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Summer term 2021

Algebraic Geometry II – Exam 04.08.2021

Begin: 09:00 **End:** 12:00

Technical details:

– Upload a scan of your solutions (max. 500kb per page) until 12:00 to the eCampus folder 'Exams/Exam submission'. Solutions uploaded any later than 12:00 will not be considered which results in you failing the exam.

– The exam will be posted at 9:00 on the usual website `https://www.math.uni-bonn.de/people/gmartin/AlgebraicGeometrySS2021.htmpl` and will also be available in the folder 'Exams' in eCampus (where this document can be found already now).

– Please sign the declaration of honor below and scan it together with your solutions. Without it your solutions will not be considered.

– If at all possible, upload your solution in one(!) pdf document with the declaration of honor as the first page. Name the file **firstname.name.pdf**. Please, write your name on each sheet.

– In case of technical problems (but only then), you can also send the scan to gmartin@math.uni-bonn.de (until 12:00 am).

– Allow at least 30-45min for scanning and uploading your solutions.

Guidelines:

– The exam consists of applying your knowledge of algebraic geometry to fill in the missing arguments in the text below. In particular, reading and understanding the arguments that are given is part of the task.

– It is recommended to work through the exercises in the given order.

– The exam will only lead to pass or fail. It will not be graded. A complete solution will be posted on eCampus, so that you can find out yourself how well you did in the exam.

– All arguments have to be justified. Apart from standard material from commutative algebra, you have to deduce everything by only using results explicitly stated or used, either in the lectures, on the exercise sheets or in the text below.

– You may use available resources to solve the exercise. You are allowed to consult the notes of the class, the exercise sheets, online books, etc. **You are not allowed** to contact any other person during the exam (by email, phone, social media, etc.) or to discuss your answers with anyone before successfully uploading them to eCampus.

I hereby swear that I completed the examination detailed above completely on my own and without any impermissible external assistance or through the use of non-permitted aids. I am aware that cheating during the execution of an examination (as detailed in §63 Para 5 of the Higher Education Act NRW) is a violation of the legal regulations for examinations and an administrative offense. The submission of false affirmation in lieu of an oath is a criminal offense.

Signed at: **Date:**

Name: **Student ID:** **Signature:**

1. Spreading Let X be a projective scheme over a field K.

Exercise 1. (i) Consider a subfield $k \subset K$. Show that there exists an integral finite type k-scheme S and a flat, projective morphism $f\colon \mathcal{X} \longrightarrow S$ of k-schemes such that the function field of S is a sub-extension, i.e. $k \subset K(S) \subset K$, and the base change $(\mathcal{X}_\eta)_K$ of the generic fibre $\mathcal{X}_\eta$ considered as a scheme over $k(\eta) = K(S)$ is isomorphic to X.

(ii) If we drop 'flatness' in (i), prove that one can choose S to be projective.

(iii) Assume X is a smooth K-scheme. Can one choose $\mathcal{X} \longrightarrow S$ to be smooth (and projective)?

(iv) Consider the curve $X = V_+(F) \subset \mathbb{P}^2_K$ with $F = x_0^2x_2 - x_1^3 + tx_1x_2^2$ and $K = \mathbb{Q}(t)$. Try to find an explicit flat family $f\colon \mathcal{X} \longrightarrow S$ as above with S projective. Study one singular fibre.

2. Curves and surfaces For simplicity we assume that k is an algebraically closed field. Add simplifying assumptions on k, e.g. on the characteristic, when needed.

Exercise 2. Consider the surfaces $S_t \coloneqq V_+(x_0^4 + \cdots + x_3^4 + 4t\prod x_i) \subset \mathbb{P}^3_k$ depending on a parameter $t \in k$.

(i) Determine for which value of t the surface S_t is smooth. How would you define S_∞ and what are its properties?

(ii) Compute the Hilbert polynomial of the surfaces S_t and explain how to view the surfaces S_t as the fibres over k-rational points of a flat morphism $\mathcal{S} \longrightarrow \mathbb{P}^1$.

(iii) Consider the projection $\pi\colon S_0 \longrightarrow \mathbb{P}^2$, $[x_0 : x_1 : x_2 : x_3] \longmapsto [x_0 : x_1 : x_2]$ from the point $[0 : 0 : 0 : 1]$ onto the plane $V_+(x_3) \cong \mathbb{P}^2$. Is π flat? Describe the locus of points in $\mathbb{P}^2$ over which π is étale. Is there are a line $L \subset \mathbb{P}^2$ such that its pre-image in S_0 is smooth? If there is, how many such lines are there? Can you find one explicitly?

(iv) In the above situation, determine the higher direct image sheaves $R^i\pi_*\mathcal{O}_{S_0}$ and $\pi^!\mathcal{O}(-3)$. Describe the sheaf $\Omega_{S_0/\mathbb{P}^2}$.

Exercise 3. Let $L \subset \mathbb{P}^2$ and $C \subset \mathbb{P}^2$ be a line and a smooth conic (i.e. a plane curve of degree two), respectively. In the following we shall consider $\mathbb{P}^2$ embedded as a hyperplane in $\mathbb{P}^3$.

(i) Assume L is contained in a smooth quartic $S \subset \mathbb{P}^3$, i.e. in a hypersurface of degree four. Show that for its self intersection number as a curve on S we have $(L.L) = -2$. Similarly, what is $(C.C)$ if the conic C is contained in S?

(ii) Assume L and C are contained in the same hyperplane $\mathbb{P}^2 \subset \mathbb{P}^3$ and in the same smooth quartic $S \subset \mathbb{P}^3$. Show that then $(L.C) = 2$. What can you say when L and C are contained in S but not necessarily in the same hyperplane?

(iii) In the situation of (i), describe the restriction of $\Omega_{S/k}$ to $L \subset S$. Compute $h^0(\Omega_S|_L)$.

(iv) Is it possible that a smooth quartic $S \subset \mathbb{P}^3$ contains a smooth elliptic curve $E \subset S$ (for example a complete intersection)?

3. Base change Let us consider a projective morphism $f\colon X \longrightarrow Y$ and a coherent sheaf $\mathcal{F}$ on X.

Exercise 4. We wish to explore the limits of the base change theorems discussed in class.

(i) Describe an example where $\mathcal{F}$ is not f-flat and the function $y \longmapsto h^i(X_y, \mathcal{F}_y)$ is not upper semi-continuous.

(ii) Describe a concrete example of a coherent sheaf $\mathcal{F}$ on X for which the fibre of $R^1 f_*\mathcal{F}$ at a closed point $y \in Y$ is not isomorphic to $H^1(X_y, \mathcal{F}_y)$. Can you even find an example with f flat? Can this happen for f flat of relative dimension one?

(iii) We proved in class that for $\mathcal{F} \in \mathrm{Coh}(X)$ f-flat the function $y \longmapsto \chi(X_y, \mathcal{F}_y)$ is locally constant and $y \longmapsto h^i(X_y, \mathcal{F}_y)$ is upper-semicontinuous. What about the functions $y \longmapsto \sum(-1)^i \dim_{k(y)}(R^i f_*\mathcal{F} \otimes k(y))$ and $y \longmapsto \dim_{k(y)}(R^i f_*\mathcal{F} \otimes k(y))$?

We suggest to consider the second projection from a product $E \times E$ with E an elliptic curve and $\mathcal{F} = \mathrm{pr}_1^*\mathcal{O}(x)(-\Delta)$ for some closed point $x \in E$.