Exercises, Algebraic Geometry II – Week 10

Exercise 51. Components of fibres (3 points)
Let \( f : X \to Y \) be a projective morphism with \( Y \) locally Noetherian. Show that the connected components of a fibre \( X_y \) are in bijection to the maximal ideals of \((f_*\mathcal{O}_X)_y\).

Exercise 52. Globally generated line bundles (3 points)
Let \( L \) be a globally generated invertible sheaf on a projective scheme \( X \) over a field \( k \). Consider the induced morphism \( \varphi_L : X \to \mathbb{P}^N_k \). Show that the morphism \( \varphi \) can be decomposed as \( \varphi = g \circ \varphi' \) with \( \varphi' : X \to Z \) projective with connected fibre and \( g : Z \to \mathbb{P}^N_k \) finite such that:

(i) \( \deg(L|_C) = 0 \) for a complete integral curve \( C \subset X \) if and only if \( \varphi(C) = \text{pt.} \)

(ii) If \( X \) is normal, then \( Z \) is normal.

Assume that for every complete curve \( C \subset X \) we have \( \deg(L|_C) \neq 0 \). Show that then \( L \) is ample.

Exercise 53. Connectedness (3 points)
Let \( f : X \to Y \) be a surjective projective morphism of Noetherian schemes. Assume that \( Y \) and all fibres \( X_y \) are connected. Show that then also \( X \) is connected. Compare this with Exercise 26.

Exercise 54. Rigidity I (4 points)
Let \( f : X \to Y \) and \( g : X \to Z \) be projective morphisms of varieties (i.e. integral schemes of finite type over a field) with \( \mathcal{O}_Y \cong f_*\mathcal{O}_X \) and such that \( g \) contracts each fibre of \( f \) (i.e. \( g(f^{-1}(y)) \) is a point for each \( y \in Y \)). Show that there exists a morphism \( h : Y \to Z \) with \( h \circ f = g \).

(Hint: Study the image of the morphism \((f, g) : X \to Y \times Z\).)

Exercise 55. Rigidity II (4 points)
Let \( X, Y, \) and \( Z \) be varieties over an algebraically closed field \( k \), and let \( X \) be proper over \( k \).
Let \( f : X \times Y \to Z \) be a morphism. Assume that there exists a closed point \( y_0 \in Y \), such that \( f(X \times \{y_0\}) \) is a single point in \( Z \). Prove that there exists \( g : Y \to Z \), such that \( f = g \circ \pi_Y \) where \( \pi_Y : X \times Y \to Y \) is the projection.