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Summer term 2021

# Exercises, Algebraic Geometry II – Week 1

**Exercise 1.** *Sober topological spaces* (5 points)
A topological space $X$ is called *sober* if every closed irreducible set $Y \subset X$ contains a unique generic point. A *Zariski topological space* is a sober topological space that is also Noetherian.

(i) Show that the topological space underlying an arbitrary scheme is sober.

(ii) Show that the topological space underlying a Noetherian scheme is Zariski.

(iii) Show that for any topological space $X$ the space $t(X)$ of all closed irreducible sets with the topology defined in class is sober.

(iv) Let $(sobTop) \subset (Top)$ be the full subcategory of sober topological spaces of the category of all topological spaces. Consider $X \mapsto t(X)$ as a functor $t\colon (Top) \to (sobTop)$. Show that it is left adjoint to the inclusion.

(v) Give a nice example of a topological space that is not sober.

**Exercise 2.** *Maximal open sets of definition of rational functions/maps* (4 points)
Consider varieties $X$ and $Y$ (over an algebraically closed field $k$).

(i) Show that for every rational function $f \in K(X)$ there exists a maximal open subset on which $f$ is regular.

(ii) Show that for every rational map $f\colon X \dashrightarrow Y$ there exists a maximal open subset on which $f$ is a morphism.

(iii) Suppose $f$ in (ii) is a birational map. Is then the restriction $f|_U$ to the maximal open subset on which $f$ is regular always injective?

**Exercise 3.** (4 points) *Cremona transformations*
A birational map from $\mathbb{P}^2_k$ ($k$ an algebraically closed field) to itself is called *plane Cremona transformation*. One example is the *quadratic transformation* $\varphi$ given by

$$[a_0 : a_1 : a_2] \mapsto [a_1a_2 : a_0a_2 : a_0a_1].$$

(i) Show that $\varphi$ is birational and its own inverse.

(ii) Find non-empty open subsets $U, V \subseteq \mathbb{P}^2_k$ such that $\varphi$ induces an isomorphism $U \to V$.

(iii) Describe the maximal open subset $U$ on which $\varphi$ is defined.

(iv) Describe the automorphism of $K(\mathbb{P}^2_k)$ induced by $\varphi$.

Due Friday 23 April 2021.

**Exercise 4.** *Dominant rational maps* (3 points)
Are there any dominant rational maps $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k$?

**Exercise 5.** *Blow-up* (4 points)
Let $Y$ be the cuspidal cubic curve $y^2 = x^3$ in $\mathbb{A}^2_k$ ($k$ an algebraically closed field). Blow up $\mathbb{A}^2_k$ in the point $O = (0, 0)$. Let $E$ be the exceptional curve, and let $\tilde{Y}$ be the strict transform of $Y$. Show that $E$ meets $\tilde{Y}$ in one point, and that $\tilde{Y} \cong \mathbb{A}^1_k$. In this case the morphism $\varphi : \tilde{Y} \to Y$ is a homeomorphism, but it is not an isomorphism.