\mathbb{P}^2 \subset \mathbb{P}^3$ and in the same smooth quartic $S \subset \mathbb{P}^3$. Show that then $(L.C) = 2$. What can you say when L and C are contained in S but not necessarily in the same hyperplane?

(iii) In the situation of (i), describe the restriction of $\Omega_{S/k}$ to $L \subset S$. Compute $h^0(\Omega_S|_L)$.

(iv) Is it possible that a smooth quartic $S \subset \mathbb{P}^3$ contains a smooth elliptic curve $E \subset S$ (for example a complete intersection)?

Solution (i) By adjunction formula, the dualizing (canonical) sheaf of S is $\omega_S \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}(4))|_S \cong \mathcal{O}_S$. For any smooth curve $D \subset S$ the adjunction formula says $\omega_D \cong (\omega_S \otimes \mathcal{O}(D))|_D \cong \mathcal{O}(D)|_D$. Hence, $2g(D) - 2 = \deg(\omega_D) = \deg(\mathcal{O}(D)|_D)$. Applied to the line $D = L$ or to the conic $D = C$, which are both smooth rational curves, this yields $(L.L) = -2 = (C.C)$.

(ii) We know that $(L.C) = \dim(\mathcal{O}_{L\cap C})$ which is 2, as $L, C \subset \mathbb{P}^2$ intersect in two points (counted with multiplicities). This does not hold when L and C are not contained in the same hyperplane. In fact, in this case we have $(L.C) \leq 1$.

(iii) We use the short exact sequence $0 \longrightarrow \mathcal{T}_L \longrightarrow \mathcal{T}_S|_L \longrightarrow \mathcal{O}_L(L) \longrightarrow 0$. Using $L \cong \mathbb{P}^1$, adjunction formula tells us $\mathcal{O}_L(L) \cong \mathcal{O}(-2)$. Furthermore, $\mathcal{T}_L \cong \mathcal{O}(2)$. However, any extension $0 \longrightarrow \mathcal{O}(2) \longrightarrow E \longrightarrow \mathcal{O}(-2) \longrightarrow 0$ on $\mathbb{P}^1$ splits. Therefore, $\mathcal{T}_S|_L \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)$ and, in particular, $h^0(S, \mathcal{T}_S|_L) = 3$. Since $\Omega_S \cong \mathcal{T}_S \otimes \omega_S \cong \mathcal{T}_S$, this also answers the question for Ω_S.

(iv) Yes, this is possible or at least the adjunction formula only shows that for an elliptic curve $E \subset S$ one has $\mathcal{O}_E \cong \omega_E \cong (\omega_S \otimes \mathcal{O}(E))|_E$ and, therefore, $\mathcal{O}(E)|_E \cong \mathcal{O}_E$, which does not imply the contradiction $\mathcal{O}(E) \cong \mathcal{O}_S$. In fact, the quartic $S = V_+(x_0^4 - x_1^4 + x_2^4 - x_3^4)$ contains the elliptic curve given as the complete intersection $V_+(x_0^2 - x_1^2 + x_2^2 + x_3^2, x_0^2 + x_1^2 + x_2^2 - x_3^2)$.

3. Base change Let us consider a projective morphism $f\colon X \longrightarrow Y$ and a coherent sheaf $\mathcal{F}$ on X.

Exercise 4. We wish to explore the limits of the base change theorems discussed in class.

(i) Describe an example where $\mathcal{F}$ is not f-flat and the function $y \longmapsto h^i(X_y, \mathcal{F}_y)$ is not upper semi-continuous.

(ii) Describe a concrete example of a coherent sheaf $\mathcal{F}$ on X for which the fibre of $R^1f_*\mathcal{F}$ at a closed point $y \in Y$ is not isomorphic to $H^1(X_y, \mathcal{F}_y)$. Can you even find an example with f flat? Can this happen for f flat of relative dimension one?

(iii) We proved in class that for $\mathcal{F} \in \mathrm{Coh}(X)$ f-flat the function $y \longmapsto \chi(X_y, \mathcal{F}_y)$ is locally constant and $y \longmapsto h^i(X_y, \mathcal{F}_y)$ is upper-semicontinuous. What about the functions $y \longmapsto \sum(-1)^i \dim_{k(y)}(R^if_*\mathcal{F} \otimes k(y))$ and $y \longmapsto \dim_{k(y)}(R^if_*\mathcal{F} \otimes k(y))$?

We suggest to consider the second projection from a product $E \times E$ with E an elliptic curve and $\mathcal{F} = \mathrm{pr}_1^*\mathcal{O}(x)(-\Delta)$ for some closed point $x \in E$.

Solution (i) Let $f\colon X := \mathrm{Bl}_0(\mathbb{A}^2) \longrightarrow Y := \mathbb{A}^2$ be the blow-up of the origin with exceptional divisor E. Then $\mathcal{F} := \mathcal{O}(E)$ is locally free on X but not f-flat (since f is not flat). For $y \neq 0$ we have $h^0(X_y = \mathrm{Spec}(k(y)), \mathcal{F}_y \cong k(y)) = 1$ and for $y = 0$ one finds $h^0(X_y = E, \mathcal{F}_y \cong \mathcal{O}(-1)) = 0$.

(ii) We start with the standard example $g\colon X' := \mathrm{Bl}_0(\mathbb{A}^2) \longrightarrow Y := \mathbb{A}^2$ and $\mathcal{F}' := \mathcal{O}(E)$. The direct image of the exact sequence $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{O}(E)|_E \longrightarrow 0$ yields

$$0 \longrightarrow \mathcal{O} \cong g_*\mathcal{O} \longrightarrow g_*\mathcal{O}(E) \longrightarrow g_*(\mathcal{O}(E)|_E) = 0$$

and, hence, $g_*\mathcal{O}(E) \cong \mathcal{O}$ with fibre $k(0) \neq 0$ at 0 but $H^0(E \cong \mathbb{P}^1, \mathcal{O}(E)|_E \cong \mathcal{O}(-1)) = 0$. Eventually, let $X := X' \times C$, where C is an elliptic curve and $\mathcal{F}$ the pull-back of $\mathcal{F}'$.

It cannot happen for f flat of relative dimension one: In this case, we showed in Lecture 19 that the vanishing $H^2(X_y, \mathcal{F}_y) = 0$ (for dimension reasons) implies $R^1f_*\mathcal{F} \otimes k(y) \cong H^1(X_y, \mathcal{F}_y)$.

(iii) Since the second projection $f = \mathrm{pr}_2$ is flat and $\mathcal{F}$ is locally free, it is f-flat. The short exact sequence

$$0 \longrightarrow \mathcal{O}(-\Delta) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

on $E \times E$ tensored with $\mathrm{pr}_1^*\mathcal{O}(x)$ yields the long exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathrm{pr}_1^*\mathcal{O}(x) \xrightarrow{\eta} f_*(\mathcal{O}_\Delta \otimes \mathrm{pr}_1^*\mathcal{O}(x)) \longrightarrow R^1f_*\mathcal{F} \longrightarrow R^1\mathrm{pr}_1^*\mathcal{O}(x).$$

Now, $f_*\mathcal{F}$ has support contained in $\{x\}$ and by Künneth

$$f_*\mathrm{pr}_1^*\mathcal{O}(x) \cong H^0(E, \mathcal{O}(x)) \otimes \mathcal{O} \text{ and } R^1f_*\mathrm{pr}_1^*\mathcal{O}(x) \cong H^1(E, \mathcal{O}(x)) \otimes \mathcal{O} = 0.$$

Thus, as $f_*\mathcal{F} \subset f_*\mathrm{pr}_1^*\mathcal{O}(x)$, we must have $f_*\mathcal{F} = 0$. Furthermore, $f_*(\mathcal{O}_\Delta \otimes \mathrm{pr}_1^*\mathcal{O}(x)) \cong \mathcal{O}(x)$ and the map η is the evaluation map $H^0(E, \mathcal{O}(x)) \otimes \mathcal{O} \longrightarrow \mathcal{O}(x)$, which is nothing but the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}(x)$, whose cokernel is $k(x)$. We conclude that $f_*\mathcal{F} = 0$ and $R^1f_*\mathcal{F} \cong k(x)$. In particular, $y \longmapsto \sum(-1)^i \dim_{k(y)} R^if_*\mathcal{F}$ is constant zero on $E \setminus \{x\}$ and takes the value -1 at x, so it is not upper-semicontinuous let alone locally constant.

Since by Serre the sheaves $R^if_*\mathcal{F}$ are coherent and for any coherent sheaf $\mathcal{G}$ the map $y \longmapsto \dim_{k(y)}(\mathcal{G} \otimes k(y))$ is upper-semicontinuous, the function $y \longmapsto \dim_{k(y)} R^if_*\mathcal{F} \otimes k(y)$ is indeed upper semi-continuous.