GRADUATE SEMINAR ON CUBIC HYPERSURFACES

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1. Introduction

After projective space itself, the simplest examples of projective varieties are hypersurfaces. While hypersurfaces of degree 1 can be described via linear algebra and hypersurfaces of degree 2 are geometric incarnations of bilinear algebra, general hypersurfaces of degree 3 and higher cannot be described using these methods. In other words, starting from degree 3, it is necessary to use algebraic geometry in order to understand hypersurfaces.

The goal of this seminar is to learn standard techniques in (complex) algebraic geometry, such as deformation theory, moduli spaces, Hodge theory, period spaces, correspondences, and derived categories, and apply them to the case of cubic hypersurfaces of arbitrary dimension. For example, one of the key objects that combines these concepts and that arises in the study of a cubic hypersurface X is its Fano variety of lines, the moduli space of lines on X.

In the second part of the seminar, we will focus on cubic threefolds and fourfolds. In particular, we will prove the irrationality of smooth cubic threefolds following Clemens and Griffiths. After that, we will study cubic fourfolds and explore connections between these fourfolds and Hyperkähler geometry. We will also investigate the relation between their derived categories and rationality.

All talks are based on the book [1].

Prerequisites: Algebraic Geometry on the level of Hartshorne Chapter II and III, familiarity with singular cohomology. Some knowledge in complex geometry and Hodge theory is very helpful. [1]

Registration: If you are interested in participating, please send an email to gmartin@math.uni-bonn.de before August 31. Describe your background in algebraic and complex geometry and give a list of three possible talks you would like to give. We will then distribute the talks according to your preferences.

Meeting: Two weeks before your talk, we will meet to discuss the content of your talk in detail. For the meeting, you should already be familiar with the content of the sections listed in the description of your talk below.

2. List of talks

(1) **Hypersurfaces:** Invariants of hypersurfaces. *Section 1.1.* Hodge numbers, Discriminant divisor, Singular cohomology and cohomology of cotangent sheaf, Intersection form.
(2) **Classical constructions:** Linear subspaces and quadric fibrations. *Section 1.5.1 - 1.5.2.* If time permits: hyperplane sections and maximal variations, triple covers. *Sections 1.5.3, 1.5.5–1.5.6.*
(3) **Automorphisms and deformations.** Basics of deformation theory. Discreteness and finiteness of automorphism group, no automorphisms generically. Dimension of deformation space. *Section 1.3.*

Date: Winter Term 2023.

[1] In case you do not have any background in Hodge theory, it is highly recommended to attend Prof. Engels class on Hodge theory this term.

(4) **Moduli spaces:** *Sections 3.* GIT description, moduli functors in general, dimension and geometry of the moduli spaces. *Sections 3.1.1–3.1.2, 3.1.5–3.1.6, 3.2.2.*
(5) **Fano of lines I:** Fano variety of linear subspaces, dimensions. *Sections 2.1.* Mention the Fano correspondence. *Section 2.5.*
(6) **Fano of lines II:** Lines of first/second type, geometric Torelli. *Sections 2.2, 2.3.* Cohomology and degree. *Section 2.4.3.*
(7) **Cubic threefolds I:** Invariants, lines, conic fibration and Prym. Singularities of Pryms. *Sections 5.0.1–5.0.2, 5.1, 5.3.2,* [2].
(8) **Cubic threefolds II:** Intermediate Jacobian and irrationality. *Sections 5.3.1–5.3.2, 5.4.5.*
(9) **Cubic fourfolds I:** Invariants, geometry of special cubic fourfolds. *Sections 6.0.1–6.0.2, 6.1.1–6.1.3, 6.2.* If time permits: geometry of very general cubic fourfolds. *Sections 6.4.*
(10) **Cubic fourfolds II:** Fano of lines as a hyperkähler fourfold. Associated K3 surfaces. *Sections 6.3, 6.5.*
(11) **Derived categories I**:[2] Reminders on derived categories. Derived torelli for cubic threefolds.
(12) **Derived categories II:**[2] Rationality conjecture, Kuznetsov component of special cubics (containing a plane / Pfaffian).

References

[1] D. Huybrechts, The geometry of cubic hypersurfaces, http://www.math.uni-bonn.de/people/huybrech/Notes.pdf

[2] D. Mumford, Prym Varieties I, Contributions to Analysis, Academic Press, 1974, Pages 325-350

[2] If you want to apply for this talk, you should ideally have some background knowledge on derived categories.