Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin Summer term 2021

Exercises, Algebraic Geometry II – Week 8

Exercise 41. *Étale vs. Zariski locally trivial* (4 points)
Let $f\colon \mathcal{C} \to |\mathcal{O}(2)|_{\rm sm}$ be the universal family of smooth conics in $\mathbb{P}^2_k$ with $k = \bar{k}$ and $\mathrm{char}(k) \neq 2$. Then each closed fibre is isomorphic to $\mathbb{P}^1_k$. Show that f is locally trivial in the étale topology but not in the Zariski topology.

Exercise 42. *Plane quartics are not hyperelliptic* (4 points)
Let $C \subset \mathbb{P}^2_k$ with $k = \bar{k}$ be a smooth curve of degree 4. Assume $\mathrm{char}(k) \neq 2$.

(i) Show that every divisor in $|K_C|$ is of the form $C \cap L$ for some line $L \subset \mathbb{P}^2_k$. Deduce from this that for every point $P \in C$, there is a unique effective divisor D such that $2P + D \in |K_C|$.

(ii) Let D be an effective divisor on C with $2D \in |K_C|$. Show that $h^0(C, \mathcal{O}(D)) = 1$. Conclude that C is not hyperelliptic.

Exercise 43. *(Dis)connected fibres* (4 points)
Consider a morphism $f\colon X \to Y$.

(i) Show that the property $\mathcal{O}_Y \cong f_*\mathcal{O}_X$ (under the natural map) is stable under flat base change.

(ii) Show that the property $\mathcal{O}_Y \cong f_*\mathcal{O}_X$ (under the natural map) is not necessarily stable under arbitrary base change.

(iii) Show that having connected fibres is not even stable under flat base change.

Exercise 44. *Flatness of the Frobenius* (3 points)
Let X be a Noetherian scheme over $\mathbb{F}_p$ and let $F_X : X \to X$ be the absolute Frobenius. Show that F_X is flat if X is regular.

(By a theorem of Kunz, the converse holds as well: If F_X is flat, then X is regular.)

Exercise 45. *Analytically isomorphic singularities* (4 points)
Let k be an algebraically closed field of characteristic $\neq 2$. Show that the completion of the local rings at the unique singular points of the curves $\mathrm{Spec}(k[x,y]/(xy))$ and $\mathrm{Spec}(k[x,y]/(x^2 + x^3 - y^2))$ are isomorphic. What about the singularity of $\mathrm{Spec}(k[x,y]/(y^2 - x^4))$?

Due Friday 18 June 2021.