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Summer term 2021

Exercises, Algebraic Geometry II – Week 6

Exercise 30. *Explicit Bertini* (4 points)
(i) Show that Bertini's theorem implies that for an algebraically closed field k of characteristic 0 there exist smooth hypersurfaces of any degree in $\mathbb{P}^n_k$.

(ii) Can you prove the statement without the assumptions on k by writing down explicit equations? (Dropping the first hypothesis is easier.)

(iii) Give an example of a smooth hypersurface $X \subset \mathbb{P}^n_k$ such that all hyperplane sections of X are smooth. Is this a general feature?

Exercise 31. *Flat covering of the node* (4 points)
Let k be a field of characteristic different from 2. Consider the affine nodal cubic curve $C := V(x_2^2 - x_1^2(x_1+1)) \subset \mathbb{A}^2_k$. Construct a curve C' together with a flat proper morphism $f\colon C' \to C$ such that C' has two irreducible components and the restriction of f to each of them describes the normalization of C. Draw a picture!

(Hint: Use the map $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $x_1 \mapsto y_2^2 - 1, x_2 \mapsto y_1 y_2$ and apply Exercise 24.)

Exercise 32. *Bertini with base points* (2 points)
Let X be a smooth projective variety over an algebraically closed field of characteristic 0. Assume that $\dim(X) \geq 1$. Show that any two closed points $x, y \in X$ are contained in a smooth (at least away from x and y) irreducible curve on X.

Exercise 33. *Representing invertible sheaves by smooth divisors* (3 points)
Let X be a smooth projective variety over an algebraically closed field of characteristic 0. Show that for an invertible sheaf $\mathcal{L}$ one always finds two smooth effective divisors $D, E \subset X$ with smooth intersection $D \cap E$ such that $\mathcal{L} \cong \mathcal{O}(D - E)$. Try to generalize this statement appropriately to the case of two invertible sheaves.

Exercise 34. *Hurwitz formula* (4 points)
Prove the following special case of the Hurwitz formula which will be proved in full generality later: Assume $f\colon C \to C'$ is a finite morphism of degree two between irreducible smooth projective curves over an algebraically closed field of characteristic 0. Assume $U = C' \setminus \{x_1, \ldots, x_r\}$ (with pairwise distinct points x_i) is the maximal open set over which f is smooth. Show that then $\deg(\omega_C) = 2\deg(\omega_{C'}) + r$. Deduce from this that r is even.

Please turn over.

Due Friday 04 June 2021.

Easy questions to test your reflexes[1]

1. Let X be a scheme. For which points $x \in X$ is $\mathrm{Spec}(k(x)) \to X$ a flat morphism.
2. Are there varieties that are neither projective nor quasi-projective nor quasi-affine?
3. Describe an example of a birational morphism $f\colon X \to Y$ whose image is neither open nor closed.
4. Write down an example of a field extension $K_1 \subset K_2$ with K_2/K_1 algebraic but $\Omega_{K_2/K_1} \neq 0$.
5. Let A be a k-algebra. Compare $\Omega_{k[x_1,\ldots,x_n]/k}$ with $\Omega_{A[x_1,\ldots,x_n]/A}$.
6. Let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ and $x \in \mathrm{Spec}(k[x_1, \ldots, x_n])$. Where does the Jacobian J_x live?
7. Let X be a scheme and $x \in X$. Compare $\dim_{k(x)} T_{X,x}$ and $\dim \mathcal{O}_{X,x}$.
8. Let X be a scheme over a field k. What is the relation between smoothness of X over k and regularity of $X_{\bar{k}}$?
9. Consider morphisms of schemes $f\colon X \to Y$ and $g\colon Y \to Z$. Is the natural morphism $f^*\Omega_{Y/Z} \to \Omega_{X/Y}$ always injective?
10. In what sense is Sard's theorem in algebraic geometry much better than in differential topology?
11. Let X be an irreducible scheme of finite type over a field k. Is X smooth over k if $\Omega_{X/k}$ is locally free?
12. What is the link between the function field $K(X)$ and rational maps $X \to \mathbb{P}^1$?
13. Spot the intruder: $\mathbb{P}^2$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, $V_+(x_0^2 + \cdots + x_3^2) \subset \mathbb{P}^3$, $V_+(x_0^4 + \cdots + x_3^4) \subset \mathbb{P}^3$, $\mathbb{P}^1 \times V_+(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^1 \times \mathbb{P}^2$.
14. Let X be an integral scheme of finite type over a field k and $x \in X$. Compare $\dim_{K(X)} \Omega_{K(X)/k}$ and $\dim_{k(x)}(\Omega_{X/k} \otimes k(x))$.
15. Find an example of a DVR $(A, \mathfrak{m})$ and an A-module M such that $\dim_{Q(A)}(M \otimes_A Q(A)) \neq \dim_{A/\mathfrak{m}}(M \otimes_A A_{\mathfrak{m}})$.
16. Describe the modules $\Omega_{k/k}$, $\Omega_{k[x]/(x^2)/k}$ and $\Omega_{k(x)/k}$.
17. Find an example of a non-empty, integral, finite type k-scheme X for which there exists no non-empty open subset $U \subset X$ which is smooth over k.
18. Is there a natural map $d\colon \mathcal{O}_X \to \Omega_{X/k}$?
19. Compare the two notions of smoothness, for morphisms $X \to Y$ and for a k-scheme.
20. Is a flat morphism always surjective? Can the image of a flat morphism be closed?
21. What is the canonical bundle $\omega_{X/k}$ of $X = \mathbb{P}^n \times_k \mathbb{P}^m$?
22. Is there any relation between $\dim(X)$, $\mathrm{rk}(\Omega_{X/k})$, and $\mathrm{trdeg}(K(X)/k)$?
23. What causes the problems when one wants to compare the Zariski tangent space $T_{X,x}$ and the fibre $\Omega_{X/k} \otimes k(x)$?
24. Give an example of a regular ring and of a non-regular ring. What is the easiest/standard example of a regular non-smooth k-scheme? Can you think of one other example?
25. What is the Hilbert polynomial of a reduced k-scheme consisting of two points? Does the answer depend on the field k or on the projective embedding?
26. Give examples of modules that are flat and of those that are not over the two ring $k[x_1, x_2]$ and $k[x]/x^3$.
27. Are the function fields of the Fermat curves $V_+(x_0^d + x_1^d + x_2^d) \subset \mathbb{P}^2$ isomorphic for all d?
28. Can $\omega_{X/k}$ be a trivial invertible sheaf without $\Omega_{X/k}$ being a trivial locally free sheaf?
29. What is meant by the phrase 'flatness' is preserved by base change?
30. Which of the following properties of a scheme over a field are preserved by passing to a field extension: smooth, regular, integral, irreducible, reduced? What if the field has characteristic 0?
31. Which of the following properties of a flat morphism f can you check by checking it on fibers over closed points: smooth, proper, finite? Can you also do that without assuming that f is flat?
32. Is non-flatness always detected by the jump in the fibre dimension?
33. What is the relation between the Zariski tangent space $T_{X,x}$ and the fibre of the tangent sheaf $\mathcal{T}_X$ at $x \in X$? What are the assumptions on X?
34. Recall the two standard right exact sequences involving $\Omega_{B/A}$ and the corresponding sheaf versions.
35. How do you pass from the ideal sheaf of the diagonal $\Delta\colon X \to X \times_Y X$ to $\Omega_{X/Y}$?
36. Have you memorized the Euler sequence?

[1] Do not submit solutions. If you cannot answer all the questions (without much thinking) the first time, do it again.