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Dr. Gebhard Martin Summer term 2021

Exercises, Algebraic Geometry II – Week 5

Exercise 24. *Flatness of finite morphisms* (4 points)
Let $f\colon X \to Y$ be a finite morphism with Y Noetherian.

(i) Show that f is flat if and only if $f_*\mathcal{O}_X$ is locally free.

(ii) If Y is integral, this is equivalent to $\dim_{k(y)}(f_*\mathcal{O}_X \otimes k(y)) \equiv \text{const}$.

This in particular proves the fact mentioned in class that the normalization of a non-normal Noetherian integral scheme is not flat.

Exercise 25. *Flatness of projections* (4 points)
Let k be a field of characteristic different from 2. Consider the projection $\pi\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$, $(a_1, a_2) \mapsto a_1 + a_2$. Decide whether the restriction of π to $X \subset \mathbb{A}^2_k$ is flat or smooth, where X is:

(i) $X = V(x_1^2 - x_2^2)$;

(ii) $X = V(x_1^2 + x_2^2 + 2x_1x_2 - x_2 + x_1)$;

(iii) $X = \mathbb{A}^2_k \setminus V(x_1 - x_2)$;

(iv) $X = V((x_1 - x_2)(x_1 - 1), (x_1 - x_2)(x_1 + x_2))$.

Exercise 26. *Irreducibility and flat morphisms* (4 points)
Describe an (interesting) example of a surjective morphism $f\colon X \to Y$ of finite type k-schemes such that f is surjective, Y is irreducible, all fibres X_y are irreducible (even geometrically), but X is not irreducible.

Show that if f is in addition flat, then X has to be irreducible as well.

Exercise 27. *Canonical bundle of projective bundles* (4 points)
Use the relative Euler sequence (see Exercise 12) to compute the canonical bundle $\omega_{\mathbb{P}(\mathcal{E})}$ of a projective bundle $X = \mathbb{P}(\mathcal{E}) \to Y$, where $\mathcal{E}$ is a locally free sheaf on a scheme Y which is smooth over a field k. Can $\omega_{\mathbb{P}(\mathcal{E})}$ ever be ample? What about its dual? Compute $\omega_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}^{\oplus k})}$ and $\omega_{\mathbb{P}(\mathcal{T}_{\mathbb{P}^n})}$.

Exercise 28. *Hilbert polynomials of curves* (4 points)
Let k be an algebraically closed field, let $n \geq 2$, and consider an integral projective curve $C \subset \mathbb{P}^n_k$. Let $P(C, \mathcal{O}_C)(t)$ be the Hilbert polynomial of $\mathcal{O}_C$ considered as a coherent sheaf on $\mathbb{P}^n_k$.

(i) Show that $P(C, \mathcal{O}_C)(t) = dt + e$ with $d, e \in k$ and give an upper bound on e.

Due Friday 21 May 2021.

(ii) By a theorem of Bertini (which will be proved next week), there exists a hyperplane $H \subset \mathbb{P}^n_k$ such that $C \cap H$ is a finite reduced scheme, or, equivalently, there exists a section $s \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$ that yields a short exact sequence of sheaves on C

$$0 \to \mathcal{O}_C \xrightarrow{\cdot s} \mathcal{O}_C(1) \to \bigoplus_{i=1}^{r} k_{p_i} \to 0.$$

for some $r > 0$, where the p_i are pairwise distinct closed points and k_{p_i} is the skyscraper sheaf with value k at p_i. Show that $r = d$.

(iii) Show that the map $H^0(C, \mathcal{O}_C(m)) \to k^{\oplus d}$ induced by the above sequence is surjective for $m \geq d - 1$.

(iv) Deduce that $H^1(C, \mathcal{O}_C(d-2)) = 0$ and use this to give a lower bound on e depending on d.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 29. *Group schemes* (3 points)
A group scheme over a scheme S consists of a morphism of schemes $G \to S$ together with S-morphisms

$$c\colon G \times_S G \to G \text{ (group operation)}, \; i\colon G \to G \text{ (inverse), and } e\colon S \to G \text{ (unit)}$$

satisfying the usual group axioms. Some of the most important examples are provided by

(i) $\mathbb{G}_a := \mathrm{Spec}(\mathbb{Z}[T]) \to \mathrm{Spec}(\mathbb{Z})$ with c given by $\mathbb{Z}[T] \to \mathbb{Z}[T_1, T_2]$, $T \mapsto T_1 + T_2$.

(ii) $\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[T, T^{-1}]) \to \mathrm{Spec}(\mathbb{Z})$ with c given by $\mathbb{Z}[T, T^{-1}] \to \mathbb{Z}[T_1, T_1^{-1}, T_2, T_2^{-1}]$, $T \mapsto T_1 T_2$.

(iii) μ_n the closed subscheme $\mathrm{Spec}(\mathbb{Z}[T, T^{-1}]/(T^n - 1)) \subset \mathbb{G}_m$.

Show that $\mathbb{G}_a$ and $\mathbb{G}_m$ are smooth over $\mathrm{Spec}(\mathbb{Z})$ and that μ_n is not. Is it flat over $\mathrm{Spec}(\mathbb{Z})$? Determine the maximal open set $U \subset \mathrm{Spec}(\mathbb{Z})$ over which μ_n is smooth.