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Summer term 2021

Exercises, Algebraic Geometry II – Week 11

Exercise 56. *Trivial direct image* (3 points)
Let $f\colon X \to Y$ be a projective morphism with Y Noetherian, integral and of positive dimension. Let $\mathcal{F} \in \mathrm{Coh}(X)$ be flat over Y and assume that there exists at most one $y \in Y$ with $H^0(X_y, \mathcal{F}_y) \neq 0$. Show that then $f_*\mathcal{F} = 0$. Find an example to show that the flatness of $\mathcal{F}$ can not be dropped.

Exercise 57. *Hodge bundles* (3 points)
Let $f\colon X \to Y$ be a smooth projective morphism with Y Noetherian and $\dim X = \dim Y + 1$. Show that the Hodge bundles $R^q f_*(\Omega^p_{X/Y})$ are locally free sheaves. (In characteristic zero, this holds true without the assumption on the dimension.)

Exercise 58. *Projection formula* (3 points)
Let $f\colon X \to Y$ be a morphism and $\mathcal{F} \in \mathrm{Qcoh}(X)$. Show that for a locally free $\mathcal{G} \in \mathrm{Coh}(Y)$ one has

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{G}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{G}.$$

Exercise 59. *Künneth formula* (5 points)
Let X_1 and X_2 be separated schemes over a field k. One writes $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ for the sheaf $p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2$ on $X_1 \times_k X_2$ for quasi-coherent sheaves $\mathcal{F}_1$ on X_1 and $\mathcal{F}_2$ on X_2. Here, $p_i\colon X \times_k X \to X$ is the i-th projection.

(i) Show that

$$H^n(X_1 \times_k X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong \bigoplus_{i+j=n} H^i(X_1, \mathcal{F}_1) \otimes_k H^j(X_2, \mathcal{F}_2).$$

(ii) Prove that $R^n p_{1*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong \mathcal{F}_1 \otimes_k H^n(X_2, \mathcal{F}_2)$. Compare this with the assumptions on the previous exercise.

Exercise 60. *Hodge numbers of products* (2 points)
For a smooth projective variety X over a field k one defines the *Hodge numbers*

$$h^{p,q}(X) \coloneqq \dim_k H^q(X, \Omega^p_{X/k}).$$

Compute the Hodge numbers $h^{p,q}(X_1 \times_k X_2)$ of a product of two smooth projective varieties X_1, X_2 in terms of $h^{p,q}(X_i)$.

Exercise 61. *Projective bundles* (2 points)
Let Y be a Noetherian scheme and $\mathcal{E}$ a locally free sheaf of rank $n+1$ on Y. Let $\pi\colon X \coloneqq \mathbb{P}(\mathcal{E}) \to Y$ and $\mathcal{O}(1)$ be the corresponding invertible sheaf on X. Denote by $\omega_{X/Y} \coloneqq \Omega^n_{X/Y}$ the relative canonical sheaf. Using the isomorphism $R^n\pi_*\omega_{X/Y} \cong \mathcal{O}_Y$ (which one can deduce from Serre duality) and the relative Euler sequence (see Exercise 12) prove that for any $\ell \in \mathbb{Z}$ we have $R^n\pi_*\mathcal{O}(\ell) \cong (\pi_*\mathcal{O}(-\ell-n-1))^* \otimes (\bigwedge^{n+1} E)^*$.

Due Friday 9 July 2021.