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Research statement

1 Introduction

My research lies mainly in harmonic analysis. Up to now, I have worked on three different problems. The first two are variations on Carleson's theorem on almost everywhere convergence of the Fourier series. The last one is related to sparse operators: a topic that has emerged in the recent decade in connection to weighted inequalities.

For the first problem, I consider a perturbation of the classical trigonometric system. One can replace the factors z of the Taylor expansion of a function holomorphic in the unit disk by other Möbius transforms mapping the unit disk to itself. If the zeros of these Möbius transforms coincide with the zeros of the function, this decomposition leads to the so-called *non-linear phase unwinding decomposition*. Due to very fast convergence at numerical experiments [Nah00], the latter has been in the center of some recent activity [CS17; CP19]. On the other hand, freezing the zeros of Möbius transforms generates a complete orthonormal system for the Hardy spaces H^p that is called the Malmquist-Takenaka system. In [Mna22a], I study the question of almost everywhere convergence for the Malmquist-Takenaka series applying the techniques of the polynomial Carleson theorem of Lie [Lie11] and Zorin-Kranich [Zor21].

The second, my more recent, project is about the non-linear Fourier transform (NLFT), an object at the junction of many areas of analysis and PDE such as the Schrödinger operator, the KdV equation, orthogonal polynomials, etc. In a recent breakthrough paper [Pol21], Alexei Poltoratski proved almost everywhere convergence of the NLFT for L^2 potentials. This gave a partial answer to the question of Muscalu, Tao and Thiele [MTT02] whether the corresponding Carleson operator is bounded on weak-L^2. Poltoratski's methods are qualitative and rely on complex analysis, in particular, on the so-called de Branges functions. In [Mna22a], I quantify parts of Poltoratski's proof by bringing into play the Hardy-Littlewood maximal function of the spectral density to control the error terms in the asymptotic estimates. These estimates allow us to push a little more towards the conjectured weak-L^2 bound of the Carleson operator. They also lead to an interesting structural result about the zeros of the de Branges functions.

My third project is about weighted estimates of sparse operators. The sparse operators were introduced by Lerner [Ler12] a decade ago in connection to the A_2 theorem. Due to their simplicity and majorization properties, the sparse operators are the modern tool for proving sharp weighted inequalities. In [Mna22c], I consider a version of a sparse operator, where the averages are replaced by maximal averages. I prove weighted L^2 bounds for this operators. The difficulty in this situation is proving the sharpness of the easy upper bound. The novelty of my result is the construction of a weight that does the job.

Next, I will elaborate on each of the projects in a little more detail.

2 Perturbations of Fourier series

The Fourier series can be though of as the Taylor expansion of an H^p function restricted to the unit circle. For $F \in H^p$, we can consider

$$F_1(z) := \frac{F(z) - F(0)}{z}, \tag{2.1}$$

then iterate the procedure on F_1 to obtain

$$F(z) = F(0) + F_1(0)z + \cdots + F_n(0)z^n + \cdots.$$

Let us change the procedure (2.1) with the Blaschke factorization, i.e. factoring out all the zeros. Let

$$F(z) - F(0) = B_1(z)F_1(z), \tag{2.2}$$

where $F_1 \in H^p$ does not have any zeros in the unit disk. Iterating this procedure leads to the formal series

$$F(z) = F(0) + F_1(0)B_1(z) + \cdots + F_n(0)B_1(z)\dots B_n(z) + \cdots, \tag{2.3}$$

called the non-linear phase unwinding decomposition. Numerical simulations from Nahon's dissertation [Nah00], where he introduced this object, suggest that for a generic function the series should converge very fast. Most of the few mathematically rigorous results known for this object are obtained by Coifman and Steinerberger [CS17], who, in particular, prove the convergence in fractional Sobolev spaces if the original function is slightly better. The unwinding series was also independently discovered and studied by Qian [Qia14] and others who apply it to Information theory. The following questions remain open.

Question 1. *Does the unwinding series (2.3) converge almost everywhere for H^p functions?*

Question 2. *Does the unwinding series converge back to the initial function "fast" for "most functions"?*

The intuitive reason of the expected fast convergence is that the zeros of the decomposing functions are adapted to the decomposed function. Now let us freeze those zeros. Let a_n be a sequence of points in the unit disk, then we replace the procedure (2.1) by

$$F_n(z) - F_n(a_n) = \frac{z - a_n}{1 - \overline{a_n}z} F_{n+1}(z). \tag{2.4}$$

The latter generates the Blaschke products and the Malmquist-Takenaka (MT) system defines as

$$B_n(z) := \prod_{j=1}^{n} \frac{z - a_n}{1 - \overline{a_n}z}, \quad \phi_n(z) := B_n(z) \cdot \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n}z}, \quad n = 0, 1, 2, \dots. \tag{2.5}$$

If $a_n \equiv 0$, the MT system $(\phi_n)_{n=0}^{\infty}$ turns into the classical trigonometric system. In general, it is still an orthonormal system and, as shown by Coifman and Péyriere [CP19] is complete in H^p if

$$\sum (1 - |a_n|) = \infty. \tag{2.6}$$

In [Mna22a], I study the following question.

Question 3. *For which sequences $(a_n)_{n=0}^{\infty}$, does the MT series converge almost everywhere for H^p functions, i.e.*

$$\sum_{j=0}^{n}\langle f,\phi_n\rangle\phi_n \xrightarrow{n\to\infty} f \text{ a.e., for all } f\in H^p? \tag{2.7}$$

By standard techniques this question follows from the corresponding estimates for the maximal partial sum operator, which is equivalent to the operator

$$T^{(a_n)}f(e^{ix}) = \sup_N \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(e^{iy})\frac{B_N(e^{iy})^{-1}}{\sin\frac{x-y}{2}}dy\right|. \tag{2.8}$$

The operator (2.8) shares many similarities with the polynomial Carleson operator defined as

$$T_d f(x) = \sup_{\deg P\le d}\left|\int_{\mathbb{R}} f(y)e^{-iP(y)}\frac{dy}{x-y}\right|, \tag{2.9}$$

where the sup is taken over all polynomials up to the degree d. In 2011, Lie [Lie11], proved the boundedness of the polynomial Carleson operator in L^p, $1<p<\infty$. Then, in 2017, Zorin-Kranich [Zor21] simplified the proof.

In [Mna22a], I observe that the phases of the modulations of the two aforementioned Carleson-type operators, namely, the polynomials and the phases of the Blaschke products, share certain uniformity/compactness properties, that turn out to be sufficient for the boundedness of the Carleson-type operators. Thus, applying the technique of the polynomial Carleson theorem, I give the following partial answers to Question 3.

Theorem 4. *Let $0<r<1$ and let $(a_n)_{n=1}^{\infty}$ be an arbitrary sequence such that $|a_n|\le r$ for all n, then*

$$\|T^{(a_n)}\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})}\lesssim \sqrt{\log\frac{1}{1-r}}. \tag{2.10}$$

Moreover, this estimate is sharp, i.e. there exists a sequence (a_n) with $|a_n|\le r$ such that

$$\|T^{(a_n)}\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})}\gtrsim \sqrt{\log\frac{1}{1-r}}. \tag{2.11}$$

Theorem 5. *Let a_n be inside the triangle with vertices $(1,0),(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},-\frac{1}{2})$ for all n, then*

$$\|T\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})}\lesssim 1. \tag{2.12}$$

In [Mna22a], we also prove certain L^p bounds and a conformally invariant version of Theorem 4.

3 The non-linear Fourier transform

Another way to look at the Fourier transform is as a solution to an ODE. More precisely, let $f \in L^1(\mathbb{R})$ and consider the equation

$$\begin{cases} \partial_t G(t,x) = e^{-2ixt} f(t) G(t,x) \\ G(-\infty, x) = 1. \end{cases} \tag{3.1}$$

There exists a unique solution to (3.1) and it almost recovers the classical Fourier transform of f, i.e.

$$\exp(\hat{f}(x)) = G(\infty, x).$$

One can consider matrix-valued analogs of the above equation such as

$$\begin{cases} \partial_t G(t,x) = \begin{pmatrix} 0 & e^{-2ixt} f(t) \\ e^{2ixt}\overline{f(t)} & 0 \end{pmatrix} G(t,x), \\ G(-\infty, x) = I \end{cases} \tag{3.2}$$

where G is a 2×2 matrix. We call the unique solution to the above equation the non-linear Fourier transform (NLFT) of f. It is well-known that the solution of (3.2) takes values in $SU(1,1)$, that is

$$G(t,x) = \begin{pmatrix} \overline{a(t,x)} & \overline{b(t,x)} \\ b(t,x) & a(t,x) \end{pmatrix} \text{ and } |a(t,x)|^2 - |b(t,x)|^2 = 1. \tag{3.3}$$

The NLFT has many similarities with the classical Fourier transform. For example, it takes translations to certain modulations and vice versa. It also scales in L^1 but, unlike the classical one, it does not scale in other L^p's. A fundamental property of the NLFT is the analog of the Plancherel identity, that is

$$\| \sqrt{\log |a(\infty, \cdot)|} \|_{L^2(\mathbb{R})} = \sqrt{\frac{\pi}{2}} \|f\|_{L^2(\mathbb{R})}. \tag{3.4}$$

From the work of Christ and Kiselev [CK02] on the spectral theory of the one-dimensional Schrödinger operator, one can deduce [TT12] Hausdorff-Young and Menshov-Paley-Zygmund inequalities for the NLFT for $1 \leq p < 2$. For $p = 2$ the analogues estimate, the non-linear version of the Carleson theorem, was conjectured by Muscalu, Tao and Thiele in [MTT02], where they proved it for the Cantor group model of the NLFT.

Conjecture 6. *Let $f \in L^2(\mathbb{R})$, then*

$$|\{x \in \mathbb{R} : \sup_t \sqrt{\log |a(t,x)|} > \lambda\}| \lesssim \frac{1}{\lambda^2} \|f\|_2^2. \tag{3.5}$$

In connection to Conjecture 6, there has been a recent breakthrough by Poltoratski.

Theorem 7 ([Pol21])**.** *Let $f \in L^2(\mathbb{R}_+)$ be real-valued, then $|a(t,\cdot)|, |b(t,\cdot)|$ and $\frac{b(t,\cdot)}{a(t,\cdot)}$ converge almost everywhere as $t \to +\infty$.*

Poltoratski uses complex analysis and, in particular, de Branges functions which are the continuous analogs of orthogonal polynomial on the unit circle [Den06]. His methods are purely qualitative and do not address Conjecture 6. He shows that the zeros of the de Branges function

$$E(t,z) := e^{-itz}(a(t,z) + b(t,z)) \tag{3.6}$$

near the point $x \in \mathbb{R}$ asymptotically control the behavior of $E(t,x)$. In particular, $|E(t,x)|, |a(t,x)|$ and $|b(t,x)|$ converge if the closest zero of $E(t,\cdot)$ to x converges to infinity in the time-scaled distance. Then, he proves that this happens for almost every point $x \in \mathbb{R}$.

In my paper [Mna22b], I quantify the arguments of Poltoratski and obtain estimates of the de Branges function through the Hardy-Littlewood maximal function of the spectral density. For $f \in L^1$, the spectral density is defined as

$$w(x) := \Re\left(\frac{1 - b(\infty,x)/a(\infty,x)}{1 + b(\infty,x)/a(\infty,x)}\right). \tag{3.7}$$

If $\|f\|_1 \leq 1$, then it is not difficult to check that

$$\|w-1\|_\infty \leq \|f\|_1, \quad \|w-1\|_2 \leq \|f\|_2.$$

One of the two main estimates in [Mna22b] is the following.

Theorem 8. *There exist $\delta, D > 0$ such that the following holds. Let $x \in \mathbb{R}$ and $t > 0$. If $f \in L^1(\mathbb{R}_+)$ with compact support is such that $\|f\|_1 \leq \delta$ and if z_0 is the closest zero of $E(t,\cdot)$ to the point x, then*

$$\left||E(t,x)| - \frac{1}{\sqrt{w(x)}}\right| \leq D(M(w-1)(x)|\log M(w-1)(x)|)^{\frac{1}{4}} + \frac{D}{\sqrt{t|z_0 - x|}}, \tag{3.8}$$

where M denotes the Hardy-Littlewood maximal function.

The above theorem gives a good estimate of $E(t,x)$ when its zeros are far away from x. We also have an estimate when the zeros are close to x, but as it looks more technical we do not state it here. As a corollary to our estimates for E, we obtain the following structural result about its zeros.

Corollary 9. *There exists $\delta > 0$ such that, if $\|f\|_1 \leq \delta$, then there are no zeros of $E(t,\cdot)$ in the region $\{z : |\Im z| \leq \frac{1}{t}\}$.*

My estimates quantify the first part of Poltoratski's proof of Theorem 7. At the moment, I work on the quantification of the second part, which leads to the following non-optimal weak-type bounds for the NLFT.

Conjecture 10. *There exists $\delta > 0$ and $D > 0$ such that if $f \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ with $\|f\|_1 \leq \delta$, then for $0 < \lambda < 1$*

$$|\{x \in \mathbb{R} : \sup_t \sqrt{\log|a(t,x)|} > \lambda\}| \leq \frac{D}{\lambda^{33}}\|f\|_2^2. \tag{3.9}$$

As the NLFT scales exclusively in L^1, the inequality (3.9) does not imply the optimal inequality (3.5) like in the linear case. Furthermore, even though all our results assume the smallness of the L^1 norm of the potential, Conjecture 6 with this restriction is still strong enough to imply the linear Carleson theorem.

4 Weighted estimates for Sparse operators

It is well known from 1970's works of Muckenhoupt [Muc72] and others, that the Hardy-Littlewood maximal function, Caldéron-Zygmund operators, etc. are bounded on weighted L^p spaces if and only if the underlying weight is in the Muckenhoupt's A_p class. The question of the sharp dependence of those operator norms on the weight has a more recent history. For the maximal function, Buckley [Buc93] showed in 1993 that the following inequality is sharp in the dependence on the A_p characteristic of the weight

$$\|M\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}. \tag{4.1}$$

For CZ operators the same question was known under the name of the A_2 conjecture and, after a number of contributions by many authors, was finally settled in 2010 by Hytönen [Hyt10] who proved

$$\|T\|_{L^p(w)\to L^p(w)} \le [w]_{A_p}^{\max(1,\frac{1}{p-1})}. \tag{4.2}$$

In 2012 Lerner [Ler12] introduced the sparse operators to give a simplified proof of the A_2 theorem. We call a family $\mathcal{S}$ of balls in $\mathbb{R}^n$ (γ-)sparse, for some $0<\gamma<1$, if there exists pairwise disjoint subsets $E_B \subset B \in \mathcal{S}$, such that $|E_B| \ge \gamma |B|$. For a sparse family the associated sparse operator is the defined as

$$\mathcal{A}_{\mathcal{S}} f(x) := \sum_{B\in\mathcal{S}} \left(\frac{1}{|B|}\int_B |f|\right) \cdot \mathbf{1}_B(x). \tag{4.3}$$

It is fairly easy to prove the inequality (4.2) for the sparse operators, namely,

$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} \le [w]_{A_p}^{\max(1,\frac{1}{p-1})}. \tag{4.4}$$

On the other hand, the sparse operators pointwise dominate CZ operators [LN15; CR16]. More specifically, for a CZ operator T and $f \in L^1_{loc}$ there exist sparse families $(S_j)_{j=1}^{3^n}$ such that

$$|Tf(x)| \lesssim \sum_j \mathcal{A}_{\mathcal{S}_j} f(x) \text{ for a.e. } x. \tag{4.5}$$

The combination of these two facts immediately implies the A_2 theorem.

In the paper [Mna22c], I study the sharp weighted estimates of the following version of the sparse operators. Let

$$M_B(f) := \sup_{A\supset B} \frac{1}{|A|}\int_A |f|,$$

and let $\mathcal{S}$ be a sparse family, then we define the associated strong-sparse operator as

$$\mathcal{A}^*_{\mathcal{S}} f(x) := \sum_{B\in\mathcal{S}} M_B(f) \cdot \mathbf{1}_B(x). \tag{4.6}$$

Question 11. *What are is the sharp dependence of $\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)}$ on the A_p characteristic of w?*

One trivially has the estimate

$$\mathcal{A}^*_{\mathcal{S}} f(x) \le \mathcal{A}_{\mathcal{S}} M f(x) \tag{4.7}$$

for any x. Hence, the sharp weighted estimates for the maximal and sparse operators (4.1) and (4.4) imply

$$\|\mathcal{A}_{\mathcal{S}}^*\|_{L^2(w)\to L^2(w)} \leq \|\mathcal{A}_{\mathcal{S}}\|_{L^2(w)\to L^2(w)} \|M\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}^2. \tag{4.8}$$

It turns out that (4.8) is sharp, however the classical examples of A_2 weights such as the power weights are not good enough to realize the upper bound.

Theorem 12. *There exists an A_2 weight that realizes the upper bound of (4.8). That is, the following inequalit is sharp*

$$\|\mathcal{A}_{\mathcal{S}}^*\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}^2. \tag{4.9}$$

In [Mna22c], I also prove the weighted weak bounds, as well as some improved strong bounds for sparse families with certain restrictions.

5 Future directions

My main current project is concerned with Conjecture 10 as a step towards Conjecture 6. Also by the end of my PhD, I hope to have an answer for Question 1.

I consider all three projects described above to be entry points for me into very exciting areas of harmonic analysis and not only. I am generally open to anything interesting and whether I will immerse myself into those areas deeper or move on to something new depends on the opportunities that will come along.

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