Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Summer term 2021

Exercises, Algebraic Geometry II – Week 2

Exercise 6. *Conics* (4 points)
Let $C \subset \mathbb{P}^2_k$ be a geometrically integral plane curve over an arbitrary field k defined by a quadratic equation (a 'smooth conic').

(i) Show that $C \cong \mathbb{P}^1_k$ if and only if $C(k) \neq \emptyset$.

(ii) Show that there always exists a quadratic field extension K of k such that C_K is rational.

(iii) Find one example of k and C where C is not rational (i.e. $C \ncong \mathbb{P}^1$).

Exercise 7. *Unirational varieties* (4 points)
A variety X over a field k is *unirational* if there exists a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$. Now, assume for simplicity that k is algebraically closed. Show that the following conditions are equivalent:

(i) The variety X is unirational.

(ii) There exists a generically finite dominant rational map $f\colon Y \dashrightarrow X$ with Y rational.

(iii) The function field $K(X)$ of X admits a finite extension that is purely transcendental over k.

Exercise 8. *Conic bundles* (4 points)
A *rational conic bundle* is a morphism between varieties $X \to S$ such that there exists an open set $\emptyset \neq U \subset S$ and an embedding $X_U := X \times_S U \hookrightarrow \mathbb{P}^2_U$ identifiying the fibers of $X_U \to U$ with conics in $\mathbb{P}^2$. Assume $T \subset X$ is a subvariety dominating S. Prove the following assertions:

(i) If $T \to S$ is birational and T is rational, then X is rational.

(ii) If T is unirational, then so is X.

Exercise 9. *Curves and function fields* (4 points)
Let k be a field of characteristic different from 2 and 3. Consider the two function fields $K_1 := k(x_1, y_1)$ and $K_2 := k(x_2, y_2)$ with $y_1^2 = x_1^3 + 4x_1^2 + 3x_1$ and $y_2^4 + x_2^4 + 1 = 0$, respectively. Use geometry to show that they are not isomorphic.

Exercise 10. *Cotangent sheaf of product* (4 points)
Let X and Y be schemes over a scheme S. Show that $\Omega_{X\times_S Y/S}$ is isomorphic to the direct sum of the pull-backs of $\Omega_{X/S}$ and $\Omega_{Y/S}$, i.e.

$$\Omega_{X\times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}.$$

Due Friday 30 April 2021.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 11. *Tsen's theorem* (4 points)
Let K be a function field over an algebraically closed field k.

(i) Show *Tsen's theorem:* Every conic in $\mathbb{P}^2_K$ has a K-rational point.

(Hint: Use that for an ample invertible sheaf $\mathcal{L}$ on a regular projective curve C of genus $g(C)$ over k with function field K one has $h^0(C, \mathcal{L}^m) = m \cdot \deg(\mathcal{L}) + 1 - g(C)$ for $m \gg 0$ and consider the map $H^0(C, \mathcal{L}^m)^3 \to H^0(C, \mathcal{L}^{2m})$ defined by the quadratic homogeneous polynomial $F \in K[x_0, x_1, x_2]$ defining the conic.)

(ii) Deduce that every rational conic bundle $\pi : X \to S$ over k, where S is an integral curve, admits a rational section, i.e. a section over an open subset of S.

More generally, one can show that function fields are C_1-fields (also called quasi-algebraically closed fields), i.e. every hypersurface in $\mathbb{P}^{n-1}_K$ of degree d has a K-rational point if $n > d$. For example, the *Chevalley–Warning theorem* asserts that finite fields are C_1-fields.

(iii) Give an example of a field that is not C_1.