Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Summer term 2021

Exercises, Algebraic Geometry II – Week 9

Exercise 46. *Testing isomorphisms on local rings* (3 points)
Let $f\colon X \to Y$ be a morphism of finite type with Y locally Noetherian. Suppose that for a point $y \in Y$ the induced morphism $X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}) \to \operatorname{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism. Show that then there exists an open neighbourhood $y \in U \subset Y$ such that the induced morphism $X \times_Y U \to U$ is an isomorphism. See [Liu, Ex. 3.2.5] for a relative version of this statement.

Exercise 47. *Stein factorization and geometrically reduced fibers* (4 points)
Let k be a field, X a normal variety over k and C an integral normal curve over k. Let $f\colon X \to C$ be a proper surjective morphism over k and let $f'\colon C' \to C$ be the finite morphism in the Stein factorization $X \to C' \to C$ of f.

(i) Show that C' is normal and f' is flat.

(ii) Let $c' \in C'$ be a ramification point of f'. Show that the corresponding fiber $f^{-1}(f'(c'))$ is geometrically non-reduced. In particular, note that if all fibers of f are geometrically reduced, then f' is étale.

(iii) Show that if $C = \mathbb{P}^1_k$ and all fibers of f are geometrically reduced, then $f_*\mathcal{O}_X = \mathcal{O}_C$. In particular, the fibers of f are connected.

Exercise 48. *(Dis)connected fibres* (2 points)
Find examples of morphisms $f\colon X \to Y$ between integral schemes with $\mathcal{O}_Y \ncong f_*\mathcal{O}_X$ with all fibres being (geometrically) connected resp. all fibres being disconnected.

Exercise 49. *Morphisms between curves* (3 points)
Let $f\colon C \to D$ be a finite morphism between reduced projective curves over a field k. Show that f is an isomorphism if and only if the natural map $\mathcal{O}_D \to f_*\mathcal{O}_C$ is an isomorphism. Compare this to Hurwitz formula.

Exercise 50. *Local analytic invariants* (4 points)
Let C be an integral curve over a field k and let $f\colon \tilde{C} \to C$ be its normalization. For a closed point $x \in C$ define $\delta_x := \operatorname{length}(\mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x})$, where $\mathcal{O}_{C,x} \subset \mathcal{O}_{\tilde{C},x} \subset K(C)$ is the normalization. Show that for two curves (C,x) and (C',x') with $\widehat{\mathcal{O}}_{C,x} \cong \widehat{\mathcal{O}}_{C',x'}$ one has $\delta_x = \delta_{x'}$.

Information from the Student Council:

This year the Maths Summer Party will take place virtually on Friday, June 25, starting at 18 c.t.. Latest information can be found on

https://fsmath.uni-bonn.de/events-detail/events/sommerfest-kopie.html.

Come by!

Due Friday 25 June 2021.