Exercise 30. Explicit Bertini (4 points)
(i) Show that Bertini’s theorem implies that for an algebraically closed field $k$ of characteristic 0 there exist smooth hypersurfaces of any degree in $\mathbb{P}^n_k$.
(ii) Can you prove the statement without the assumptions on $k$ by writing down explicit equations? (Dropping the first hypothesis is easier.)
(iii) Give an example of a smooth hypersurface $X \subset \mathbb{P}^n_k$ such that all hyperplane sections of $X$ are smooth. Is this a general feature?

Exercise 31. Flat covering of the node (4 points)
Let $k$ be a field of characteristic different from 2. Consider the affine nodal cubic curve $C := V(x_2^2 - x_1^2(x_1 + 1)) \subset \mathbb{A}^2_k$. Construct a curve $C'$ together with a flat proper morphism $f: C' \to C$ such that $C'$ has two irreducible components and the restriction of $f$ to each of them describes the normalization of $C$. Draw a picture!
(Hint: Use the map $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $x_1 \mapsto y_2^2 - 1, x_2 \mapsto y_1y_2$ and apply Exercise 24.)

Exercise 32. Bertini with base points (2 points)
Let $X$ be a smooth projective variety over an algebraically closed field of characteristic 0. Assume that $\dim(X) \geq 1$. Show that any two closed points $x, y \in X$ are contained in a smooth (at least away from $x$ and $y$) irreducible curve on $X$.

Exercise 33. Representing invertible sheaves by smooth divisors (3 points)
Let $X$ be a smooth projective variety over an algebraically closed field of characteristic 0. Show that for an invertible sheaf $L$ one always finds two smooth effective divisors $D, E \subset X$ with smooth intersection $D \cap E$ such that $L \cong \mathcal{O}(D - E)$. Try to generalize this statement appropriately to the case of two invertible sheaves.

Exercise 34. Hurwitz formula (4 points)
Prove the following special case of the Hurwitz formula which will be proved in full generality later: Assume $f: C \to C'$ is a finite morphism of degree two between irreducible smooth projective curves over an algebraically closed field of characteristic 0. Assume $U = C' \setminus \{x_1, \ldots, x_r\}$ (with pairwise distinct points $x_i$) is the maximal open set over which $f$ is smooth. Show that then $\deg(\omega_C) = 2 \deg(\omega_{C'}) + r$. Deduce from this that $r$ is even.

Please turn over.
1. Let $X$ be a scheme. For which points $x \in X$ is $\text{Spec}(k(x)) \to X$ a flat morphism.
2. Are there varieties that are neither projective nor quasi-projective nor quasi-affine?
3. Describe an example of a birational morphism $f : X \to Y$ whose image is neither open nor closed.
4. Write down an example of a field extension $K_1 \subset K_2$ with $K_2/K_1$ algebraic but $\Omega_{K_2/K_1} \neq 0$.
5. Let $A$ be a $k$-algebra. Compare $\Omega_{k[x_1, \ldots, x_n]/k}$ with $\Omega_{A[x_1, \ldots, x_n]/A}$.
6. Let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ and $x \in \text{Spec}(k[x_1, \ldots, x_n])$. Where does the Jacobian $J_x$ live?
7. Let $X$ be a scheme and $x \in X$. Compare $\dim_{k(x)} T_{X,x}$ and $\dim O_{X,x}$.
8. Let $X$ be a scheme over a field $k$. What is the relation between smoothness of $X$ over $k$ and regularity of $X_k$?
9. Consider morphisms of schemes $f : X \to Y$ and $g : Y \to Z$. Is the natural morphism $f^* \Omega_{Y/Z} \to \Omega_{X/Y}$ always injective?
10. In what sense is Sard’s theorem in algebraic geometry much better than in differential topology?
11. Let $X$ be an irreducible scheme of finite type over a field $k$. Is $X$ smooth over $k$ if $\Omega_{X/k}$ is locally free?
12. What is the link between the function field $K(X)$ and rational maps $X \to \mathbb{P}^1$?
13. Spot the intruder: $\mathbb{P}^2$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, $V_+(x_0^2 + \cdots + x_3^2) \subset \mathbb{P}^3$, $V_+(x_0^2 + \cdots + x_2^2) \subset \mathbb{P}^3$, $\mathbb{P}^1 \times V_+(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^1 \times \mathbb{P}^2$.
14. Let $X$ be an integral scheme of finite type over a field $k$ and $x \in X$. Compare $\dim_{K(X)} \Omega_{K(X)/k}$ and $\dim_{k(x)} (\Omega_{X/k} \otimes k(x))$.
15. Find an example of a DVR $(A, m)$ and an $A$-module $M$ such that $\dim_{Q(A)} (M \otimes_{A} Q(A)) \neq \dim_{A/m} (M \otimes_{A} A_m)$.
16. Describe the modules $\Omega_{k/k}$, $\Omega_{k[x]/(x^2)/k}$ and $\Omega_{k(x)/k}$.
17. Find an example of a non-empty, integral, finite type $k$-scheme $X$ for which there exists no non-empty open subset $U \subset X$ which is smooth over $k$.
18. Is there a natural map $d : O_X \to \Omega_{X/k}$?
19. Compare the two notions of smoothness, for morphisms $X \to Y$ and for a $k$-scheme.
20. Is a flat morphism always surjective? Can the image of a flat morphism be closed?
21. What is the canonical bundle $\omega_{X/k}$ of $X = \mathbb{P}^1 \times_k \mathbb{P}^m$?
22. Is there any relation between $\dim(X)$, $\text{rk}(\omega_{X/k})$, and $\text{trdeg}(K(X)/k)$?
23. What causes the problems when one wants to compare the Zariski tangent space $T_{X,x}$ and the fibre $\Omega_{X/k} \otimes k(x)$?
24. Give an example of a regular ring and of a non-regular ring. What is the easiest/standard example of a regular non-smooth $k$-scheme? Can you think of one other example?
25. What is the Hilbert polynomial of a reduced $k$-scheme consisting of two points? Does the answer depend on the field $k$ or on the projective embedding?
26. Give examples of modules that are flat and of those that are not over the two ring $k[x_1, x_2]$ and $k[x]/x^3$.
27. Are the function fields of the Fermat curves $V_+(x_0^d + x_1^d + x_2^d) \subset \mathbb{P}^2$ isomorphic for all $d$?
28. Can $\omega_{X/k}$ be a trivial invertible sheaf without $\Omega_{X/k}$ being a trivial locally free sheaf?
29. What is meant by the phrase ‘flatness’ is preserved by base change?
30. Which of the following properties of a scheme over a field are preserved by passing to a field extension: smooth, regular, integral, irreducible, reduced? What if the field has characteristic 0?
31. Which of the following properties of a flat morphism $f$ can you check by checking it on fibers over closed points: smooth, proper, finite? Can you also do that without assuming that $f$ is flat?
32. Is non-flatness always detected by the jump in the fibre dimension?
33. What is the relation between the Zariski tangent space $T_{X,x}$ and the fibre of the tangent sheaf $\mathcal{T}_X$ at $x \in X$? What are the assumptions on $X$?
34. Recall the two standard right exact sequences involving $\Omega_{V/A}$ and the corresponding sheaf versions.
35. How do you pass from the ideal sheaf of the diagonal $\Delta : X \to X \times_X X$ to $\Omega_{X/Y}$?
36. Have you memorized the Euler sequence?

1Do not submit solutions. If you cannot answer all the questions (without much thinking) the first time, do it again.