Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Summer term 2021

Exercises, Algebraic Geometry II – Week 7

Exercise 35. *Unramified morphisms* (5 points)
In class we proved that a morphism locally of finite type $f\colon X \to Y$ between locally Noetherian schemes is unramified if and only if $\Omega_{X/Y} = 0$. Prove that this is also equivalent to the diagonal morphism $\Delta\colon X \to X \times_Y X$ being an open immersion.

(Hint: You can check whether f is unramified on geometric fibers.)

Exercise 36. *Composition of étale and unramified morphisms* (3 points)
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms such that $g \circ f$ is étale and g is unramified. Show that then also f is étale.

Exercise 37. *Étale morphisms* (4 points)
Decide which of the following morphisms is étale or at least unramified.

(i) $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k \setminus \{0\}$, $t \mapsto t^2$.

(ii) $\mathbb{P}^n_k \to \mathbb{P}^n_k$, $[x_0 : \cdots : x_n] \mapsto [x_0^\ell : \cdots : x_n^\ell]$ where $l > 1$.

(iii) $\mathrm{Spec}(\mathcal{O}_{\mathbb{Q}(\sqrt{5})}) \to \mathrm{Spec}(\mathcal{O}_{\mathbb{Q}})$.

(iv) $\mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x,y]/(x^3 - y^2))$ given by $x \mapsto t^2$, $y \mapsto t^3$.

Exercise 38. *Ramification divisor of a blow-up* (3 points)
Let X be a smooth variety over an algebraically closed field k and let $p \in X$ be a closed point. Calculate the ramification divisor of the blow-up $\pi : \mathrm{Bl}_p(X) \to X$.

Exercise 39. *Taking roots of sections* (5 points)
Let X be a smooth variety over a field k. Fix a section $0 \neq s \in H^0(X, \mathcal{L}^n)$, where $\mathcal{L}$ is an invertible sheaf on X. Let $\tilde{\pi}\colon \mathbb{V}(\mathcal{L}^*) := \mathrm{Spec}(\bigoplus_{i\leq 0} \mathcal{L}^i)$ be the vector bundle associated with $\mathcal{L}^*$ (see last semester) and define $Y = V(\tilde{\pi}^* s - \tilde{t}^n)$, where $\tilde{t} \in H^0(\mathbb{V}(\mathcal{L}), \tilde{\pi}^*\mathcal{L}) = H^0(X, \mathcal{L} \otimes \tilde{\pi}_*\mathcal{O}_{\tilde{X}}) = H^0(X, \mathcal{L} \otimes \bigoplus_{i\leq 0} \mathcal{L}^i)$ is the natural trivializing section of $\mathcal{L} \otimes \mathcal{L}^*$ and $V(\tilde{\pi}^* s - \tilde{t}^n)$ is the vanishing locus of the section $\tilde{\pi}^* s - \tilde{t}^n$. In particular, note that $t := \tilde{t}|_Y$ satisfies $t^n = \pi^* s \in H^0(Y, \pi^*\mathcal{L}^n)$, where $\pi : Y \to X$ is the restriction of $\tilde{\pi}$ to Y. Assume that $\mathrm{char}(k)$ and n are coprime.

(i) Show that if $X = \mathrm{Spec}\ A$ is affine and $\mathcal{L} = \mathcal{O}_X$, then $Y \cong \mathrm{Spec}\ A[t]/(s - t^n)$ and π is induced by the natural inclusion $A \hookrightarrow A[t]/(s - t^n)$. Deduce that, for general X and $\mathcal{L}$, π is finite, separable, and surjective of degree n.

(ii) If $n > 1$, show that π is ramified precisely over $V(s)$.

Due Friday 11 June 2021.

(iii) Show that Y is smooth if and only if $V(s)$ is smooth (or empty).

(iv) If Y is smooth and integral, calculate the ramification divisor of π and describe the canonical bundle of Y in terms of X and $\mathcal{L}$.

(v) Write down a finite flat morphism $\pi : Y \to \mathbb{P}^2_{\mathbb{C}}$ of degree 2 such that Y is smooth and ω_Y is trivial.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 40. *Double covers* (4 points)
Assume that X is a complete, irreducible, smooth variety over an algebraically closed field k of characteristic $\neq 2$. Let $\pi : Y \to X$ be a finite flat morphism of degree 2.

(i) Show that the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ splits. In particular, note that $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{L}^{-1}$ for some $\mathcal{L} \in \operatorname{Pic}(X)$.

(Hint: Construct a retraction of the natural map by gluing trace maps.)

(ii) Deduce that π arises via the construction of Exercise 39 for some $s \in H^0(X, \mathcal{L}^{\otimes 2})$.

(iii) Assume that $V(s)$ is empty. Show that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$, $\pi^*\mathcal{L} \cong \mathcal{O}_Y$. Show that Y is integral if and only if $\mathcal{L} \not\cong \mathcal{O}_X$.

(iv) Conclude that every étale double cover of projective space $\mathbb{P}^n_k$ is trivial, that is, isomorphic to two disjoint copies of $\mathbb{P}^n_k$.

(In fact, the analogous statement is true for étale covers of $\mathbb{P}^n_k$ of arbitrary degree, that is, $\mathbb{P}^n_k$ is *algebraically simply connected*)