

Exercises, Algebraic Geometry II – Week 9

Exercise 46. Testing isomorphisms on local rings (3 points)
Let $f : X \to Y$ be a morphism of finite type with $Y$ locally Noetherian. Suppose that for a point $y \in Y$ the induced morphism $X \times_Y \text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism. Show that then there exists an open neighbourhood $y \in U \subset Y$ such that the induced morphism $X \times_Y U \to U$ is an isomorphism. See [Liu, Ex. 3.2.5] for a relative version of this statement.

Exercise 47. Stein factorization and geometrically reduced fibers (4 points)
Let $k$ be a field, $X$ a normal variety over $k$ and $C$ an integral normal curve over $k$. Let $f : X \to C$ be a proper surjective morphism over $k$ and let $f' : C' \to C$ be the finite morphism in the Stein factorization $X \to C' \to C$ of $f$.

(i) Show that $C'$ is normal and $f'$ is flat.

(ii) Let $c' \in C'$ be a ramification point of $f'$. Show that the corresponding fiber $f^{-1}(f'(c'))$ is geometrically non-reduced. In particular, note that if all fibers of $f$ are geometrically reduced, then $f'$ is étale.

(iii) Show that if $C = \mathbb{P}^1_k$ and all fibers of $f$ are geometrically reduced, then $f_* \mathcal{O}_X = \mathcal{O}_C$. In particular, the fibers of $f$ are connected.

Exercise 48. (Dis)connected fibres (2 points)
Find examples of morphisms $f : X \to Y$ between integral schemes with $\mathcal{O}_Y \neq f_* \mathcal{O}_X$ with all fibres being (geometrically) connected resp. all fibres being disconnected.

Exercise 49. Morphisms between curves (3 points)
Let $f : C \to D$ be a finite morphism between reduced projective curves over a field $k$. Show that $f$ is an isomorphism if and only if the natural map $\mathcal{O}_D \to f_* \mathcal{O}_C$ is an isomorphism. Compare this to Hurwitz formula.

Exercise 50. Local analytic invariants (4 points)
Let $C$ be an integral curve over a field $k$ and let $f : \tilde{C} \to C$ be its normalization. For a closed point $x \in C$ define $\delta_x := \text{length}(\mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x})$, where $\mathcal{O}_{C,x} \subset \mathcal{O}_{\tilde{C},x} \subset K(C)$ is the normalization. Show that for two curves $(C, x)$ and $(C', x')$ with $\tilde{\mathcal{O}}_{C,x} \cong \tilde{\mathcal{O}}_{C',x'}$ one has $\delta_x = \delta_{x'}$.

Information from the Student Council:
This year the Maths Summer Party will take place virtually on Friday, June 25, starting at 18 c.t.. Latest information can be found on https://fsmath.uni-bonn.de/events-detail/events/sommerfest-kopie.html. Come by!

Due Friday 25 June 2021.