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Summer term 2021

Exercises, Algebraic Geometry II – Week 10

Exercise 51. *Components of fibres* (3 points)
Let $f\colon X \to Y$ be a projective morphism with Y locally Noetherian. Show that the connected components of a fibre X_y are in bijection to the maximal ideals of $(f_*\mathcal{O}_X)_y$.

Exercise 52. *Globally generated line bundles* (3 points)
Let $\mathcal{L}$ be a globally generated invertible sheaf on a projective scheme X over a field k. Consider the induced morphism $\varphi_{\mathcal{L}}\colon X \to \mathbb{P}^N_k$. Show that the morphism φ can be decomposed as $\varphi = g \circ \varphi'$ with $\varphi'\colon X \to Z$ projective with connected fibre and $g\colon Z \to \mathbb{P}^N_k$ finite such that:

(i) $\deg(\mathcal{L}|_C) = 0$ for a complete integral curve $C \subset X$ if and only if $\varphi(C) = \text{pt}$.

(ii) If X is normal, then Z is normal.

Assume that for every complete curve $C \subset X$ we have $\deg(L|_C) \neq 0$. Show that then $\mathcal{L}$ is ample.

Exercise 53. *Connectedness* (3 points)
Let $f\colon X \to Y$ be a surjective projective morphism of Noetherian schemes. Assume that Y and all fibres X_y are connected. Show that then also X is connected. Compare this with Exercise 26.

Exercise 54. *Rigidity I* (4 points)
Let $f\colon X \to Y$ and $g\colon X \to Z$ be projective morphisms of varieties (i.e. integral schemes of finite type over a field) with $\mathcal{O}_Y \cong f_*\mathcal{O}_X$ and such that g contracts each fibre of f (i.e. $g(f^{-1}(y))$ is a point for each $y \in Y$). Show that there exists a morphism $h\colon Y \to Z$ with $h \circ f = g$.

(Hint: Study the image of the morphism $(f, g) : X \to Y \times Z$.)

Exercise 55. *Rigidity II* (4 points)
Let X, Y, and Z be varieties over an algebraically closed field k, and let X be proper over k. Let $f\colon X \times Y \to Z$ be a morphism. Assume that there exists a closed point $y_0 \in Y$, such that $f(X \times \{y_0\})$ is a single point in Z. Prove that there exists $g\colon Y \to Z$, such that $f = g \circ \pi_Y$ where $\pi_Y\colon X \times Y \to Y$ is the projection.

Due Friday 2 July 2021.