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Summer term 2021

Exercises, Algebraic Geometry II – Week 3

Exercise 12. *Relative Euler sequence* (4 points)
Let $\mathcal{E}$ be a locally free sheaf on a scheme Y and let $\pi\colon X := \mathbb{P}(\mathcal{E}) = \mathrm{Proj}(S^*(\mathcal{E})) \to Y$ be the associated projective bundle. Show that there exists a natural exact sequence

$$0 \to \Omega_{X/Y} \to \pi^*\mathcal{E} \otimes \mathcal{O}_\pi(-1) \to \mathcal{O}_X \to 0.$$

Here, $\mathcal{O}_\pi(-1)$ is the invertible sheaf associated with $S^*(\mathcal{E})(-1)$.

Exercise 13. *Plurigenera of smooth plane curves* (4 points)
Consider a smooth curve $C \subset \mathbb{P}^2_k$ defined by a polynomial of degree d. In particular, $\Omega_{C/k}$ is an invertible sheaf.

(i) Show that $\Omega_{C/k} \cong \mathcal{O}(d-3)|_C$.

(ii) Compute the n-th plurigenus $h^0(C, \Omega_{C/k}^{\otimes n}) := \dim H^0(C, \Omega_{C/k}^{\otimes n})$.

(iii) Compare $h^0(C, \Omega_{C/k})$ with the arithmetic genus of C.

Exercise 14. *Flatness of morphisms* (4 points)
Recall the definition of flatness of a morphism. Try to find examples and non-examples of flat morphisms. In particular, answer the following questions:

(i) Let $f\colon X \to \mathbb{A}^n_k$ be the blow-up of the origin. Is f flat?

(ii) Consider $X = V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Z}}$ and the natural projection $f\colon X \to \mathrm{Spec}(\mathbb{Z})$. Is f flat?

(iii) Let $X \subset \mathbb{A}^2_{\mathbb{Z}}$ be the closure of $V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Q}}$ in $\mathbb{A}^2_{\mathbb{Z}}$. Is the projection $f\colon X \to \mathbb{A}^2_{\mathbb{Z}}$ flat?

Exercise 15. *Examples of Kähler differentials* (3 points)
Let k be a field.

(i) Let $A = k[x,y]/(y^2 + xy - x^3)$. Compute $\Omega_{A/k}$ and show that its torsion submodule $(\Omega_{A/k})_{tors}$ is annihilated by (x,y) and satisfies $\dim_k(\Omega_{A/k})_{tors} = 1$.

(ii) Show that the conormal sequence for $\mathfrak{a} = (x^2, y^2) \subset k[x,y]$ is not exact on the left.

(iii) Let $A = k[x]/(x^n)$ for some $n > 0$. Calculate $\Omega_{A/k}$ and observe that $\Omega_{A/k}$ is a free A-module if and only if $\mathrm{char}(k) \mid n$.

(iv) Compute the support of $\Omega_{\mathbb{Z}[i]/\mathbb{Z}}$.

Due Friday 07 May 2021.

The last two exercises are not strictly necessary for the understanding of the lectures at this point.

Exercise 16. *The Jouanolou trick.* (5 points)
Prologue: Let X be a projective variety over a field k (algebraically closed for simplicity). Is it feasible that there exists a surjective morphism $f\colon Y \to X$ with Y affine and all fibres isomorphic to affine spaces $\mathbb{A}^n_k$ (of constant dimension)? Think about this question for ten minutes before doing the following exercise.

(i) Let V be a vector space of dimension $n+1$ and V^* its dual. We write $\mathbb{P}(V) \coloneqq \mathrm{Proj}(S^*(V^*))$ and $\mathbb{P}(V^*) = \mathrm{Proj}(S^*(V))$ (and think of them as the projective space of lines $\ell \subset V$ resp. hyperplanes $H \subset V$). Consider the 'incidence variety'

$$\Gamma \coloneqq \{(\ell, H) \mid \ell \subset H\} \subset \mathbb{P}(V) \times \mathbb{P}(V^*),$$

which is defined by the equation obtained from the dual pairing $V \times V^* \to k$.

(ii) Use the Segre embedding to show that $Y \coloneqq \mathbb{P}(V) \times \mathbb{P}(V^*) \setminus \Gamma$ is affine.

(iii) Show that the fibres of the first projection $\pi\colon Y \to \mathbb{P}(V)$ are isomorphic to affine spaces $\mathbb{A}^n$.

(iv) Show that there exists an open covering $\mathbb{P}(V) = \bigcup U_i$ and isomorphisms $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^{n-1}$ compatible with the projections.

(v) Use the above to prove the following statement: For any projective variety X there exists an affine variety Y and a morphism $\pi\colon Y \to X$ which is a Zariski locally trivial $\mathbb{A}^n$-bundle.

Epilogue: So, by passing to an $\mathbb{A}^n$-bundle, any projective variety X becomes affine.

The construction can be performed over $\mathrm{Spec}(\mathbb{Z})$. Moreover, $Y \to X$ as above exists for any scheme X smooth over $\mathrm{Spec}(A)$ with A a Noetherian and regular ring (Thomason's extension). Topologists phrase this result as: 'Up to $\mathbb{A}^1$-weak equivalence, any smooth A-scheme is an affine scheme smooth over A'.

Exercise 17. *The Jouanolou trick: Matrix version* (4 points)
Let k be an algebraically closed field. Consider the set Y of all matrices $A \in M(n+1, n+1, k)$ of rank one satisfying $A^2 = A$.

(i) Show that Y is naturally an affine variety.

(ii) Show that the fibres of the morphism $\pi\colon Y \to \mathbb{P}^n_k$, $A \mapsto \mathrm{Im}(A)$ are isomorphic to $\mathbb{A}^n_k$.

(iii) Compare this construction with the one in the previous exercise.