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Summer term 2021

Exercises, Algebraic Geometry II – Week 12

Exercise 62. *Enriques–Severi–Zariski* (5 points)
Let X be a normal and projective scheme over a field k with an ample invertible sheaf $\mathcal{O}(1)$. Assume $\dim(X) \geq 2$.

(i) Let $\mathcal{F}$ be a locally free sheaf on X. Show that $H^1(X, \mathcal{F}(-m)) = 0$ for $m \gg 0$ (Observe that this follows from Serre duality if X is Cohen–Macaulay).

(ii) Assume k is algebraically closed and X is integral. Show that if $Y \subseteq X$ is an effective Cartier divisor with $\mathcal{O}_X(Y) \cong \mathcal{O}(1)$, then Y is connected. In particular, by Bertini's theorem, if X is smooth, $\mathcal{O}(1)$ is very ample, and Y is generic, then Y is smooth and irreducible.

Exercise 63. *Triviality of invertible sheaves* (2 points)
Assume X is an integral, smooth and projective scheme over an algebraically closed field k of dimension n and let $\mathcal{L}$ be an invertible sheaf with $H^0(X, \mathcal{L}^m) \neq 0$ for some $m > 0$. Show that $H^n(X, \omega_X \otimes \mathcal{L}) \neq 0$ implies $\mathcal{L}$ is trivial.

Exercise 64. *Norm map for finite morphisms of curves* (5 points)
Let X and Y be smooth, proper, and integral curves over an algebraically closed field k. Let $f : X \to Y$ be a finite morphism of degree n. Consider the map $f_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ given by $f_*(\sum n_i P_i) = \sum n_i f(P_i)$.

(i) Recall that if $\mathcal{E}$ is a locally free sheaf of rank r on Y, then $\det \mathcal{E} := \wedge^r \mathcal{E} \in \mathrm{Pic}(Y)$. Show that for every divisor D on X, we have

$$\det(f_*\mathcal{O}_X(D)) = \det(f_*\mathcal{O}_X) \otimes \mathcal{O}_Y(f_*D).$$

(ii) Deduce that f_* induces a well-defined homomorphism $f_* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. Show that f_*f^* coincides with multiplication by n on $\mathrm{Pic}(Y)$.

(iii) Use duality to show that

$$\det(f_*\mathcal{O}_X) \otimes \det(f_*\Omega_X) \cong \Omega_Y^{\otimes n}.$$

(iv) Assume that f is separable with ramification divisor R. Let $B = f_*R$ be the *branch divisor* of f. Show that

$$(\det(f_*\mathcal{O}_X))^{\otimes 2} \cong \mathcal{O}_Y(-B).$$

Due Friday 16 July 2021. This is the last sheet that counts towards the necessary 50 % needed for the exam.

Exercise 65. *Adjunction for effective Cartier divisors* (3 points)
Assume X is a Gorenstein, projective, and equidimensional scheme over k. Let $D \subset X$ be a Cartier divisor. Prove the adjunction formula

$$\omega_D \cong (\omega_X \otimes \mathcal{O}(D))|_D.$$

Here, ω_D is the dualizing sheaf of the Cohen–Macaulay scheme D. Note that we know this formula already for smooth X and D.

Exercise 66. *A glimpse of mirror symmetry* (3 points)
Assume X is a smooth projective variety over a field k. Assume furthermore that $\omega_X \cong \mathcal{O}_X$. (This is sometimes called a *Calabi–Yau variety.*)

(i) The space $H^1(X, \mathcal{T}_X)$ parametrizes first order deformations of X (see Example 9.13.2 in Hartshorne's book). Show that its dimension is a Hodge number of X. Which one?[1]

(ii) Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. For which d is X a Calabi–Yau variety? What is the dimension of $H^1(X, \mathcal{T}_X)$?

[1] Mirror symmetry roughly says that for every Calabi–Yau variety X of dimension three, there exists a dual Calabi–Yau variety X' with $(h^1(\mathcal{T}_X), h^{1,1}(X)) = (h^{1,1}(X'), h^1(\mathcal{T}_{X'}))$.