The Extended Affine Lie Algebra Associated with a Connected Non-negative Unit Form

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Connected positive definite unit forms (corank 0).
Connected positive definite unit forms (corank 0) $\leftrightarrow$ Simply-laced simple Lie algebras of finite type.
Connected **positive definite unit** forms (corank 0). ↔ Simply-laced simple Lie algebras of **finite** type.

Connected **non-negative unit** forms of corank 1.
Connected **positive definite** unit forms (corank 0).

Connected **non-negative** unit forms of corank 1.

\[\leftrightarrow\] Simply-laced simple Lie algebras of **finite** type.

\[\leftrightarrow\] Simply-laced Kac-Moody algebras of **affine** type.
Connected positive definite unit forms (corank 0) $\leftrightarrow$ Simply-laced simple Lie algebras of finite type.

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Connected non-negative unit forms of corank $\geq 2$. 
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Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative unit form unit form.

Connected positive definite unit forms (corank $0$) $\leftrightarrow$ Simply-laced simple Lie algebras of finite type.

Connected non-negative unit forms of corank $1$ $\leftrightarrow$ Simply-laced Kac-Moody algebras of affine type.

Connected non-negative unit forms of corank $\geq 2$. ?
Let $q : \mathbb{Z}^n \rightarrow \mathbb{Z}$ be a connected non-negative unit form.

We associate with $q$ a matrix $C$ given by

$$C_{ij} = q(c_i + c_j) - q(c_i) - q(c_j).$$

Where $\{c_1, \ldots, c_n\}$ denotes the standard basis of $\mathbb{Z}^n$. 
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We call \( R = R^0 \cup R^\times \) the root system of \( q \).
Construction [Barot, Kussin, Lenzing]

Let $FL$ be the free Lie algebra with $3n$ generators

$$e_{-i}, h_i, e_i \quad i \in \{1, \ldots, n\}$$

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Gustavo Jasso Ahuja (UNAM)
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The Algebra $\tilde{G}(q)$

Let $\text{corank } q \in \mathbb{Z}_{\geq 0}$ and let $G(q)$ be the quotient of $FL$ by the ideal generated by the following generalized Serre relations:

Construction [Barot, Kussin, Lenzing]

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(S1) $[h_i, h_j] = 0$.

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(S3) \quad [e_{\varepsilon i}, e_{-\varepsilon i}] &= \varepsilon h_i. \\
(S\infty) \quad [e_{\varepsilon_1 i_1}, \ldots, e_{\varepsilon_t i_t}] &= 0,
\end{align*}

whenever $q(\sum_{k=1}^{t} \varepsilon_k c_k) > 1$, for $\varepsilon_k = \pm 1$ and $i_k \in \{1, \ldots, n\}$. 

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Definition

$$\tilde{G}(q) := G(q) \oplus (\text{rad } q)^*.$$
Properties of $\tilde{G}(q)$

*(EA2)* It has a finite dimensional abelian subalgebra $H$ which equals its own centralizer in $\tilde{G}(q)$ and such that $\text{ad} \ h$ is diagonalizable for all $h \in H$.

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Extended Affine Lie Algebras

[Høegh-Krohn & Torresani. Allison, Azam, Berman, Gao, Pianzola]

An extended affine Lie algebra (EALA) is a complex Lie algebra which satisfies axioms (EA2)-(EA5) together with the following axiom:

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\tilde{\mathcal{G}}(q)
\]

Properties of $\tilde{\mathcal{G}}(q)$

- **(EA2)** It has a finite dimensional abelian subalgebra $H$ which equals its own centralizer in $\tilde{\mathcal{G}}(q)$ and such that $\text{ad } h$ is diagonalizable for all $h \in H$.
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**EA1** The algebra has a non-degenerate invariant symmetric bilinear form.

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If \( \text{corank } q \geq 2 \), then the algebra \( \tilde{G}(q) \) is not an EALA.

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If \( \text{corank } q \geq 2 \), then the algebra \( \tilde{G}(q) \) is not an EALA.

One cannot define a \textit{non-degenerate} symmetric invariant bilinear form on \( \tilde{G}(q) \).
How do we fix it?
The algebra $\tilde{G}(q)$ is an $H^*$-graded $H$-module, hence it contains a unique maximal ideal $I$ with respect to $I \cap H = \{0\}$.

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The algebra $E(q) := \tilde{G}(q)/I$ is an EALA.

Remark
If corank $q \geq 2$, then the algebra $\tilde{G}(q)$ is not an EALA.
One cannot define a non-degenerate symmetric invariant bilinear form on $\tilde{G}(q)$. 
Main result

Theorem

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative unit form with associated root system $R$. Then the Lie algebra $E(q)$ is a centerless tame EALA with root system $R$.

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Remark
In order to show that $E(q)$ is an EALA it is useful to introduce an alternative construction of this algebra.

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Theorem
Let $q : \mathbb{Z}^n \rightarrow \mathbb{Z}$ be a connected non-negative unit form with associated root system $R$. Then the Lie algebra $E(q)$ is a centerless tame EALA with root system $R$. Furthermore, if $q'$ is a connected non-negative unit form which is equivalent to $q$ then $E(q)$ and $E(q')$ are isomorphic as EALAs.
The Algebra $\hat{E}(q)$

Construct a Lie algebra $\hat{E}(q)$

Remark
In order to show that $E(q)$ is an EALA it is useful to introduce an alternative construction of this algebra.
The Algebra $\hat{E}(q)$

such that there is a projection

$$\hat{E}(q) \xleftarrow{p} \tilde{G}(q)$$

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which factors

\[
\begin{array}{c}
\hat{E}(q) & \xrightarrow{p} & \tilde{G}(q) \\
\downarrow & & \uparrow \\
\tilde{G}(q)/\ker p & & \\
\end{array}
\]

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exactly through $E(q)$.

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\[ \hat{E}(q) \leftarrow^{p} \tilde{G}(q) \leftarrow \hat{E}(q) \]

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The Algebra $\hat{E}(q)$

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An let $\hat{E}(q) = \bigoplus_{\alpha \in R} \hat{E}(q)_\alpha$ as a vector space.

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Let $\alpha, \beta \in R^\times$, $\sigma, \tau \in R^0$, $v, w \in \mathbb{C}^n$:

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Let $\alpha, \beta \in R^\times$, $\sigma, \tau \in R^0$, $v, w \in \mathbb{C}^n$:

\[(B1) \quad [\pi_\sigma(v), \pi_\tau(w)] = q(v, w)\pi_{\sigma+\tau}(\sigma).\]

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$$[e_\alpha, e_\beta] = \begin{cases} 
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Let $\beta \in R^\times$, $\tau \in R^0$, $w \in \mathbb{C}^n$ and $\xi, \zeta \in (\text{rad } q)^*$:

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Let $\beta \in R^\times$, $\tau \in R^0$, $w \in \mathbb{C}^n$ and $\xi, \zeta \in (\text{rad } q)^*$:

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(B5) $[\xi, \pi_\tau(w)] = \xi \rho(\beta) \pi_\tau(w)$.

The Algebra $\hat{E}(q)$

Let $\alpha, \beta \in R^\times$, $\sigma, \tau \in R^0$, $v, w \in \mathbb{C}^n$:

(B1) $[\pi_\sigma(v), \pi_\tau(w)] = q(v, w) \pi_{\sigma + \tau}(\sigma)$.

(B2) $[\pi_\sigma(v), e_{\beta}] = q(v, \beta) e_{\beta + \sigma}$.

(B3) $[e_\alpha, e_{\beta}] = \begin{cases} 
\epsilon(\alpha, \beta) e_{\alpha + \beta} & \text{if } \alpha + \beta \in R^\times, \\
\epsilon(\alpha, \beta) \pi_{\alpha + \beta}(\alpha) & \text{if } \alpha + \beta \in R^0, \\
0 & \text{otherwise}.
\end{cases}$
The Algebra $\hat{E}(q)$

Let $\beta \in R^\times$, $\tau \in R^0$, $w \in \mathbb{C}^n$ and 
$\xi, \zeta \in (\text{rad } q)^*$:

(B4) $\ [\xi, e_\beta] = -[e_\beta, \xi] = \xi \rho(\beta) e_\beta.$

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(B6) $\ [\xi, \zeta] = 0.$

The Algebra $\hat{E}(q)$

Let $\alpha, \beta \in R^\times$, $\sigma, \tau \in R^0$, $v, w \in \mathbb{C}^n$:

(B1) $\ [\pi_\sigma(v), \pi_\tau(w)] = q(v, w) \pi_{\sigma+\tau}(\sigma).$

(B2) $\ [\pi_\sigma(v), e_\beta] = q(v, \beta) e_{\beta+\sigma}.$

(B3) $\ [e_\alpha, e_\beta] = \begin{cases} 
\epsilon(\alpha, \beta) e_{\alpha+\beta} & \text{if } \alpha + \beta \in R^\times, \\
\epsilon(\alpha, \beta) \pi_{\alpha+\beta}(\alpha) & \text{if } \alpha + \beta \in R^0, \\ 0 & \text{otherwise.} 
\end{cases}$
Thanks for your attention!

The Algebra $\hat{E}(q)$

Let $\beta \in R^\times$, $\tau \in R^0$, $w \in \mathbb{C}^n$ and $\xi, \zeta \in (\text{rad } q)^*$:

(B4) $[\xi, e_\beta] = -[e_\beta, \xi] = \xi \rho(\beta)e_\beta$.
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(B6) $[\xi, \zeta] = 0$. 

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