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# Exercises, Algebraic Geometry II – Week 4

**Exercise 18.** *Complete intersection* (4 points)
Consider a complete intersection $X \subset \mathbb{P}^n_k$ of type $(d_1, \ldots, d_r)$, i.e. $X = V_+(f_1, \ldots, f_r)$ with $f_i \in k[x_0, \ldots, x_n]_{d_i}$ and $\dim(X) = n - r$. Assume $n > r$.

(i) Show that the restriction maps $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m)|_X)$ are surjective.

(ii) Show that $X$ is geometrically connected.

(iii) Assume $X$ is smooth over $k$ and show $\omega_{X/k} \cong \mathcal{O}(\sum d_i - n - 1)$.

**Exercise 19.** *Flatness of morphisms* (4 points)
Recall the definition of flatness of a morphism. Try to find examples of flat and non-flat morphisms. In particular, decide whether the following morphisms are flat.

(i) The blow-up of the origin $f \colon X \to \mathbb{A}^n_k$ .

(ii) The natural projection $f \colon X \to \operatorname{Spec}(\mathbb{Z})$ where $X = V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Z}}$.

(iii) The natural projection $f \colon X \to \mathbb{A}^2_{\mathbb{Z}}$ where $X \subset \mathbb{A}^2_{\mathbb{Z}}$ is the closure of $V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Q}}$.

**Exercise 20.** *Smoothness of morphisms* (4 points)
Let $k$ be a field of characteristic $\neq 2$. Decide whether the following morphisms are smooth and determine all singular (i.e. non-smooth) fibers.

(i) The morphism $f \colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $k[x] \to k[y]$, $x \mapsto y^2$.

(ii) The natural projection $f \colon X \to \operatorname{Spec}(\mathbb{Z})$, where $X = V(y^2z - x^3 + x^2z + xz^2 - 10z^3) \subset \mathbb{P}^2_{\mathbb{Z}}$.

(iii) The natural projection $f \colon X \to \mathbb{A}^1_k$, $(x, y, t) \mapsto t$, where $X \subset \mathbb{A}^3_k$ is defined by $x^2 + y^2 = t$.

**Exercise 21.** *Universal family* (4 points)
Consider the 'incidence variety' $X \subset \mathbb{P}^n_k \times \mathbb{P}^N_k$ of all degree $d$ hypersurfaces given by the equation $\sum_I a_I x^I$, where $x^I = x_0^{i_0} \ldots x_n^{i_n}$ and the sum is over all $I = (i_0, \ldots, i_n)$ with $\sum i_k = d$. (So, $N = \dim |\mathcal{O}_{\mathbb{P}^n_k}(d)| - 1$ or, in other words, $\mathbb{P}^N_k = |\mathcal{O}_{\mathbb{P}^n_k}(d)|$.) The second projection $f \colon X \to |\mathcal{O}_{\mathbb{P}^n_k}(d)|$ is called the *universal family of hypersurfaces* of degree $d$.

(i) Show that $f$ is flat.

(ii) Describe the first projection $X \to \mathbb{P}^n_k$.

(iii) Consider the fibres $X_{(a_I)}$, $(a_I) \in \mathbb{P}^N_k$. Are the dimensions $h^i(X_{(a_I)}, \mathcal{O}_{X_{(a_I)}})$ constant? What about $\chi(X_{(a_I)}, \mathcal{O}_{X_{(a_I)}})$ or $\chi(X_{(a_I)}, \mathcal{O}_{X_{(a_I)}}(m))$?

Due Friday 14 May 2021.

The last two exercises are not strictly necessary for the understanding of the lectures at this point.

**Exercise 22.** *Koszul complex* (4 points)
Let $A$ be a ring and $a_1, \ldots, a_n \in A$ a sequence of elements. Recall that a sequence is called *regular*, if for any $1 \leq i \leq n$ the element $a_i$ is not a zero-divisor in $A/(a_1, \ldots, a_{i-1})$. Consider the complex

$$K_i^\bullet := \Big( \cdots \to 0 \to A \xrightarrow{a_i} A \to 0 \to \cdots \Big)$$

concentrated in degrees $-1$ and $0$, where the only non-zero differential is multiplication by $a_i$. Let $K^\bullet$ be the tensor product $K^\bullet := K_1^\bullet \otimes_A \otimes_A \cdots \otimes_A K_n^\bullet$, so that $K^\bullet$ is concentrated in degrees $-n, \ldots, 0$.

Assuming that the sequence $a_1, \ldots, a_n$ is regular, prove that $H^i(K^\bullet) = 0$ for $i < 0$ and $H^0(K^\bullet) = A/(a_1, \ldots, a_n)$. The complex $K^\bullet$ is called *Koszul complex* and it gives a free resolution of the $A$-module $A/(a_1, \ldots, a_n)$.

Now let $\mathcal{E}$ be a locally free sheaf of finite rank $r$ on a scheme $X$ and $s \in H^0(X, \mathcal{E})$ a section which is locally given by a regular sequence of elements in the corresponding ring (this is true if the zero locus of $s$ has codimension equal to the rank of $E$).

Let $Y \subset X$ be the zero locus of $s$. Observe that the local Koszul complexes glue into a global locally free resolution of $\mathcal{O}_Y$ (which is also called Koszul complex):

$$0 \to \bigwedge^r \mathcal{E}^* \to \bigwedge^{r-1} \mathcal{E}^* \to \cdots \to \bigwedge^2 \mathcal{E}^* \to \mathcal{E}^* \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

The differentials are given by convolution with $s$.

**Exercise 23.** *Beilinson resolution* (3 points)
Consider the projective space $\mathbb{P}_k^n := \mathrm{Proj}(S^*V^*)$, where $V^*$ is a vector space of dimension $n+1$ over a field $k$.

Let $s \in H^0(\mathbb{P}_k^n \times \mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1) \boxtimes \mathcal{T}_{\mathbb{P}_k^n})$ be the global section given by the map

$$\mathcal{O}_{\mathbb{P}_k^n}(-1) \boxtimes \Omega_{\mathbb{P}_k^n/k} \to \mathcal{O}_{\mathbb{P}_k^n \times \mathbb{P}_k^n}$$

which at a point $(\ell, \ell') \in \mathbb{P}_k^n \times \mathbb{P}_k^n$ is given by $(x, \varphi) \mapsto \varphi(x)$. Here, $\ell$ and $\ell'$ are thought of as lines in $V$, $x \in \ell$, and $\varphi \in \Omega_{\mathbb{P}_k^n/k} \otimes k(\ell')$ is viewed as a linear form on $V/\ell'$ via the Euler sequence.

Show that the Koszul complex for $s$ describes a locally free resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset \mathbb{P}_k^n \times \mathbb{P}_k^n$. Observe that all sheaves in this resolution are exterior products of locally free sheaves on the two factors. *Warning:* Typically, the structure sheaf of the diagonal $\Delta \subset X \times X$ of a variety does not posses any easy locally free resolution.