Exercises, Algebraic Geometry II – Week 12

Exercise 62. Enriques–Severi–Zariski (5 points)
Let $X$ be a normal and projective scheme over a field $k$ with an ample invertible sheaf $\mathcal{O}(1)$. Assume $\dim(X) \geq 2$.

(i) Let $\mathcal{F}$ be a locally free sheaf on $X$. Show that $H^1(X, \mathcal{F}(-m)) = 0$ for $m \gg 0$ (Observe that this follows from Serre duality if $X$ is Cohen–Macaulay).

(ii) Assume $k$ is algebraically closed and $X$ is integral. Show that if $Y \subseteq X$ is an effective Cartier divisor with $\mathcal{O}_X(Y) \cong \mathcal{O}(1)$, then $Y$ is connected. In particular, by Bertini’s theorem, if $X$ is smooth, $\mathcal{O}(1)$ is very ample, and $Y$ is generic, then $Y$ is smooth and irreducible.

Exercise 63. Triviality of invertible sheaves (2 points)
Assume $X$ is an integral, smooth and projective scheme over an algebraically closed field $k$ of dimension $n$ and let $\mathcal{L}$ be an invertible sheaf with $H^0(X, \mathcal{L}^m) \neq 0$ for some $m > 0$. Show that $H^n(X, \omega_X \otimes \mathcal{L}) \neq 0$ implies $\mathcal{L}$ is trivial.

Exercise 64. Norm map for finite morphisms of curves (5 points)
Let $X$ and $Y$ be smooth, proper, and integral curves over an algebraically closed field $k$. Let $f : X \to Y$ be a finite morphism of degree $n$. Consider the map $f_* : \text{Div}(X) \to \text{Div}(Y)$ given by $f_*(\sum n_i P_i) = \sum n_i f(P_i)$.

(i) Recall that if $\mathcal{E}$ is a locally free sheaf of rank $r$ on $Y$, then $\det \mathcal{E} := \bigwedge^r \mathcal{E} \in \text{Pic}(Y)$. Show that for every divisor $D$ on $X$, we have

$$\det(f_* \mathcal{O}_X(D)) = \det(f_* \mathcal{O}_X) \otimes \mathcal{O}_Y(f_* D).$$

(ii) Deduce that $f_*$ induces a well-defined homomorphism $f_* : \text{Pic}(X) \to \text{Pic}(Y)$. Show that $f_* f^*$ coincides with multiplication by $n$ on $\text{Pic}(Y)$.

(iii) Use duality to show that

$$\det(f_* \mathcal{O}_X) \otimes \det(f_* \Omega_X) \cong \Omega_Y^\otimes n.$$

(iv) Assume that $f$ is separable with ramification divisor $R$. Let $B = f_* R$ be the branch divisor of $f$. Show that

$$\left(\det(f_* \mathcal{O}_X)\right)^\otimes 2 \cong \mathcal{O}_Y(-B).$$
Exercise 65. **Adjunction for effective Cartier divisors** (3 points)
Assume $X$ is a Gorenstein, projective, and equidimensional scheme over $k$. Let $D \subset X$ be a Cartier divisor. Prove the adjunction formula

$$\omega_D \cong (\omega_X \otimes \mathcal{O}(D))|_D.$$ 

Here, $\omega_D$ is the dualizing sheaf of the Cohen–Macaulay scheme $D$. Note that we know this formula already for smooth $X$ and $D$.

Exercise 66. **A glimpse of mirror symmetry** (3 points)
Assume $X$ is a smooth projective variety over a field $k$. Assume furthermore that $\omega_X \cong \mathcal{O}_X$. (This is sometimes called a *Calabi–Yau variety*.)

(i) The space $H^1(X, \mathcal{T}_X)$ parametrizes first order deformations of $X$ (see Example 9.13.2 in Hartshorne’s book). Show that its dimension is a Hodge number of $X$. Which one?¹

(ii) Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d$. For which $d$ is $X$ a Calabi–Yau variety? What is the dimension of $H^1(X, \mathcal{T}_X)$?

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¹Mirror symmetry roughly says that for every Calabi–Yau variety $X$ of dimension three, there exists a dual Calabi–Yau variety $X'$ with $(h^1(\mathcal{T}_X), h^{1,1}(X)) = (h^{1,1}(X'), h^1(\mathcal{T}_{X'}))$. 

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