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Exercises, Algebraic Geometry II – Week 13

Exercise 67. *Blowing up surfaces*
Let $\pi^*\colon \tilde{S} \to S$ be the blow-up of a closed point $x \in S$ in a smooth projective surface and denote by E the exceptional line. In class it was mentioned that the natural map $\pi^*\Omega_S \to \Omega_{\tilde{S}}$ is injective with cokernel ω_E.

(i) Prove the details of this assertion.

(ii) Furthermore, deduce from the resulting short exact sequence $0 \to \pi^*\Omega_S \to \Omega_{\tilde{S}} \to \omega_E \to 0$ that $\omega_{\tilde{S}} \cong \pi^*\omega_S \otimes \mathcal{O}(E)$.

(iii) Also, prove directly (without evoking the general result), that $P_m(S) = P_m(\tilde{S})$ (equality of plurigenera), $\mathrm{kod}(S) = \mathrm{kod}(\tilde{S})$ (equality of Kodaira dimensions), and $R(S) \cong R(\tilde{S})$ (isomorphism of canonical rings).

(iv) Compare the Riemann–Roch formulas for invertible sheaves on the two surfaces.

(v) Can the blow-up $\tilde{S}$ be isomorphic to a product of two curves?

Exercise 68. *Curves on a surface*
Assume $C \subset S$ is a smooth curve in a smooth projective surface S.

(i) Recall the adjunction formula and apply it to compute the canonical bundle of C as a restriction of an invertible sheaf on S.

(ii) Express the arithmetic genus of C as an intersection number on S. More precisely, show $2p_a(C) - 2 = (\mathcal{O}(C).\mathcal{O}(C) \otimes \omega_S)$.

(iii) Describe an example in which $\mathrm{Pic}(S) \to \mathrm{Pic}(C)$ is not surjective.

Exercise 69. $\mathrm{c}_1(S)^2$

(i) Let $S \subset \mathbb{P}^3$ be a smooth hypersurface of degree d. Show that $(\omega_S.\omega_S) = d(d-4)^2$.

(ii) Let $S = C_1 \times C_2$. Show that $(\omega_S.\omega_S) = 8(g(C_1) - 1)(g(C_2) - 1)$.

Find explicit examples of surfaces that do not fall in either of the two cases. (Be ambitious!)

Exercise 70. *Ampleness under numerical equivalence*
Assume $\mathcal{L}, \mathcal{M}$ are two numerically equivalent invertible sheaves on a smooth projective surface. Show that $\mathcal{L}$ is ample if and only if $\mathcal{M}$ is ample.

Exercise 71. *Very ample invertible sheaves on curves*
Let C be a smooth projective irreducible curve of genus g and let $\mathcal{L}$ be an invertible sheaf on C. Recall that if $\deg(\mathcal{L}) \geq 2g+1$, then $\mathcal{L}$ is very ample.

(i) Assume $g \leq 2$. Show that if $\mathcal{L}$ is very ample, then $\deg(\mathcal{L}) \geq 2g+1$.

(ii) Assume $g \geq 3$. Show that there exist both very ample and non very ample divisors of degree $2g$ on C.

(iii) Conclude that one cannot replace "ample" by "very ample" in Exercise 70.