EXISTENCE AND REGULARITY OF STEADY-STATE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS ARISING FROM IRREGULAR DATA

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Abstract. We analyze the forced incompressible stationary Navier-Stokes flow in $\mathbb{R}^n_+$, $n \geq 2$. Existence of a unique solution satisfying a global integrabilty property measured in tent spaces is established for small data in homogenous Sobolev space with $-\frac{1}{2}$-degree of smoothness. Moreover, the pressure and the first-order derivative of the velocity field are shown to be locally Hölder continuous.
Our approach is based on the analysis of the inhomogeneous Stokes system for which we derive a new solvability result involving Dirichlet data in Triebel-Lizorkin classes with negative amount of smoothness and is of independent interest.

Contents

1. Introduction

The steady state (forced) incompressible Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is the following system

$$(\mathbf{NS}) \qquad \begin{cases} -\Delta u + \nabla \pi + u \cdot \nabla u = F \text{ in } \Omega \\ \operatorname{div} u = 0 \text{ in } \Omega \end{cases}$$

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where $u : \Omega \to \mathbb{R}^n$ is the unknown velocity field, $\pi : \Omega \to \mathbb{R}$ is the unknown scalar pressure and $F : \Omega \to \mathbb{R}^n$ is a given external force. This system is supplemented by the boundary condition

$$u = f \;\text{ on }\; \partial\Omega \tag{1.1}$$

where $f = (f_1, ..., f_n)$ is a prescribed vector field satisfying (in the case Ω smooth bounded) the compatibility condition $\displaystyle\int_{\partial\Omega} f \cdot \mathbf{N} d\sigma(Q)$ with $\mathbf{N} = (N_1, \cdots, N_n)$ being the outer unit normal vector at the boundary.

Probably, the first striking result regarding the solvability of the Dirichlet problem for the Navier-Stokes equations was obtained by Leray [19]. In a bounded three dimensional domain, he showed the existence of a weak solution $(u, \pi) \in W^{1,q}(\Omega) \times L^q(\Omega)$ provided $f \in W^{1-1/q,q}(\partial\Omega)$ and $F \in W^{-1,q}(\Omega)$, $q \in [2, \infty)$. Although $n = 3$ is the physically most relevant dimension, this result remains true in any dimension. Also, any weak solution is known to be smooth (see e.g. [25] in the case $n = 2, 3$ and nonhomogeneous smooth data, [12] in the case $n = 4$ and $f = 0$ on $\partial\Omega$ and the monograph [10] for a complete theory). As for uniqueness of weak solutions, it seems that a smallness assumption on the given data is necessary and a recent result by Luo [20] predicts that this condition cannot be dropped.

There have been growing interest in recent years in the analysis of the Navier-Stokes equations subject to low regularity data. By this, we mean that f satisfies a weaker regularity than that needed in order for generalized weak solutions to exit. In such a context, does problem (**NS**)-(1.1) admits a solution? In the affirmative case, what are the qualitative properties of such solutions? Prescribing boundary value with low regularity forces one to consider a notion of solution weaker than weak solutions. A good candidate, roughly speaking is obtained by testing (**NS**) against a suitable divergence-free smooth vector field and performing two successive integration by parts. This idea to the best of our knowledge first appeared in [2]. When $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ is a C^2 regular bounded domain, the author in [21] constructed such a solution in $L^{2n/(n-1)}(\Omega)$ provided $f \in L^2(\partial\Omega)$ (with arbitrarily large norm) and $F \in W^{-1,2n/(n-1)}(\Omega)$. Existence, uniqueness and regularity of very weak solutions (u, π) in the class $L^q(\Omega) \times W^{-1,q}(\Omega)$ have been obtained in [7, 11] under certain smallness conditions on $f \in W^{-1/q,q}(\partial\Omega)$ and $F \in W^{-1,r}(\Omega)$ with $1 < r \leq q < \infty$, $1/r \leq 1/n + 1/q$. These results were generalized in [18] where the author gave a complete theory for very weak solutions of (**NS**)-(1.1). In particular, refining the definition of very weak solutions and using some ideas from the preceding references, the author showed the existence of $(u, \pi) \in L^n(\Omega) \times W^{-1,n}(\Omega)$ for arbitrary large data f and forcing term F for $n = 3, 4$. In two-dimensions, he proved existence of $(u, \pi) \in L^{q_0}(\Omega) \times W^{-1,1/q_0}(\Omega)$, $2 < q_0 < 3$. Moreover, he investigated the regularity of these solutions and also derived uniqueness results under suitable smallness assumptions. The existence theory for very weak solutions in unbounded domains (the half-space, exterior domains, ect.) seems to be more subtle. In general, similar methods as those employed for instance in [18] which rely on duality arguments and functional analytic tools cannot be carried out. We refer the reader to [9] for an interesting discussion pertaining to generalized (weak) solutions – we point out however, some recent existence results for the linear Stokes problem in half-space domain [8] and in exterior domains [17].

This paper aims at establishing the solvability theory for (**NS**) in the case $\Omega = \mathbb{R}^n_+$ (with possible adaptation of the ideas to the case of bounded smooth domains) by means of novel ideas. The techniques employed here are new and complement those introduced by H. Koch and the author in [30] for the analysis of elliptic "critical" problems subject to low regularity data. In addition, they can be invoked to study similar questions for other semilinear elliptic equations.

Assume $\Omega = \mathbb{R}^n_+$, we seek for velocity field of the form $u = v + w$ where v solves the linear Stokes equation with Dirichlet data f while w solves the inhomogeneous Stokes problem with zero boundary data and source term $F + u \cdot \nabla u$. Odqvist [24] proved that v assumes an integral representation, it is the Stokes extension of f to $\mathbb{R}^n_+$ (see Section 2). We look for f in a large class of distributions on $\mathbb{R}^{n-1}$ for which v is well-defined and has f as trace in a suitable sense. On one hand, (**NS**) is scaling (and translation) invariant with respect to the maps

$$u_\lambda(x) = \lambda u(\lambda x), \quad \pi_\lambda(x) = \lambda^2 \pi(\lambda x), \quad \lambda > 0$$

for appropriately rescaled external force and boundary data. On the other hand, we want to have u in the local Lebesgue space L^2_{loc} in order to make sense of the equation. From these observations, we are led to the consideration of v in $T^{2(n-1),2}$, a scale of tent spaces introduced by Coifmann, Meyer and Stein in [6]. Thanks to the work by Triebel [28], we know that v must have a distributional trace f in the homogeneous negative Sobolev space $\dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})$. By the same token, the pressure π is sought for in the weighted tent space $T^{2(n-1),2}_{-1/(n-1)}$ (see below for the definition of weighted tent spaces). Tent spaces naturally arise in the analysis of linear elliptic equations and systems, see e.g. [4, 15] and references therein. We quote the recent work [30] where these spaces are used in the context of nonlinear systems.

Our main result states that there exits a unique solution of (**NS**)-(1.1) in a suitable framework under a smallness condition on $f \in \dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})$. A more general statement involving Dirichlet data in homogeneous Triebel-Lizorkin spaces is obtained. In both cases, the solutions constructed satisfy some global integrability property, expressed in terms of tent norms and it is further shown that these solutions enjoy a higher regularity locally. This latter feature is derived from the pointwise decay rate of the velocity field near the boundary. In order to derive all the previous results, we study the inhomogeneous Stokes problem in $\mathbb{R}^n_+$ (which plays a fundamental role in the analysis of (**NS**) when the flow takes place in an exterior region, a channel or a pipe) and derived key estimates of the solution for prescribed data in the homogeneous Triebel-Lizorkin class with negative amount of smoothness.

1.1. **Tent spaces and functional settings.** Throughout, a point $x \in \mathbb{R}^n_+$ will typically be denoted by (x', x_n), $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$. For $R > 0$, $B_R(x')$ is the closed ball with radius $R > 0$ and center at $x' \in \mathbb{R}^{n-1}$. Given $\alpha > 0$, define the cone (nontangential region) with vertex at $x' \in \mathbb{R}^{n-1}$ by

$$\mathit{\Gamma}_\alpha(x') := \{(y', y_n) \in \mathbb{R}^n_+ : |x' - y'| < \alpha y_n\}.$$

We simply use the notation $\mathit{\Gamma}$ when $\alpha = 1$. Given a ball $B = B_R(x')$, we denote by $T(B) = B_R(x') \times (0, \mathrm{diam}(B_R(x')))$ the Carleson box over $B_R(x')$. For $q \in [1, \infty)$, consider the functionals $\mathcal{A}_q$, $\mathscr{C}_q$ defined for F measurable in $\mathbb{R}^n_+$ by

(1.2)

$$\mathcal{A}_q F(x') = \left(\iint_{\mathit{\Gamma}(x')} |F(y', y_n)|^q s^{-(n-1)} dy' dy_n \right)^{1/q}, \ \mathcal{A}_\infty F(x') = \operatorname*{ess\,sup}_{(y',y_n) \in \mathit{\Gamma}(x')} |F(y', y_n)|$$

$$\mathscr{C}_q F(x') = \sup_{B \ni x'} \left(\iint_{T(B)} |F(y', y_n)|^q dy_n dy' \right)^{1/q}. \tag{1.3}$$

The membership of each of these functionals in a Lebesgue space gives rise to a scale of functions space first introduced by Coifman, Meyer and Stein [6]. We point out here the use of a different normalization in (1.2) and (1.3). Let $p, q \in [1, \infty)$. The tent space $T^{p,q}$

collects all functions $F \in L^q_{loc}(\mathbb{R}^n_+)$ for which $\mathcal{A}_q F \in L^p(\mathbb{R}^{n-1})$. We equip this space with the norm

$$\|F\|_{T^{p,q}} := \|\mathcal{A}_q F\|_{L^p(\mathbb{R}^{n-1})}. \tag{1.4}$$

When $p = \infty$, the space $T^{\infty,q}$ is defined by

$$T^{\infty,q} = \{F \in L^q_{loc}(\mathbb{R}^n_+) : \mathscr{C}_q \in L^\infty(\mathbb{R}^{n-1})\}.$$

The space $T^{\infty,q}$ is intrinsically linked to Carleson measures. In fact, it is the space of functions $F \in L^q_{loc}(\mathbb{R}^n_+)$ for which $d\mu(y', y_n) = |F|^q dy' dy_n$ is a Carleson measure in $\mathbb{R}^n_+$. For any $q \in [1, \infty)$ and $p \in (1, \infty]$, $T^{p,q}$ is a Banach space having the space of functions in $L^q(\mathbb{R}^n_+)$ with compact support as a dense subspace. This property together with the completeness of $T^{p,q}$ for any $q \in [1, \infty]$ follows from Lemma 1.5 below.

Lemma 1.5. *Let K be a compact set in $\mathbb{R}^n_+$ and assume that $F \in T^{p,q}$ for $p, q \in [1, \infty)$. Then*

$$C_1 \|\mathbf{1}_K F\|_{T^{p,q}} \le \|F\|_{L^q(K)} \le C_2 \|F\|_{T^{p,q}} \tag{1.6}$$

where the constant C_1, C_2 only depend on p, q, n and K.

Out of convenience, we defer the proof of Lemma 1.5 to the Appendix.

Remark 1.7. We also remark that change of aperture in the cone does not affect the tent norm. In other words, if

$$\mathcal{A}^\alpha_q F(x') := \left(\iint_{\Gamma_\alpha(x')} |F(y', y_n)|^q s^{-(n-1)} dy' dy_n \right)^{1/q}, \quad \alpha > 0$$

then

$$\|\mathcal{A}^\alpha_q\|_{L^p(\mathbb{R}^{n-1})} \approx \|\mathcal{A}^\beta_q F\|_{L^p(\mathbb{R}^{n-1})} \tag{1.8}$$

where the implicit constant depends on p, q and $\alpha, \beta \in (0, \infty)$. See [6, Proposition 4, p. 309] which remains valid for $q \neq 2$.

For $s \in \mathbb{R}$, we say that $F : \mathbb{R}^n_+ \to \mathbb{R}$ belongs to the weighted tent spaces [1, 15], which we denote by $T^{p,q}_s$ if

$$(y', y_n) \mapsto y_n^{-(n-1)\nu} F(y', y_n) \in T^{p,q}.$$

We easily verify that $\|F\|_{T^{p,q}_\nu} := \left\| y_n^{(1-n)s} F \right\|_{T^{p,q}}$ defines a norm on $T^{p,q}_s$. Moreover, for $s_1, s_2 \in \mathbb{R}$ such that $s_2 < s_1$ and $1 \le p_1 < p_2 \le \infty$, $q \in (0, \infty]$ the following continuous embedding holds (see [1, Lemma 2.21])

$$T^{p_1 q}_{s_1} \subset T^{p_2 q}_{s_2} \tag{1.9}$$

provided $s_2 - s_1 = \dfrac{1}{p_2} - \dfrac{1}{p_1}$. Recall Hölder's inequality in weighted tent spaces.

Lemma 1.10. *Let $p_i, q_i, r_i \in [1, \infty]$, $s_i \in \mathbb{R}$ and denote by p'_i the conjugate exponent of p_i, $i \in \{1, 2, 3\}$. If $f \in T^{p_1,q_1}_{s_1}$ and $g \in T^{p_2,q_2}_{s_2}$, then $fg \in T^{p_0,q_0}_{s_0}$ and it holds that*

$$\|fg\|_{T^{p_0,q_0}_{s_0}} \le C \|f\|_{T^{p_1,q_1}_{s_1}} \|g\|_{T^{p_1,q_2}_{s_2}} \tag{1.11}$$

provided $s_0 = s_1 + s_2$.

This lemma can be proved via a direct argument – there is also another strategy relying on factorization of tent spaces, see [14]. It is long-established that there is an intrinsic connection between weighted tent spaces and Triebel-Lizorkin spaces which we now recall its definition.

Let us denote by $\mathcal{S}(\mathbb{R}^{n-1})$ the class of Schwartz (smooth rapidly decreasing) functions on $\mathbb{R}^{n-1}$ and $\mathcal{S}'(\mathbb{R}^{n-1})$ its topological dual space endowed with the weak-$\star$ topology. Define the space

$$\dot{\mathcal{S}}(\mathbb{R}^{n-1}) = \{f \in \mathcal{S}(\mathbb{R}^{n-1}) \mid \int x^\gamma f(x)dx = 0,\ \forall \gamma \in \mathbb{N}^n\}$$

which inherits the topology of $\mathcal{S}(\mathbb{R}^{n-1})$ as subspace. This space may be identified with the space of Schwartz functions whose Fourier transforms vanish together with all their derivatives at the origin. Its dual space is denoted by $\dot{\mathcal{S}}'(\mathbb{R}^{n-1})$. Let φ be a cut-off function given by

$$\varphi(\xi) = \begin{cases} 1 \ \text{if} \ |\xi| \le 1 \\ \text{smooth} \ \text{if} \ 1 < |\xi| \le 2 \\ 0 \ \text{if} \ |\xi| > 2. \end{cases}$$

Let $\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$ and define $\psi_j(\xi) = \psi(2^{-j}\xi)$, $j \in \mathbb{Z}$ so that

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1,\ \ \xi \in \mathbb{R}^{n-1} \setminus \{0\}.$$

Let us denote by $\mathcal{F}f$ the Fourier transform of f on $\mathbb{R}^{n-1}$ and $\dot{\Delta}_j = \mathcal{F}^{-1}(\psi_j \mathcal{F})$ the homogeneous Littlewood-Paley operator. Let $f \in \dot{\mathcal{S}}'(\mathbb{R}^{n-1})$. For $s \in \mathbb{R}$; $p, q \in [1, \infty)$ we say that f belongs to the Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^{n-1})$ if

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^{n-1})} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{-jsq} |\dot{\Delta}_j f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n-1})} < \infty.$$

This space is of Banach type and is equivalent to the Sobolev space $\dot{H}^{s,p}(\mathbb{R}^{n-1})$ whenever $q = 2$ and $1 < p < \infty$. Moreover, for $1 \le q_1, q_2 \le \infty$ and $-\infty < s_2 < s_1 < \infty$ we have the continuous inclusion

$$\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^{n-1}) \subset \dot{F}^{s_2}_{p_2,q_2}(\mathbb{R}^{n-1})$$

provided $p_1, p_2 \in (1, \infty)$ with $s_1 - \frac{n-1}{p_1} = s_2 - \frac{n-1}{p_2}$.

Definition 1.12. For $q \in (\frac{n}{n-1}, \infty)$, $n \ge 2$ we define $\mathbf{X}^q$ as the space of vector fields $u : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfying $\|u\|_{\mathbf{X}^q} < \infty$ and $\mathbf{Z}^q := \{\pi : \mathbb{R}^n_+ \to \mathbb{R} \mid \|\pi\|_{\mathbf{Z}^q} < \infty\}$ where

$$\|u\|_{\mathbf{X}^q} = \sup_{x_n>0} x_n^{\frac{1}{q-1}} \|u(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \|u\|_{T^{p,q}}$$

and

$$\|\pi\|_{\mathbf{Z}^q} := \|\pi\|_{T^{p,q}_{s_0}},\ \ s_0 = -\frac{1}{n-1},\ p = (n-1)(q-1)q.$$

Definition 1.13. Let $1 \le \eta < \tau < \infty$. We say that $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ belongs to $\mathbf{Y}^{\tau,\eta}$ if

$$\|F\|_{\mathbf{Y}^{\tau,\eta}} = \sup_{x_n>0} x_n^{\frac{1}{\eta}+\frac{n-1}{\tau}} \|F(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \|F\|_{T^{\tau,\eta}}$$

is finite.

Note that either of the expression $\|\cdot\|_{\mathbf{X}^q}$ or $\|\cdot\|_{\mathbf{Z}^q}$ defines a norm on $\mathbf{X}^q$ and $\mathbf{Z}^q$ respectively. It can also be easily verified that they are Banach spaces. For convenience, when $q = 2$, we will specially denote the spaces $\mathbf{X}^q$ and $\mathbf{Z}^q$ by $\mathbf{X}$ and $\mathbf{Z}$, respectively.

1.2. **Main results.** Our first result deals with the well-posedness theory. In what follows, the dimension is assumed larger or equals to 3 unless otherwise stated.

Theorem 1.14. *Assume that* $F = 0$*. System* (**NS**) *has a unique solution* (u, π) *in a small closed ball of* $\mathbf{X} \times \mathbf{Z}$ *provided the data* f *has a sufficiently small* $[\dot{H}^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})]^n$*-norm.*

In presence of the forcing term, our main finding reads as follows.

Theorem 1.15. *Let* $1 < \eta < \tau < \infty$ *such that* $\dfrac{1}{\eta} + \dfrac{n-1}{\tau} = 3$*. There exist* $\varepsilon > 0$ *and* $\kappa := \kappa(\varepsilon) > 0$ *such that for every* $f \in [\dot{H}^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})]^n$ *and* $F \in \mathbf{Y}^{\tau,\eta}$ *satisfying* $\|f\|_{\dot{H}^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau,\eta}} < \varepsilon$*, Eq.* (**NS**) *has a solution* (u, π) *in* $\mathbf{X} \times \mathbf{Z}$ *which is the only one among those satisfying the condition* $\|u\|_{\mathbf{X}} + \|\pi\|_{\mathbf{Z}} \leq 2\kappa$*.*

Existence of solutions in $\mathbf{X}^q \times \mathbf{Z}^q$ for any $2 < q < \infty$ is a consequence of Theorem 1.15 together with an improved regularity result. In more details, the statement reads as follows.

Theorem 1.16. *Let* $2 < q < \infty$*,* $\eta > 1$ *as in Theorem 1.15 and* $1 < \eta_1 < \tau_1 < \infty$*. Given* f *in* $\left[\dot{H}^{-\frac{1}{2},2(n-1)} \cap \dot{F}^s_{p,q}(\mathbb{R}^{n-1})\right]^n$ *and* $F \in \mathbf{Y}^{\tau,\eta} \cap \mathbf{Y}^{\tau_1,\eta_1}$*, there exist* $\varepsilon_q \in (0, \varepsilon)$ *and* $\kappa_q > 0$ *such that if* $\|f\|_{\dot{H}^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau,\eta}} < \varepsilon_q$ *then there exists a solution* (u, π) *of* (**NS**) *in the space* $\mathbf{X}^q \times \mathbf{Z}^q$ *which is unique in the ball*

$$B_{2\kappa_q}(\mathbf{0}) = \{(u, \pi) \in \mathbf{X} \times \mathbf{Z} : \|u\|_{\mathbf{X}} + \|\pi\|_{\mathbf{Z}} \leq 2\kappa_q\}$$

provided $\frac{1}{\eta_1} + \frac{n-1}{\tau_1} = 2 + \frac{1}{q-1}$*,* $s = -\frac{1}{q}$ *and* $p = (n-1)(q-1)q$*.*

The uniqueness of the pressure as claimed in the previous results should be understood up to an additive constant. We also record the following regularity result which arises as a consequence of the local boundedness property of the velocity field.

Theorem 1.17. *If* $(u, \pi) \in \mathbf{X} \times \mathbf{Z}$ *is the solution of the Navier-Stokes equation* (**NS**) *constructed in Theorem 1.15 (or* $(u, \pi) \in \mathbf{X}^q \times \mathbf{Z}^q$ *obtained in Theorem 1.16), then* $(u, \pi) \in [C^{1,\alpha}_{loc}(\mathbb{R}^n_+)]^n \times C^{0,\alpha}_{loc}(\mathbb{R}^n_+)$*. In addition, if* F *is identically zero, then the solution is infinitely locally smooth,* $u \in [C^{\infty}_{loc}(\mathbb{R}^n_+)]^n \times C^{\infty}_{loc}(\mathbb{R}^n_+)$*.*

Remark 1.18. It is easy to see that Theorem 1.15 in the precise form stated above fails to hold in two dimensions. This is due to the presence of the forcing term F as the required condition on η and τ will fail to hold if $n = 2$. However, a close inspection reveals that the two-dimensional (unforced) Navier-Stokes equations is well-posed in our functional setting. Theorem 1.16 shows that if f is taken in a slightly more regular space, then the solution (u, π) has a better global integrability property.

2. Auxiliary results

This section is devoted to the analysis of the Dirichlet problem for the following system

$$(\mathbf{S}) \qquad \begin{cases} -\Delta u + \nabla \pi = F + \operatorname{div} H & \text{in } \mathbb{R}^n_+ \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n_+ \\ u = f & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

for given vector fields f, F and tensor H. Our goal is to prove that (**S**) admits a solution (u, π) in the target space $\mathbf{X}^q \times \mathbf{Z}^q$ whose norm can be estimated by the norms of f, F and H in suitable functions spaces. To this end, for better readability we simply separate the study into two parts: the homogeneous case ($f = 0$) and the inhomogeneous case ($F = 0$, $H = \mathbf{0}$).

2.1. **Homogeneous Stokes system and linear estimates.** Consider the Stokes operator L_S acting on pair of functions $(u,\pi) \in [\mathscr{D}'(\mathbb{R}^n)]^n \times \mathscr{D}'(\mathbb{R}^n)$, $n \geq 2$ and given by

$$L_S(u,\pi) = \big(-\Delta u_1 + \partial_1 \pi, \cdots, -\Delta u_n + \partial_n \pi, \sum_{i=1}^n \partial_i u_i \big).$$

A fundamental solution of the Stokes operator L_S in $\mathbb{R}^n$ is a pair $(\mathbb{E}, \mathbf{b})$ with $\mathbb{E} = (E_{ij})_{i,j=1}^n$ in $\mathcal{M}_{n\times n}[\mathcal{S}'(\mathbb{R}^n)]$ and $\mathbf{b} = (b_1, ..., b_n) \in [\mathcal{S}'(\mathbb{R}^n)]^n$ satisfying coordinate-wise the equations

$$\begin{cases} -\Delta E_{ij} + \partial_i b_j = \delta_{ij}\delta \text{ in } \mathcal{S}'(\mathbb{R}^n), \;\; i,j \in \{1,...,n\} \\ \displaystyle\sum_{k=1}^n E_{kj} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^n), \;\; j \in \{1,...,n\}. \end{cases}$$

Applying the Fourier transform to both sides of each of the above equations yields the following explicit expressions

(2.1)
$$E_{ij}(x) = \frac{1}{2\omega_{n-1}} \left[\frac{1}{(n-2)} \frac{\delta_{ij}}{|x|^{n-2}} + \frac{x_i x_j}{|x|^n} \right],\; b_j = -\frac{1}{\omega_{n-1}} \frac{x_j}{|x|^n};\; x \in \mathbb{R}^{n-1} \setminus \{0\},\; i,j \in \{1,...,n\}.$$

where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^{n-1}$. Details of explicit computations leading to (2.1) can be found in [23, Chap. 10]. Now, let us consider the homogeneous Stokes system

$$\begin{cases} -\Delta u + \nabla \pi = 0 \;\text{ in }\; \mathbb{R}^n_+ \\ \operatorname{div} u = 0 \;\text{ in }\; \mathbb{R}^n_+ \\ u = f \;\text{ on }\; \partial \mathbb{R}^n_+. \end{cases} \tag{2.2}$$

With the convolution being understood in a component wise sense, define

$$\mathcal{H}f(x',x_n) = (\mathcal{K}_{x_n} * f)(x'), \quad \mathcal{E}f(x',x_n) = (\mathrm{k}_{x_n} * f)(x') \tag{2.3}$$

where

$$\mathcal{K}_{x_n}(x') = (K_{ij}(x',x_n))_{1\leq i,j\leq n} \text{ and } \mathrm{k}_{x_n}(x') = (\mathrm{k}_1(x,x_n), ..., \mathrm{k}_n(x',x_n))$$

are commonly referred to as the Odqvist kernels [24] – each entry of the tensors assuming an explicit form in terms of (2.1) via the formulas

$$K_{ij}(x) = 2(\partial_{x_n} E_{ij} + \partial_j E_{in} + \delta_{jn} b_i) = \frac{2n}{\omega_{n-1}} \frac{x_n x_i x_j}{|x|^{n+2}} \tag{2.4}$$

and

$$\mathrm{k}_j(x) = 4\partial_j b_n = \frac{1}{\omega_{n-1}} \partial_j \frac{4x_n}{|x|^n}. \tag{2.5}$$

For the derivation of these kernels, the interested reader may as well consult the articles [24, 27]. Note that if f belongs to the weighted Lebesgue space $L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{(1+|x'|)^n}\big)$, then $u = \mathcal{H}f$ and $\pi = \mathcal{E}f$ are both meaningful as absolutely convergent integrals and (u,π) is the unique solution of Eq. (2.2) decaying at infinity. This is no longer the case if f is merely a generic distribution. In fact, the Stokes extension $\mathcal{H}$ does not map $\mathcal{S}'(\mathbb{R}^{n-1})$ into itself in general (for example, in one dimension $f(x') = x'^2 \in \mathcal{S}'(\mathbb{R})$ but $\mathcal{H}f \notin \mathcal{S}'(\mathbb{R})$). However, it can be shown that if $f \in \dot{\mathcal{S}}'(\mathbb{R}^{n-1})$, then so are $\mathcal{H}f$ and $\mathcal{E}f$. Poisson extensions of Schwartz distributions have been studied by H. Triebel [28] – they characterize almost all scale of Triebel-Lizorkin spaces on $\mathbb{R}^{n-1}$ using tent spaces. In particular, the following equivalence holds true

$$\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})} \sim \Big\| \mathcal{A}_q[\mathcal{P}_{x_n} * f] \Big\|_{L^p(\mathbb{R}^{n-1})}, \quad p = q(q-1(n-1)) \tag{2.6}$$

where $1 < q < \infty$ and $\mathcal{P}_{x_n}(x') = c_n x_n(|x'|^2 + x_n^2)^{-\frac{n}{2}}$ (with c_n normalizing constant such that $\mathcal{P}_{x_n}$ has a normalized L^1-norm equals to 1) is the Poisson kernel for the Laplacian in $\mathbb{R}^n_+$.

Lemma 2.7. *Given $q \in (\frac{n}{n-1}, \infty)$, $n \geq 2$ and set $p = (n-1)q(q-1)$. There exists a constant $C := C(n,q) > 0$ such that*

$$\|\mathcal{H}f\|_{\mathbf{X}^q} + \|\mathcal{E}f\|_{\mathbf{Z}^q} + \sup_{x_n>0} x_n^{q/(q-1)}\|\mathcal{E}f(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})} \tag{2.8}$$

for all $f \in [\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})]^n$ where $\mathcal{H}f$ and $\mathcal{E}f$ are defined as in (2.3).

We state two more auxiliary results which will be useful in the demonstration of Lemma 2.7.

Lemma 2.9 (Averaging Lemma)**.** *Assume that $F \in L^q_{loc}(\mathbb{R}^n_+)$, $q \geq 1$. We have*

$$\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} |F(y', y_n)|^q \frac{dy'dy_n}{y_n^{n-1}} dx' = \mu \int_{\mathbb{R}^n_+} |F(y)|^q dy$$

where $\mu > 0$ only depends on n, the dimension.

The proof of this identity follows from a simple application of Fubini–Tonelli's Theorem.

Lemma 2.10. *Let $K \subset \mathbb{R}^n_+$ compact set and $E(K) = \{x' \in \mathbb{R}^{n-1} : K \cap \Gamma(x') \neq \emptyset\}$. Then $E(K)$ is open, its Lebesgue measure $|E(K)|$ is finite and only depends on K.*

Proof. Let $x \in E(K)$, there exists $(y', y_n) \in \mathbb{R}^n_+$ with $(y, y_n) \in K$ and $y' \in B(x', y_n)$. Putting $R = y_n - |x' - y'| > 0$, it plainly follows that $B(x', R) \subset E(K)$. Moving on, we remark that $E(K)$ is actually bounded. Moreover, since K is compact, we may assume without loss of generality that $K = B_\theta(z') \times [a, b]$ for some $a, b, \theta > 0$ with $a < b$ and thus a simple covering argument implies that $|E(K)| \leq C\theta^n$ for some constant $C > 0$. □

Now we are ready to prove Lemma 2.7.

Proof of Lemma 2.7. By a direct computation, each coefficient of the matrix $\mathcal{K}$ satisfies the pointwise estimate $\left|\nabla^k K_{ij}(x', x_n)\right| \leq c x_n^{-k}\mathcal{P}_{x_n}(x')$, $k = 0, 1$ for each $i, j = 1, \cdots, n$. Also, from the explicit expression

$$\mathrm{k}_j(x', x_n) = \frac{4}{\omega_{n-1}} \begin{cases} \frac{x_n x_j}{|x|^{n+2}} & \text{if } j = 1, ..., n-1 \\ \frac{n x_n^2 - |x|^2}{n|x|^{n+2}} & \text{if } j = n \end{cases}$$

we verify that $|\mathrm{k}_j(x', x_n)| \leq C x_n^{-1}\mathcal{P}_{x_n}(x')$ for all $(x', x_n) \in \mathbb{R}^n_+$. Let $(\overline{u}, \overline{\pi}) = (\mathcal{H}(f), \mathcal{E}(f))$ be a solution to Eq. (2.2). For $x \in \mathbb{R}^n_+$ fixed, the interior estimate (see e.g. [26]) for the linear Stokes problem together with Lemma 2.9 allow us to write

$$\begin{aligned}
|\overline{\pi}(x)|^q &\leq C|B_{x_n/2}(x)|^{-1} \int_{B_{x_n/2}(x)} |\overline{\pi}(y', y_n)|^q dy'dy_n \\
&\leq C|B_{x_n/2}(x)|^{-1} \int_{B_{x_n/2}(x)} |y_n^{-1}(\mathcal{P}_{y_n} * f)|^q dy'dy_n \\
&\leq C|B_{x_n/2}(x)|^{-1} \int_{B_{x_n}(x')\times[x_n/3, 2x_n]} |y_n^{-1}(\mathcal{P}_{y_n} * f)|^q dy'dy_n \\
&\leq C x_n^{-(n+q)} \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{B_{y_n}(z')} \mathbf{1}_{B_{y_n}(x')\times[x_n/3, 2x_n]}(y', y_n)|\mathcal{P}_{y_n} * f|^q \frac{dy'dy_n}{y_n^{n-1}} dz'
\end{aligned}$$

$$\le C x_n^{-(n+q)} \big\| \mathcal{A}_q [\mathcal{P}_{y_n} * f] \big\|^q_{L^p(\mathbb{R}^{n-1})} \Big| E\big(B_{x_n}(x') \times [x_n/3, 2x_n]\big)\Big|^{\frac{p-q}{p}}$$
$$\le C x_n^{-1-q-\frac{(n-1)q}{p}} \|f\|^q_{\dot F^{-1/q}_{p,q}(\mathbb{R}^{n-1})}.$$

Observe that we have used Hölder's inequality and Lemma 2.10 to get the estimate before the last and the choice $p = q(n-1)(q-1)$ yields the desired bound. From the above remark on the kernel k_j, one has

$$\|\overline{\pi}\|_{T_1^{p,q}} = \left\| \left(\iint_{\Gamma(\cdot)} |y_n \overline{\pi}(y', y_n)|^q \frac{dy' dy_n}{y_n^{n-1}} \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n-1})}$$
$$\le C \left\| \left(\iint_{\Gamma(\cdot)} |\mathcal{P}_{y_n} * f|^q \frac{dy' dy_n}{y_n^{n-1}} \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n-1})} \le C \|f\|_{\dot F^{-1/q}_{p,q}(\mathbb{R}^{n-1})}.$$

The latter bound is a consequence of the extrinsic characterization (2.6). The same observation pertaining to the velocity field gives $\|\overline{u}\|_{T^{p,q}} \le C\|f\|_{\dot F^{-1/q}_{p,q}(\mathbb{R}^{n-1})}$. It then remains to establish the bound

$$\sup_{x_n>0} x_n^{\frac{1}{q-1}} \|\overline{u}(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \le C \|f\|_{\dot F^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})}. \tag{2.11}$$

Let f_j in $\dot F^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})$ and write $\overline{u}_i(x) = K_{ij}(\cdot, x_n) * f_j(x')$, $i = 1, ..., n$ with the summation convention so that by the mean value property for the velocity field [26, Theorem 4.5] and with the same notation as above

$$|\overline{u}_i(x)| \le \fint_{B_{x_n/2}(x)} |\overline{u}_i(y)| dy + \frac{1}{2} \fint_{B_{x_n/2}(x)} |\overline{\pi}(z)||z_i - x_i| dz := I + II, \ i = 1, 2, ..., n.$$

Using Hölder's inequality and Lemma 2.9, we estimate I as follows:

$$I^q \le C x_n^{-(n-1)} \int_{B_{x_n/2}(x)} |\mathcal{P}_{x_n} * f_j|^q dy$$
$$\le C x_n^{-(n-1)} \int_{\mathbb{R}^{n-1}} \iint_{\Gamma(z')} \mathbf{1}_{B_{x_n/2}(x)}(y) |\mathcal{P}_{x_n} * f_j|^q y_n^{-(n-1)} dy dz'$$
$$\le C x_n^{-(n-1)} \|\mathcal{P}_{x_n} * f_j\|^q_{T^{p,q}} \Big| E\big(B_{\frac{x_n}{2}}(x') \times [x_n/3, 2x_n]\big)\Big|^{\frac{p-q}{p}}$$
$$\le C x_n^{q/(q-1)} \|f\|^q_{\dot F^{1/q}_{p,q}(\mathbb{R}^{n-1})}. \tag{2.12}$$

In order to estimate the integral $II := \frac{1}{2} \fint_{B_{x_n/2}(x)} |\overline{\pi}(z)||z_i - x_i| dz$, we use the fact that if $z \in B_{x_n/2}(x)$, then $B_{x_n/2}(z) \subset B_{x_n}(x)$. Indeed, we have

$$II \le |B_{x_n/2}(x)|^{-1} \int_{B_{x_n/2}(x)} \left(\fint_{B_{x_n/2}(z)} |\overline{\pi}(y)| dy \right) |z_i - x_i| dz$$
$$\le C |B_{x_n}(x)|^{-1} \left(\fint_{B_{x_n}(x)} |\overline{\pi}(y)|^q dy \right)^{1/q} \int_{B_{x_n/2}(x)} |z - x| dz$$
$$\le C |B_{x_n}(x)|^{-1} \left(\fint_{B_{x_n}(x)} |\overline{\pi}(y)|^q dy \right)^{1/q} \int_0^{x_n/2} \sigma^n d\sigma$$
$$\le x_n^{-\frac{1}{q-1}} \|f\|_{\dot F^{1/q}_{p,q}(\mathbb{R}^{n-1})} \tag{2.13}$$

Combining (2.12) and (2.13), we obtain (2.11). This achieves the proof of Theorem 2.7. □

2.2. **Inhomogeneous Stokes system.** Consider the operators $\mathscr{G}$ and Ψ in $\mathbb{R}^n_+$ respectively defined by

$$\mathscr{G}(F,H)(x) = \int_{\mathbb{R}^n_+} G(x,y)F(y)dy - \int_{\mathbb{R}^n_+} \nabla_y G(x,y)H(y)dy,$$
$$\Psi(F,H)(x) = \int_{\mathbb{R}^n_+} \mathrm{g}(x,y)F(y)dy - \int_{\mathbb{R}^n_+} \nabla_y \mathrm{g}(x,y)H(y)dy$$

whenever the integrals make sense for almost every $x \in \mathbb{R}^n_+$. Recall that the kernels $G(x,y) = (G_{ij}(x,y))^n_{i,j=1}$ and $\mathrm{g}(x,y) = (g_j(x,y))^n_{j=1}$, $(x \neq y)$ are the Green tensor for the Stokes operator in $\mathbb{R}^n_+$, that is, coordinates-wise the function satisfying

$$\begin{cases} -\Delta_x G_{ij} + \partial_i g_j = \delta_x \delta_{ij} \text{ in } \mathbb{R}^n_+ \\ \partial_i G_{ij} = 0 \text{ in } \mathbb{R}^n_+ \\ G_{ij}(x,\cdot)\big|_{\partial\mathbb{R}^n_+} = 0. \end{cases} \tag{2.14}$$

in the sense of distributions where δ_x is the Dirac distribution with mass at $x \in \mathbb{R}^n_+$, Under mild assumptions on F and H, the vector-valued functions $v = \mathscr{G}(F,H)$ and $w = \Psi(F,H)$ satisfy the system of equations

$$\begin{cases} -\Delta v + \nabla w = F + \operatorname{div} H \text{ in } \mathbb{R}^n_+ \\ \operatorname{div} v = 0 \text{ in } \mathbb{R}^n_+ \\ v = 0 \text{ on } \partial\mathbb{R}^n_+ \end{cases} \tag{2.15}$$

Refined properties of Green matrices were recently obtained by the authors in [16] relying on ideas introduced earlier in the articles [22] (for $n = 2,3$) and [10] (for the general case). For our purpose we will need the following properties which include sharp pointwise decay bounds.

Lemma 2.16. *The Green tensor G is symmetric, $G_{ij}(x,y) = G_{ji}(y,x)$ for all $x,y \in \mathbb{R}^n_+$ and satisfies together with* g *the pointwise estimates*

(2.17)
$$\left|\nabla_x^\alpha \nabla_y^\beta G_{ij}(x,y)\right| \leq C_N \begin{cases} |x-y|^{-(n-2+N)} \\ \dfrac{x_n y_n}{|x-y|^{n+N}} & \text{if } \alpha_n = \beta_n = 0 \\ \dfrac{x_n}{|x-y|^{n-1+N}} & \text{if } \alpha_n = 0 \end{cases} \quad \text{for } \alpha,\beta \in \mathbb{N}^n,\ |\alpha|+|\beta| = N.$$

$$\left|\nabla^\alpha \mathrm{g}_j(x,y)\right| \leq C_\alpha |x-y|^{-(n-1)-|\alpha|}, \quad \alpha \in \mathbb{N}^n_0,\ j = 1,...,n \tag{2.18}$$

for $n \geq 2$, where the constants are independent of x and y.

These inequalities find their applicability in our next result which deals with the mapping properties of the potentials $\mathscr{G}$ and Ψ. Recall the space $\mathbf{Y}^{\tau,\eta}$ introduced in Section 1.

Proposition 2.19. *Fix $n \geq 3$ and assume that $q \in (\frac{n}{n-1}, \infty)$. Let $1 < \eta < \tau < \infty$ and $1 \leq \sigma < \Lambda < p < \infty$ satisfy the condition*

$$\frac{1}{\eta} + \frac{n-1}{\tau} = 2 + \frac{1}{q-1} = 1 + \frac{1}{\sigma} + \frac{n-1}{\Lambda}.$$

For all $F \in \mathbf{Y}^{\tau,\eta}$ and $H \in \mathbf{Y}^{\Lambda,\sigma}$ we have $\mathscr{G}(F,H) \in \mathbf{X}^q$, $\Psi(F,H) \in \mathbf{Z}^q$ and it holds that

$$\|\mathscr{G}(F,H)\|_{\mathbf{X}^q} + \|\Psi(F,H)\|_{\mathbf{Z}^q} \leq C(\|F\|_{\mathbf{Y}^{\tau,\eta}} + \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}) \tag{2.20}$$

for some constant $C := C(n,q) > 0$ independent of F and H.

Remark 2.21. The proof of the above result reveals that elliptic estimates of the form

$$\sup_{x_n>0} x_n^{\frac{1}{q}-1+|\alpha|}\big\|\partial^\alpha_{x'}u(\cdot,x_n)\big\|_{L^\infty(\mathbb{R}^{n-1})}\le C(\|F\|_{\mathbf{Y}^{\tau,\eta}}+\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}) \tag{2.22}$$

are valid for u solution of the Stokes equation (2.15) for each multi-index $\alpha\in\mathbb{N}_0^n$. However, it is not clear whether vertical derivatives of u enjoy this property. In fact, we are relying heavily on (2.17) which seems to fail in the case $\alpha_n\neq 0$ or $\beta_n\neq 0$, see [16, Remark 2.6]. We also point that in absence of the forcing term F, Proposition 2.19 holds true in two dimensions.

The proof of the proposition essentially relies on two auxiliary results, one of which deals with the mapping properties in mixed Lebesgue spaces of the operator G_β defined for $0<\beta<n$ by

$$G_\beta F(y)=\int_{\mathbb{R}^n_+}\frac{F(z)dz}{|y-z|^{n-\beta}} \tag{2.23}$$

whenever the integral exists for almost all $y\in\mathbb{R}^n_+$. For $p,q\in[1,\infty]$, let us denote by $L^pL^q(\mathbb{R}^n_+)$ the mixed Lebesgue space of function $F:\mathbb{R}^n_+\to\mathbb{R}$ with the property that $x'\mapsto F(x',\cdot)\in L^p(\mathbb{R}^{n-1})$ and $x_n\mapsto F(\cdot,x_n)\in L^q(\mathbb{R}_+)$ and equip it with the norm

$$\|F\|_{L^pL^q(\mathbb{R}^n_+)}=\big\|\|F(\cdot,x_n)\|_{L^q(\mathbb{R}_+,dx_n)}\big\|_{L^p(\mathbb{R}^{n-1})}.$$

Lemma 2.24. *Let $0<\beta<n$ and $1<\tau<\infty$. Assume that $1\le\eta\le q\le p<\infty$ are such that*

$$\frac{1}{\eta}<\beta+\frac{1}{q},\qquad \frac{n-1}{p}=\frac{n-1}{\tau}+\frac{1}{\eta}-\frac{1}{q}-\beta. \tag{2.25}$$

Then the operator G_β is bounded from $L^\tau L^\eta(\mathbb{R}^n_+)$ into $L^pL^q(\mathbb{R}^n_+)$.

Recall the Riesz potential I_α of order $\alpha\in(0,n)$, that is, the convolution operator with the kernel $|x|^{\alpha-n}$, $x\in\mathbb{R}^n\setminus\{0\}$.

Proof. Along the lines of the proof of [29, Lemma 2.2], take $F\in L^\tau L^\eta(\mathbb{R}^n_+)$ and let $\widetilde{F}$ be the zero extension of F to $\mathbb{R}^n$. For $1\le\eta<\infty$ and $1<\tau<\infty$ we have

$$\|G_\beta F\|_{L^pL^q(\mathbb{R}^n_+)}=\Big\|\big\|G_\beta F(y',\cdot)\big\|_{L^q(\mathbb{R}_+)}\Big\|_{L^p(\mathbb{R}^{n-1})}.$$

Let $x'\in\mathbb{R}^{n-1}$ and set $S(x',s)=(|x'|^2+s^2)^{-\frac{n-\beta}{2}}$, $s>0$. For $1\le\theta<\infty$ such that $\frac{1}{\eta}+\frac{1}{\theta}\ge1$ we use Minkowski's inequality to arrive at

$$\begin{aligned}
\big\|G_\beta F(y',\cdot)\big\|_{L^q(\mathbb{R}_+)}&=\bigg\|\int_{\mathbb{R}^n_+}\frac{|F(z',z_n)|dz'dz_n}{(|y'-z'|^2+|\cdot-z_n|^2)^{\frac{n-\beta}{2}}}\bigg\|_{L^q(\mathbb{R}_+)}\\
&=\bigg\|\int_{\mathbb{R}^{n-1}}\big(S(|y'-z'|,\cdot)*|\widetilde{F}|(z')\big)(y_n)dy'\bigg\|_{L^q(\mathbb{R}_+,dy_n)}\\
&\le C\int_{\mathbb{R}^{n-1}}\|(S(|y'-z'|,\cdot)*|\widetilde{F}|)(z',\cdot)\|_{L^q(\mathbb{R}_+)}dz'\\
&\le C\int_{\mathbb{R}^{n-1}}\|S(|y'-z'|,\cdot)\|_{L^\theta(\mathbb{R}_+)}\|F(z',\cdot)\|_{L^\eta(\mathbb{R}_+)}dz'\\
&\le C[I_{\beta+\frac{1}{\theta}-1}\|F(\cdot,y_n)\|_{L^\eta(\mathbb{R}_+,dy_n)}](y'),\quad y'\in\mathbb{R}^{n-1}
\end{aligned}$$

where $\frac{1}{q}+1=\frac{1}{\theta}+\frac{1}{\eta}$. Thus, if $\frac{n-1}{p}=\frac{n-1}{\tau}-(\beta+\frac{1}{\theta}-1)$, then by the boundedness of I_α in Lebesgue spaces, we find that

$$\begin{aligned}\|G_\beta F\|_{L^pL^q(\mathbb{R}^n_+)} &\leq C\big\|I_{\beta-\frac{1}{\theta}-1}\|F(y',\cdot)\|_{L^\eta(\mathbb{R}_+)}\big\|_{L^p(\mathbb{R}^{n-1},dy')}\\ &\leq C\big\|F\big\|_{L^\tau L^\eta(\mathbb{R}^n_+)}.\end{aligned}$$

□

Remark 2.26. In the sequel, we will need a weighted analogue of (2.24) of the form

$$\Big\|\big\|G_\beta F(\cdot,y_n)\big\|_{L^q(\mathbb{R}_+,y_n^q dy_n)}\Big\|_{L^p(\mathbb{R}^{n-1})} \leq C\Big\|\big\|F(\cdot,y_n)\big\|_{L^\eta(\mathbb{R}_+,y_n^{b\eta}dy_n)}\Big\|_{L^r(\mathbb{R}^{n-1})} \tag{2.27}$$

which holds for all functions F such that $(x',x_n)\mapsto x_n^bF\in L^rL^\eta(\mathbb{R}^n_+)$ under the conditions

$$\begin{cases}2+\dfrac{1}{q}=(n-1)\left(\dfrac{1}{r}-\dfrac{1}{p}\right)+\dfrac{1}{\eta}+b-(\beta-1)\\ 1<r<p<\infty,\quad b\geq 1\\ n>\beta+2+\frac{1}{q}-\frac{1}{\eta}-b.\end{cases} \tag{2.28}$$

In fact, one may use the same strategy as before to prove (2.27). If $a\geq 1$ and $\delta>1$ are such that

$$\frac{1}{\delta}+a=(n-1)\left(\frac{1}{r}-\frac{1}{p}\right)-(\beta-1),$$

then using the weighted convolution inequality [13, Theorem 1.2] for $n=1$, we obtain

$$\begin{aligned}\big\|G_\beta F(y',\cdot)\big\|_{L^q(\mathbb{R}_+,y_n^q dy_n)} &\leq C\int_{\mathbb{R}^{n-1}}\|S(|y'-z'|,\cdot)\|_{L^\delta(\mathbb{R}_+,y_n^{a\delta}dy_n)}\|F(z',\cdot)\|_{L^\eta(\mathbb{R}_+,y_n^{b\eta})}dz'\\ &\leq I_{\frac{1}{\delta}+a+\beta-1}\|F(\cdot,y_n)\|_{L^\eta(\mathbb{R}_+,y_n^{b\eta}dy_n)}(y'),\quad y'\in\mathbb{R}^{n-1}.\end{aligned}$$

This, in conjunction with (2.28) gives the desired bound.

We are now ready to prove Proposition 2.19 and we divide the proof in two steps.

Step 1. The Bound

$$\|\mathscr{G}(F,H)\|_{\mathbf{X}^q}\leq C(\|F\|_{\mathbf{Y}^{\tau,\eta}}+\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}). \tag{2.29}$$

Let $1<\eta<\infty$ and $1<\tau<\infty$ such that $\frac{1}{\eta}+\frac{n-1}{\tau}=2+\frac{1}{q-1}$. Pick F in $\mathbf{Y}^{\tau,\eta}$ and $H\in\mathbf{Y}^{\Lambda,\sigma}$. We first prove that

$$\sup_{x_n>0}x_n^{1/(q-1)}\|\mathscr{G}(F,H)(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\leq\|F\|_{\mathbf{Y}^{\tau,\eta}}. \tag{2.30}$$

Fix $x'\in\mathbb{R}^{n-1}$ and $x_n>0$ and write

$$\int_{\mathbb{R}^n_+}G(x',x_n,y)F(y)dy=J_1+J_2+J_3+J_4$$

where

$$J_1=\int_{B_{x_n}(x')}\int_0^{x_n/2}G(x,y)F(y)dy,\ J_2=\int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}G(x,y)F(y)dy,$$

$$J_3=\int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')}\int_0^{2x_n}G(x,y)F(y)dy,\ J_4=\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}G(x,y)F(y)dy.$$

Next, we estimate each of these integrals by means of the pointwise inequalities from Lemma 2.16. Indeed, starting with J_1 and using the summation convention, we have

$$\begin{aligned}
|J_1| &\le \int_{B_{x_n}(x')}\int_0^{x_n/2} |G_{ij}(x',x_n,y)||F_j(y)|dy\\
&\le C\int_{B_{x_n}(x')}\int_0^{x_n/2} \frac{|F(y)|}{(|x'-y'|^2+(x_n-y_n)^2)^{\frac{n-2}{2}}}dy_n dy'\\
&\le Cx_n^{-(n-2)}x_n^{\frac{n}{\eta'}}\left(\int_{B_{x_n}(x')}\int_0^{x_n/2}|F(y)|^\eta dy_n dy'\right)^{1/\eta}\\
&\le Cx_n^{-(n-2)+\frac{n}{\eta'}}\left(\int_{\mathbb{R}^{n-1}}\iint_{\Gamma(x')\cap B_{x_n}(x')\times(0,x_n/2)}|F(y)|^\eta y_n^{-(n-1)}dy_n dy' dz'\right)^{1/\eta}\\
&\le Cx_n^{-(n-2)+\frac{n}{\eta'}}\|F\|_{T^{\tau,\eta}}\big|E(B_{x_n/2}(x')\times(0,x_n/2))\big|^{\frac{\tau-\eta}{\tau\eta}}\\
&\le Cx_n^{2-\frac{n-1}{\tau}-\frac{1}{\eta}}\|F\|_{T^{\tau,\eta}}.
\end{aligned}$$

where we have utilized Hölder's inequality in order to derive the third and fifth bounds in the above chain of estimates and $\frac{1}{\eta'}+\frac{1}{\eta}=1$. On the other hand,

$$\begin{aligned}
|J_2| &\le \int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}|G_{ij}(x,y)||F_j(y)|dy\\
&\le C\int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}|x-y|^{-(n-2)}|F_j(y)|dy\\
&\le C\sup_{y_n>0} x_n^{\frac{n-1}{\tau}+\frac{1}{\eta}}\|F(\cdot,y_n)\|_{L^\infty(\mathbb{R}^{n-1})}\int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}\frac{y_n^{-\frac{1}{\eta}-\frac{n-1}{\tau}}dy_n dy'}{[|x'-y'|^2+(x_n-y_n)^2]^{(n-2)/2}}\\
&\le Cx_n^{-\frac{1}{\eta}-\frac{n-1}{\tau}}\|F\|_{\mathbf{Y}^{\tau,\eta}}\int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}|x'-y'|^{-(n-2)}dy_n dy'\\
&\le Cx_n^{1-\frac{1}{\eta}-\frac{n-1}{\tau}}\|F\|_{\mathbf{Y}^{\tau,\eta}}\int_{B_{x_n}(x')}|x'-y'|^{-(n-2)}dy\\
&\le Cx_n^{2-\frac{1}{\eta}-\frac{n-1}{\tau}}\|F\|_{\mathbf{Y}^{\tau,\eta}}.
\end{aligned}$$

Similarly as above, by Lemma 2.9 and Hölder's inequality, we find that

$$\begin{aligned}
|J_3| &\le \int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')}\int_0^{2x_n}|G_{ij}(x,y)||F_j(y)|dy\\
&\le Cx_n\int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')}\int_0^{2x_n}|x-y|^{-(n-1)}|F_j(y)|dy\\
&\le Cx_n\sum_{k=1}^\infty\int_{2^kB_{x_n}(x')\setminus 2^{k-1}B_{x_n}(x')}\int_0^{2x_n}|x-y|^{-(n-1)}|F_j(y)|dy\\
&\le Cx_n^{2-n+n/\eta'}\sum_{k=1}^\infty 2^{-(k-1)(n-1)+\frac{(n-1)k}{\eta'}}\left(\int_{2^kB_{x_n}(x')}\int_0^{2x_n}|F_j(y)|^\eta dy_n dy'\right)^{\frac{1}{\eta}}\\
&\le Cx_n^{2-\frac{n}{\eta}+\frac{(n-1)(\tau-\eta)}{\tau\eta}}\left[\int_{\mathbb{R}^{n-1}}\left(\iint_{\Gamma(z')}|F_j(y)|^\eta\frac{dy}{y_n^{n-1}}\right)^{\tau/\eta}dz'\right]^{\frac{1}{\tau}}\left(\sum_{k=1}^\infty 2^{-\frac{k(n-1)}{\tau}}\right)\\
&\le Cx_n^{2-\frac{1}{\eta}-\frac{n-1}{\tau}}\|F\|_{\mathbf{Y}^{\tau,\eta}}.
\end{aligned}$$

Again, by using the Green matrix bounds, we bound J_4 as follows

$$\begin{aligned}|J_4| &\le \int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}|G_{ij}(x,y)||F_j(y)|dy\\ &\le C\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}\frac{x_ny_n|F_j(y)|}{|x-y|^n}dy\\ &\le C\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}\frac{|F_j(y)|dy_ndy'}{\left[|x'-y'|^2+y_n^2\right]^{\frac{n}{2}}}\\ &\le C\sup_{x_n>0}x_n^{\frac{1}{\eta}+\frac{n-1}{\tau}}\|F(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\left(\int_{2x_n}^{\infty}x_ny_n^{-\frac{1}{\eta}-\frac{n-1}{\tau}}dy_n\right)\left(\int_{\mathbb{R}^{n-1}}\frac{dz'}{\left[|z'|^2+1\right]^{\frac{n}{2}}}\right)\\ &\le Cx_n^{2-\frac{1}{\eta}-\frac{n-1}{\tau}}\|F\|_{\mathbf{Y}^{\tau,\eta}}.\end{aligned}$$

In the same vein, we establish the weighted gradient sup-norm estimate

$$\sup_{x_n>0}x_n^{q/(q-1)}\left\|\int_{\mathbb{R}^n_+}\nabla_yG(\cdot,x_n,y)H(y)dy\right\|_{L^\infty(\mathbb{R}^{n-1})}\le C\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}. \tag{2.31}$$

Decompose the solid integral in the above estimate into four parts to get

$$L_1=\int_{B_{x_n}(x')}\int_0^{x_n/2}\nabla_yG(x,y)H(y)dy,\ L_2=\int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}\nabla_yG(x,y)H(y)dy,$$

$$L_3=\int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')}\int_0^{2x_n}\nabla_yG(x,y)H(y)dy,\ L_4=\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}\nabla_yG(x,y)H(y)dy.$$

Suppose $\frac{1}{\sigma}+\frac{n-1}{\Lambda}=\frac{1}{\eta}+\frac{n-1}{\tau}-1$. Utilizing (2.17) and Hölder's inequality, we arrive at

$$\begin{aligned}|L_1| &\le \int_{B_{x_n}(x')}\int_0^{x_n/2}|\nabla_yG(x,y)||H(y)|dy\\ &\le C\int_{B_{x_n}(x')}\int_0^{x_n/2}\frac{|H(y',y_n)|}{(|x'-y'|^2+(x_n-y_n)^2)^{\frac{n-1}{2}}}dy_ndy'\\ &\le Cx_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|\mathcal{A}_qH\|_{L^\Lambda(\mathbb{R}^{n-1})}\\ &\le Cx_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}\end{aligned}$$

Next, noticing that $|\nabla G_{ij}(x,\cdot)|$ belongs to the weak-Lebesgue space $L^{\frac{n}{n-1},\infty}(\mathbb{R}^n_+)$ uniformly for all $x\in\mathbb{R}^n_+$, it follows that

$$\begin{aligned}|L_2| &\le \int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}|\nabla_yG(x,y)H(y)|dy\\ &\le C\sup_{x_n>0}x_n^{\frac{1}{\sigma}+\frac{n-1}{\Lambda}}\|H(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n}y_n^{-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}|\nabla_yG(x,y)|dy_ndy'\\ &\le Cx_n^{-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\sup_{x_n>0}x_n^{\frac{1}{\sigma}+\frac{n-1}{\Lambda}}\|H(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\|\nabla_yG(x,\cdot)\|_{L^1(B_{x_n}(x')\times[x_n/2,2x_n])}\\ &\le Cx_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}.\end{aligned}$$

Recall here that for any $p>1$ the belonging of f to $L^{p,\infty}(\mathbb{R}^{n-1})$ is equivalent to the condition

$$\sup_{E\subset\mathbb{R}^{n-1}}|E|^{1/p-1}\int_E|f(y)|dy<\infty$$

where the supremum runs over all open set E of $\mathbb{R}^{n-1}$. We argue as before to bound L_3

$$
\begin{aligned}
|L_3| &\le \int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')}\int_0^{2x_n} |\nabla_y G(x,y)H(y)|dy_n dy' \\
&\le \sum_{k=1}^{\infty}\int_{2^kB_{x_n}(x')\setminus 2^{k-1}B_{x_n}(x')}\int_0^{2x_n} |\nabla_y G(x,y)||H(y)|dy_n dy' \\
&\le C\sum_{k=1}^{\infty}\int_{2^kB_{x_n}(x')\setminus 2^{k-1}B_{x_n}(x')}\int_0^{2x_n} |x-y|^{-n+1}|H(y)|dy_n dy' \\
&\le Cx_n^{1-\frac{n}{\sigma}}\sum_{k=1}^{\infty}2^{-(n-1)k+\frac{k(n-1)}{\sigma'}}\left(\int_{\mathbb{R}^{n-1}}\iint_{\Gamma(z')\cap[2^kB_{x_n}(x')\times(0,2x_n)]}|H(y)|^{\sigma}\frac{dy_n dy'}{y_n^{n-1}}dz'\right)^{\frac{1}{\sigma}} \\
&\le Cx_n^{-(n-1)+\frac{n}{\sigma'}+\frac{(\Lambda-\sigma)}{\Lambda\sigma}n}\sum_{k=1}^{\infty}2^{-(k-1)(n-1)+\frac{k(n-1)}{\sigma'}+k(n-1)\frac{(\Lambda-\sigma)}{\sigma\Lambda}}\|\mathcal{A}_\sigma H\|_{L^{\Lambda}(\mathbb{R}^{n-1})} \\
&\le Cx_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|H\|_{T^{\Lambda,\sigma}}\sum_{k=1}^{\infty}2^{-\frac{(n-1)}{\Lambda}k} \\
&\le Ct^{2-\frac{1}{\eta}-\frac{n-1}{\tau}}\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}.
\end{aligned}
$$

Finally, observe that for $y_n > 2x_n$, we have $y_n - x_n > \frac{1}{2}y_n$ so that by the third bound in (2.17), we find that

$$
\begin{aligned}
|L_4| &\le \int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}|\nabla_y G(x,y)||H(y)|dy \\
&\le C\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}\frac{x_n|H(y)|dy_n dy'}{\left[|x'-y'|^2+(x_n-y_n)^2\right]^{\frac{n}{2}}} \\
&\le C\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}\frac{x_n|H(y)|dy_n dy'}{\left[|x'-y'|^2+y_n^2\right]^{\frac{n}{2}}} \\
&\le C\sup_{x_n>0} x_n^{\frac{1}{\sigma}+\frac{n-1}{\Lambda}}\|H(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\left(\int_{\mathbb{R}^{n-1}}\frac{dy'}{(|y'|^2+1)^{\frac{n}{2}}}\right)\left(\int_{2x_n}^{\infty}x_n y_n^{-\frac{1}{\sigma}-\frac{n-1}{\Lambda}-1}dy_n\right) \\
&\le Ct^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}.
\end{aligned}
$$

Summing up all the above inequalities, one obtains (2.30). Next, we show that

$$
\|\mathscr{G}(F,H)\|_{T^{nq(q-1),q}} \le C(\|F\|_{T^{\tau,\eta}}+\|H\|_{T^{\Lambda,\sigma}}). \tag{2.32}
$$

Write

$$
\|\mathscr{G}(F,H)\|_{T^{p,q}} \le \left\|\int_{\mathbb{R}^n_+}G(\cdot,y)F(y)dy\right\|_{T^{p,q}} + \left\|\int_{\mathbb{R}^n_+}\nabla_y G(\cdot,y)H(y)dy\right\|_{T^{p,q}} := I + II.
$$

Fix $x'\in\mathbb{R}^{n-1}$ and $y_n>0$ and let's decompose $F\in L^{\eta}_{loc}(\mathbb{R}^n_+)$ into three parts

$$
F = F\mathbf{1}_{B_{4y_n}(x')\times(0,4y_n]} + F\mathbf{1}_{B_{4y_n}(x')\times(4y_n,\infty)} + F\mathbf{1}_{(\mathbb{R}^{n-1}\setminus B_{4y_n}(x'))\times(0,\infty)} = F_1+F_2+F_3
$$

and write

$$
I := \Sigma_1+\Sigma_2+\Sigma_3, \quad \Sigma_i = \left\|\int_{\mathbb{R}^n_+}G(\cdot,y)F_i(y)dy\right\|_{T^{p,q}}, \quad i=1,2,3.
$$

We dispose of I^3 using the following

Claim 2.33. *For all $x' \in \mathbb{R}^{n-1}$ and $y_n > 0$, there exists $C > 0$ independent on x' and y_n such that*

$$A(x', y_n) \leq CG_2\Big(\fint_{B_{y_n}(\cdot)} |F(z', \cdot)|dz'\Big)(x', y_n).$$

Here,

$$A(x', y_n) = \left(\fint_{B_{y_n}(x')} \left|\int_{\mathbb{R}^n_+} G(y,z)F_3(z)dz\right|^q dy'\right)^{\frac{1}{q}}, \quad (x', y_n) \in \mathbb{R}^n_+.$$

Proof. We have

$$\begin{aligned}
A(x', y_n) &= \left(\fint_{B_{y_n}(x')} \left(\int_{\mathbb{R}^n_+} |G(y,z)||F_3(z)|dz\right)^q dy'\right)^{1/q} \\
&\leq C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\mathbb{R}^{n-1}\setminus B_{4y_n}(x')} \frac{|F(z', z_n)|dz'dz_n}{(|y'-z'|^2 + |y_n - z_n|^2)^{\frac{n-2}{2}}}\right)^q dy'\right)^{1/q} \\
&\leq C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-z'|>4y_n\}} \frac{|F(z', z_n)|dz'dz_n}{(|y'-z'|^2 + |y_n - z_n|^2)^{\frac{n-2}{2}}} \fint_{B_{y_n}(z')} dw\right)^q dy'\right)^{\frac{1}{q}} \\
&\leq C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-w|>3y_n\}} \fint_{B_{y_n}(w)} \frac{|F(z', z_n)|dz'dwdz_n}{(|y'-z'|^2 + |y_n - z_n|^2)^{\frac{n-2}{2}}}\right)^q dy'\right)^{1/q} \\
&\leq C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-w|>3y_n\}} (|x'-w|^2 + |y_n - z_n|^2)^{\frac{-(n-2)}{2}} \cdot \right.\right. \\
&\qquad \left.\left. \fint_{B_{y_n}(w)} \left[\frac{(|x'-w|^2 + |y_n - z_n|^2)}{(|y'-z'|^2 + |y_n - z_n|^2)}\right]^{\frac{n-2}{2}} |F(z', z_n)|dz'dwdz_n\right)^q dy\right)^{\frac{1}{q}} \\
&\leq C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-w|>3y_n\}} (|x'-w|^2 + |y_n - z_n|^2)^{\frac{-(n-2)}{2}} \cdot \right.\right. \\
&\qquad \left.\left. \fint_{B_{y_n}(w)} |F(z', z_n)|dz'dwdz_n\right)^q dy'\right)^{1/q} \\
&\leq C\int_0^\infty \int_{\{|x'-w|>3y_n\}} (|x'-w|^2 + |y_n - z_n|^2)^{\frac{-(n-2)}{2}} \left(\fint_{B_{y_n}(w)} |F(z', z_n)|dz'\right)dwdz_n \\
&\leq C\int_{\mathbb{R}^n_+} (|x'-w|^2 + |y_n - z_n|^2)^{\frac{-(n-2)}{2}} \left(\fint_{B_{y_n}(w)} |F(z', z_n)|dz'\right)dwdz_n \\
&\leq CG_2\Big(\fint_{B_{y_n}(\cdot)} |F(z', \cdot)|dz'\Big)(x', y_n).
\end{aligned}$$

□

Applying Lemma 2.24 and Jensen's inequality, the above claim clearly implies that

$$\begin{aligned}
\Sigma_3 = \|A\|_{L^pL^q(\mathbb{R}^n_+)} &\leq C\left\|G_2\Big(\fint_{B_{y_n}(\cdot)} |F(z', \cdot)|dz'\Big)\right\|_{L^pL^q(\mathbb{R}^n_+)} \\
&\leq C\left\|\fint_{B_{y_n}(\cdot)} |F(z', y_n)|dz'\right\|_{L^pL^q(\mathbb{R}^n_+)} \\
&\leq C\|F\|_{T^{\tau,\eta}}.
\end{aligned}$$

To bound Σ_2, we first observe that

$$\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right| \leq Cy_n\mathcal{A}_\eta F(x')\Big(\int_{4y_n}^\infty\int_{B_{4y_n}(x')}\frac{z_n^{\frac{n-1}{\eta-1}}dz'dz_n}{(|y'-z'|^2+|y_n-z_n|^2)^{\frac{(n-1)\eta'}{2}}}\Big)^{\frac{1}{\eta'}}$$

$$\leq Cy_n^{2-\frac{1}{\eta}}\mathcal{A}_\eta F(x'), \quad x'\in B(y',y_n). \tag{2.34}$$

On the other hand, this inequality also implies the pointwise bound

$$\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right| \leq Cy_n^{-\frac{1}{q-1}}\|\mathcal{A}_\eta F\|_{L^\tau(\mathbb{R}^{n-1})}, \quad y'\in B(x',y_n). \tag{2.35}$$

Let $M>0$ to be determined later. Using (2.34) and (2.35), we find that

$$\int_0^\infty \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n \leq \int_0^M \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n +$$

$$\int_M^\infty \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n$$

$$\leq CM^{1+(2-\frac{1}{\eta})q}[\mathcal{A}_\eta F(x')]^q + M^{-\frac{1}{q-1}}\|F\|^q_{T^{\tau,\eta}}.$$

Optimizing this inequality with respect to M, that is taking $M=\left(\frac{\|F\|_{T^{\tau,\eta}}}{\mathcal{A}_\eta F(x')}\right)^{\frac{\tau}{n-1}}$, we arrive at

$$\Big(\int_0^\infty \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n\Big)^{\frac{1}{q}} \leq C\|F\|_{T^{\tau,\eta}}^{1-\frac{\tau}{q(n-1)(q-1)}}[\mathcal{A}_\eta F(x')]^{\frac{\tau}{q(n-1)(q-1)}}.$$

Taking the L^p-norm on both sides of the inequality, we conclude that

$$\Sigma_2 \leq C\|F\|_{T^{\tau,\eta}}.$$

Finally, estimating Σ_1 goes through a duality argument. Let $r=(n-1)(q-1)$ and $\varphi\in L^{r'}(\mathbb{R}^{n-1})$, $\varphi\geq 0$ and define the operator

$$M_t\varphi(x') = t^{-(n-1)}\int_{B_t(x')}\varphi(y')dy', \quad t>0.$$

If $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $L^p(\mathbb{R}^{n-1})$ and its dual $L^{p'}(\mathbb{R}^{n-1})$, then

$$\Big\langle \mathcal{A}_q^q\Big[\int_{\mathbb{R}^n_+} G(\cdot,z)F_1(z)dz\Big],\varphi\Big\rangle = \int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_1(z)dz\right|^q\frac{dy'dy_n}{y_n^{n-1}}\varphi(x')dx'$$

$$\leq C\int_{\mathbb{R}^{n-1}}\int_0^\infty\fint_{B_{y_n}(y')}\varphi(x')dx'[G_2|F|(y',y_n)]^q dy'dy_n$$

$$\leq C\int_{\mathbb{R}^{n-1}}\int_0^\infty [G_2|F|(y',y_n)]^q M_{y_n}\varphi(y')dy'dy_n$$

$$\leq C\|G_2F\|^q_{L^pL^q(\mathbb{R}^n_+)}\|M_\cdot\varphi\|_{L^{r'}L^\infty(\mathbb{R}^n_+)}$$

$$\leq C\|G_2F\|^q_{L^pL^q(\mathbb{R}^n_+)}\|\mathcal{M}\varphi\|_{L^{r'}(\mathbb{R}^{n-1})}.$$

Applying Lemma 2.24, the fact that $\int_0^\infty |F(y',y_n)|^q dy' \leq \liminf_{\alpha\to 0}[\mathcal{A}_q^\alpha F(y')]^q$ (which is a consequence of the Lebesgue differentiation Theorem and Fatou lemma), the boundedness of the Hardy-Littlewood maximal function in Lebesgue spaces successively, we obtain

$$\Big\langle \mathcal{A}_q^q\Big[\int_{\mathbb{R}^n_+} G(\cdot,z)F_1(z)dz\Big],\varphi\Big\rangle \leq \|F\|^q_{T^{\tau,\eta}}\|\varphi\|_{L^{r'}(\mathbb{R}^{n-1})} \quad \forall\varphi\in L^{r'}(\mathbb{R}^{n-1}),$$

from which it plainly follows that

$$\Sigma_1 \leq C\|F\|_{T^{\tau,\eta}}.$$

We equally estimate II splitting H into three components exactly as before and follow the same procedure (details are left to the interested reader). This yields

$$\left\| \int_{\mathbb{R}^n_+} \nabla_y G(\cdot, y) H(y) dy \right\|_{T^{p,q}} \leq C\|H\|_{T^{\Lambda,\sigma}}.$$

Summarizing, we see that (2.29) holds true. This finishes Step 1.

Step 2. The estimate

$$\|\Psi(F,H)\|_{\mathbf{Z}^q} \leq C\big(\|F\|_{\mathbf{Y}^{\tau,\eta}} + \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}\big) \tag{2.36}$$

for all $F \in \mathbf{Y}^{\tau,\eta}$ and $H \in \mathbf{Y}^{\Lambda,\sigma}$. We have

$$\|\Psi(F,H)\|_{T^{p,q}_{s_0}} \leq \left\| \int_{\mathbb{R}^n_+} \mathrm{g}(\cdot, y) F(y) dy \right\|_{T^{p,q}_{s_0}} + \left\| \int_{\mathbb{R}^n_+} \nabla_y \mathrm{g}(\cdot, y) H(y) dy \right\|_{T^{p,q}_{s_0}} := III + IV.$$

Let F_1, F_2 and F_3 as above and write correspondingly

$$III \leq III_1 + III_2 + III_3, \quad III_i = \left\| \int_{\mathbb{R}^n_+} \mathrm{g}(\cdot, y) F_i(y) dy \right\|_{T^{p,q}_{s_0}}, \quad i = 1, 2, 3.$$

Since (see proof of Claim 2.33)

$$\bigg(\fint_{B_{y_n}(x')} \left| \int_{\mathbb{R}^n_+} \mathrm{g}(y,z) F_3(z) dz \right|^q dy' \bigg)^{\frac{1}{q}} \leq c G_1 \bigg(\fint_{B_{y_n}(\cdot)} |F(z', \cdot)| dz' \bigg)(x', y_n), \quad (x', y_n) \in \mathbb{R}^n_+.$$

Now let $\tau < r < \infty$ such that $\frac{1}{r} + \frac{1}{n-1} \leq \frac{1}{\tau}$. Invoking (2.27) with $\beta = 1$ and $b = (n-1)(1/\tau - 1/r)$ together with Jensen's inequality we arrive at

$$\begin{aligned} III_3 &\leq C \left\| \Big\| G_1 \bigg(\fint_{B_{y_n}(\cdot)} |F(z', \cdot)| dz' \bigg) \Big\|_{L^q(\mathbb{R}_+, y_n^q dy_n)} \right\|_{L^p(\mathbb{R}^{n-1})} \\ &\leq C \bigg(\int_{\mathbb{R}^{n-1}} \bigg(\int_0^\infty \fint_{B_{y_n}(x')} |y_n^b F(z', y_n)|^\eta dz' dy_n \bigg)^{\frac{r}{\eta}} dx' \bigg)^{\frac{1}{r}} \\ &\leq C\|F\|_{T^{\tau,\eta}}. \end{aligned}$$

The last inequality follows from the embedding (1.9) (with $s_1 = 0$, $s_2 = -\frac{b}{n-1}$, $q = \eta$, $p_1 = \tau$ and $p_2 = r$). Moving on, we use (2.18) and Hölder's inequality to get the pointwise bound

$$\begin{aligned} \left| \int_{\mathbb{R}^n_+} \mathrm{g}(y,z) F_2(z) dz \right| &\leq C|G_1 F_2(y)| \\ &\leq C \mathcal{A}_\eta F(x') \bigg(\int_{4y_n}^\infty \int_{B_{4y_n}(x')} \frac{z_n^{\frac{n-1}{\eta-1}} dz' dz_n}{(|y' - z'|^2 + |y_n - z_n|^2)^{\frac{(n-1)\eta'}{2}}} \bigg)^{\frac{1}{\eta'}} \\ &\leq C y_n^{1-\frac{1}{\eta}} \mathcal{A}_\eta F(x'), \quad x' \in B_{y_n}(y') \end{aligned} \tag{2.37}$$

from which it follows that

$$\left| \int_{\mathbb{R}^n_+} \mathrm{g}(y,z) F_2(z) dz \right| \leq C y_n^{-\frac{q}{q-1}} \|\mathcal{A}_\eta F\|_{L^\tau(\mathbb{R}^{n-1})}, \quad y' \in B(x', y_n). \tag{2.38}$$

Therefore, for $M>0$ to be determined later, we have

$$\int_0^\infty \fint_{B_{y_n}(x')} y_n^q \left| \int_{\mathbb{R}^n_+} \mathrm{g}(y,z)F_2(z)dz \right|^q dy'dy_n \leq \int_0^M \fint_{B_{y_n}(x')} y_n^q \left| \int_{\mathbb{R}^n_+} \mathrm{g}(y,z)F_2(z)dz \right|^q dy'dy_n + \int_M^\infty \fint_{B_{y_n}(x')} y_n^q \left| \int_{\mathbb{R}^n_+} \mathrm{g}(y,z)F_2(z)dz \right|^q dy'dy_n$$
$$\leq CM^{1+(2-\frac{1}{\eta})q}[\mathcal{A}_\eta F(x')]^q + M^{-\frac{1}{q-1}} \|F\|^q_{T^{\tau,\eta}}.$$

The choice $M = \left(\dfrac{\|F\|_{T^{\tau,\eta}}}{\mathcal{A}_\eta F(x')} \right)^{\frac{\tau}{n-1}}$ yields

$$\left(\int_0^\infty \fint_{B_{y_n}(x')} y_n^q \left| \int_{\mathbb{R}^n_+} \mathrm{g}(y,z)F_2(z)dz \right|^q dy'dy_n \right)^{\frac{1}{q}} \leq C\|F\|_{T^{\tau,\eta}}^{1-\frac{\tau}{(n-1)q(q-1)}} [\mathcal{A}_\eta F(x')]^{\frac{\tau}{(n-1)q(q-1)}}.$$

Hence, (after taking the L^p-norm on both sides of the previous inequality)

$$III_2 \leq C\|F\|_{T^{\tau,\eta}}.$$

We also claim that

$$III_1 \leq C\|F\|_{T^{\tau,\eta}}.$$

In fact, setting $VF(y,y_n) = y_n \int_{\mathbb{R}^n_+} \mathrm{g}(y,z)F_1(z)dz$, for all $\phi \in L^r(\mathbb{R}^{n-1})$ we have that

$$\begin{aligned}
\left\langle \mathcal{A}_q^q(VF), \varphi \right\rangle &= \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{B_{y_n}(x')} |VF_1(y',y_n)|^q \frac{dy'dy_n}{y_n^{n-1}} \phi(x')dx' \\
&\leq C \int_{\mathbb{R}^{n-1}} \int_0^\infty \fint_{B_{y_n}(y')} \phi(x')dx' [y_n(G_1|F|)(y',y_n)]^q dy'dy_n \\
&\leq C \int_{\mathbb{R}^{n-1}} \int_0^\infty [y_n(G_1|F|)(y',y_n)]^q M_{y_n}\phi(y')dy'dy_n \\
&\leq C \big\|(y',y_n) \mapsto y_n G_1|F|\big\|^q_{L^pL^q(\mathbb{R}^n_+)} \|M.\phi\|_{L^{r'}L^\infty(\mathbb{R}^n_+)} \\
&\leq C\|F\|^q_{T^{m,\eta}_{-\frac{b}{n-1}}} \|\mathcal{M}\phi\|_{L^{r'}(\mathbb{R}^{n-1})} \\
&\leq C\|F\|^q_{T^{\tau,\eta}} \|\phi\|_{L^{r'}(\mathbb{R}^{n-1})}.
\end{aligned}$$

Note that the penultimate inequality follows from Remark 2.26 with $m \in (1,r)$ is such that $\frac{1}{\tau} \leq \frac{1}{m} - \frac{1}{n-1}$ and $b = (n-1)(\frac{1}{\tau} - \frac{1}{m})$ while the last bound comes from (1.9). Collecting and summing up all the estimates on the Σ_i's, we find that

$$\left\| \int_{\mathbb{R}^n_+} \mathrm{g}(\cdot,y)F(y)dy \right\|_{T^{p,q}} \leq C\|F\|_{T^{\tau,\eta}}.$$

The remaining estimate reads

$$\left\| \int_{\mathbb{R}^n_+} \nabla_y \mathrm{g}(\cdot,y)H(y)dy \right\|_{T^{p,q}} \leq C\|H\|_{T^{\Lambda,\sigma}}.$$

The argument used here is similar to the previous one. In fact, for $(y',y_n) \in \mathbb{R}^n_+$ we write

$$y_n \left| \int_{\mathbb{R}^n_+} \nabla_z \mathrm{g}(y,z)H(z)dz \right| \leq \sum_{k=1}^3 \Gamma_k(y',y_n),$$

with

$$\begin{aligned}\Gamma_1(y',y_n) &= y_n\int_{\mathbb{R}^{n-1}\setminus B_{4y_n}(y')}\int_0^\infty |\nabla_z g(y,z)||H(z)|dz\\ \Gamma_2(y',y_n) &= y_n\int_{B_{4y_n}(y')}\int_{4y_n}^\infty |\nabla_z g(y,z)||H(z)|dz\\ \Gamma_3(y',y_n) &= y_n\int_{B_{4y_n}(y')}\int_0^{4y_n} |\nabla_z g(y,z)||H(z)|dz.\end{aligned}$$

It is easy to see that $|\Gamma_1(y',y_n)|\le G_1H(y',y_n)$ for any $(y',y_n)\in\mathbb{R}^n_+$. Then, by Step 1 and in particular (2.32), we deduce the desired estimate. Next, we show that

$$\|\Gamma_2\|_{T^{p,q}}\le C\|H\|_{T^{\Lambda,\sigma}}. \tag{2.39}$$

To achieve this, let us primarily observe that

$$\begin{aligned}|\Gamma_2(y',y_n)| &\le Cy_n\mathcal{A}_\sigma^{\frac{5}{4}}H(x')\bigg(\int_{4y_n}^\infty\int_{B_{4y_n}(x')}\frac{z_n^{\frac{n-1}{\sigma-1}}dz'dz_n}{(|y'-z'|^2+|y_n-z_n|^2)^{\frac{n\sigma'}{2}}}\bigg)^{\frac{1}{\sigma'}}\\ &\le Cy_n^{1-\frac{1}{\sigma}}\mathcal{A}_\sigma^{\frac{5}{4}}H(x'),\quad x'\in B(y',y_n).\end{aligned}$$

Taking the Λ-power of both sides of the last inequality and integrating with respect to the variable x'

$$|\Gamma_2(y',y_n)|\le Cy_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\big\|\mathcal{A}_\sigma^{\frac{5}{4}}H\big\|_{L^\Lambda(\mathbb{R}^{n-1})}\le C\|H\|_{T^{\Lambda,\sigma}},\quad y'\in B(x',y_n).$$

Let $\delta>0$. The preceding inequalities imply

$$\begin{aligned}\int_0^\infty ⨍_{B_{y_n}(x')}\big|\Gamma_2(y',y_n)\big|^q dy'dy_n &\le \bigg(\int_0^\delta+\int_\delta^\infty\bigg)⨍_{B_{y_n}(x')}\big|\Gamma_2(y',y_n)\big|^q dy'dy_n\\ &\le C\delta^{1+\frac{q}{\sigma'}}[\mathcal{A}_\sigma H(x')]^q+\delta^{1-\left(\frac{n-1}{\Lambda}-\frac{1}{\sigma'}\right)q}\|H\|^q_{T^{\Lambda,\sigma}}.\end{aligned}$$

Optimizing with respect to δ (i.e. choosing $\delta=\Big(\|H\|_{T^{\Lambda,\sigma}}/\mathcal{A}_\sigma^{\frac{5}{4}}H(x')\Big)^{\frac{\Lambda}{n-1}}$) yields

$$\bigg(\int_0^\infty ⨍_{B_{y_n}(x')}\big|\Gamma_2(y',y_n)\big|^q dy'dy_n\bigg)^{\frac{1}{q}}\le C\|H\|_{T^{\Lambda,\sigma}}^{\frac{\Lambda}{n-1}(1+\frac{1}{q}-\frac{1}{\sigma})}\big[\mathcal{A}_\sigma^{\frac{5}{4}}H(x')\big]^{1-\frac{\Lambda}{n-1}(1+\frac{1}{q}-\frac{1}{\sigma})}.$$

Taking the L^p-norm on both sides and using Remark 1.7 gives (2.39). Finally, the $T^{p,q}$-norm of Γ_3 is controlled from above by a constant multiple of $\|H\|_{T^{\Lambda,\sigma}}$. This is derived from a simple duality argument. The proof of Proposition 2.19 is now complete.

We can now summarize the findings obtained above into a single theorem establishing the well-posedness of System (**S**) for boundary data in the scale of Triebel-Lizorkin space with negative amount of smoothness. We say that a pair (u,π) is a solution to (**S**) if u and π satisfy the relations

$$u(x)=\mathcal{H}f(x)+\mathscr{G}(F,H)(x),\quad \pi(x)=\mathcal{E}f(x)+\Psi(F,H)(x),\ \ x\in\mathbb{R}^n_+. \tag{2.40}$$

Theorem 2.41. *Assume that the positive numbers η,Λ,σ,p,q are as in Lemma 2.19. Then for any $f\in[\dot F_{p,q}^{-1/q}(\mathbb{R}^{n-1})]^n$, $F\in\mathbf{Y}^{\tau,\eta}$ and $H\in\mathbf{Y}^{\Lambda,\sigma}$, the Stokes system* (**S**) *has a solution $(u,\pi)\in\mathbf{Z}^q\times\in\mathbf{X}^q$ (in the sense made precise in (2.40)) which obeys*

$$\|u\|_{\mathbf{X}^q}+\|\pi\|_{\mathbf{Z}^q}\le C(\|f\|_{\dot F_{p,q}^{-\frac{1}{q}}(\mathbb{R}^{n-1})}+\|F\|_{\mathbf{Y}^{\tau,\eta}}+\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}) \tag{2.42}$$

for some constant $C>0$ independent of f, F and H.

Remark 2.43. Practically, Theorem 2.41 can easily be extended to the case where the vector field u is not necessarily solenoidal, i.e. $\operatorname{div} u = \phi$ using the formulation derived in [27, formula 2.32], see also [5] so that our result gives an alternative approach to the Dirichlet problem for the Stokes system (to be compared to [3] and [8] wherein the analysis is carried out in weigthed Sobolev spaces and Lebesgue spaces, respectively). These estimates of the velocity field and the pressure in tent and weighted tent framework respectively against boundary data in low regularity spaces are new and generalize well-known results. In fact, our boundary class $\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})$ contains the homogeneous Sobolev space $\dot{H}^{s,r}(\mathbb{R}^{n-1})$ for $-1/q < s$ and $\dfrac{n-1}{r} - s = \dfrac{1}{q-1}$.

3. Proofs of main results

Here, we prove Theorems 1.15 and 1.16 relying on preliminary results obtained in Section 2.

Proof of Theorem 1.15. Let $f \in [\dot{H}^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})]^n$ with $n > 2$ and assume $F \in \mathbf{Y}^{\tau,\eta}$ for $1 < \eta < \tau < \infty$ with $\dfrac{1}{\eta} + \dfrac{n-1}{\tau} = 3.$. Equip the Banach space $\mathbf{X} \times \mathbf{Z}$ by the norm $\|\cdot\| := \|\cdot\|_{\mathbf{X}} + \|\cdot\|_{\mathbf{Z}}$ and introduce the operators $\mathscr{L}$ defined by

$$\mathscr{L}(u,\pi) = \big(\mathcal{H}f + \mathscr{G}[F, u\otimes u],\ \mathcal{E}(f) + \Psi[F, u \otimes u]\big)$$

where $\mathcal{H}$ and $\mathcal{E}$ are given by (2.3). A solution of Eq. (**NS**) according to Definition 2.40 is a couple (u,π) satisfying the fixed point equation

$$(u,\pi) = \mathscr{L}(u,\pi) \ \text{ in } \mathbb{R}^n_+. \tag{3.1}$$

Using a Banach fixed point argument, we wish to show that the latter equation admits a solution in $\mathbf{X} \times \mathbf{Z}$. Take another couple $(v,\pi') \in \mathbf{X} \times \mathbf{Z}$ solution of (**NS**) associated to the same data and forcing term and use Proposition 2.19 with $q = 2$, $(\Lambda,\sigma) = (n-1,1)$ to get

$$\begin{aligned}
\big\|\mathscr{L}(u,\pi) - \mathscr{L}(v,\pi')\big\| &= \big\|\mathscr{G}(0, u\otimes u - v\otimes v)\big\|_{\mathbf{X}} + \big\|\Psi(0, u\otimes u - v \otimes v\big\|_{\mathbf{Z}} \\
&\le C\|u\otimes u - v\otimes v\|_{\mathbf{Y}^{n-1,1}} \\
&\le C(\|u\otimes(u-v)\|_{\mathbf{Y}^{n-1,1}} + \|(u-v)\otimes v\|_{\mathbf{Y}^{n-1,1}}) \\
&\le C\big(\sup_{x_n>0} x_n^2\big\|[u\otimes(u-v)](\cdot,x_n)\big\|_{L^\infty(\mathbb{R}^{n-1})} + \|u\otimes(u-v)\|_{T^{n-1,1}} + \\
&\qquad \sup_{x_n>0} x_n^2\|[(u-v)\otimes v](\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \|(u-v)\otimes v\|_{T^{n-1,1}}\big) \\
&\le C\big(\sup_{x_n>0} x_n\|u(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \sup_{x_n>0} x_n\|(u-v)(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \\
&\qquad \sup_{x_n>0} x_n\|(u-v)(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \sup_{x_n>0} x_n\|v(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \\
&\qquad \|u\|_{T^{2(n-1),2}}\|u-v\|_{T^{2(n-1),2}} + \|u-v\|_{T^{2(n-1),2}}\|v\|_{T^{2(n-1),2}}\big) \\
&\le C\|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}}).
\end{aligned} \tag{3.2}$$

In light of Lemma 2.7 (applied with $q = 2$) we find that

$$\begin{aligned}
\big\|\mathscr{L}(u,\pi)\big\| &\le K\big(\|u\|^2_{\mathbf{X}} + \|\mathcal{H}f\|_{\mathbf{X}} + \|\mathscr{G}[F,0]\|_{\mathbf{X}} + \|\mathcal{E}(f)\|_{\mathbf{Z}} + \|\Psi[F,0]\|_{\mathbf{Z}}\big) \\
&\le K(\|u\|^2_{\mathbf{X}} + \|f\|_{\dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau,\eta}}).
\end{aligned} \tag{3.3}$$

Now pick $\varepsilon > 0$ such that $\|f\|_{\dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau,\eta}} \le \varepsilon$. If ε is sufficiently small, then it readily follows from (3.2) and (3.3) that $\mathscr{L}$ has a unique fixed point in a closed ball of $\mathbf{X}\times\mathbf{Z}$ centered at the origin with radius $c\varepsilon$ for some $c > 0$. $\square$

Proof of Theorem 1.16. Let $2 < q < \infty$ and put $p = (n-1)q(q-1)$. Further let $1 < \eta_1 < \tau_1 < \infty$ and $1 < \sigma < \varLambda$ such that

$$\frac{1}{\eta_1} + \frac{n-1}{\tau_1} = 1 + \frac{1}{\sigma} + \frac{n-1}{\varLambda} = 2 + \frac{1}{q-1}. \tag{3.4}$$

Assume $f \in \dot{H}^{-\frac{1}{2},2(n-1)} \cap \dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})$ and $F \in \mathbf{Y}^{\tau_1,\eta_1} \cap \mathbf{Y}^{\tau,\eta}$. We remark that the solution found above may be realized as the unique limit in $\mathbf{X} \times \mathbf{Z}$ of the following sequence of approximations given by

$$(u_1, \pi_1) = (\mathcal{H}(f), \mathcal{E}(f));\ (u_{j+1}, \pi_{j+1}) = (\mathscr{G}[F, u_j \otimes u_j] + u_1, \Psi[F, u_j \otimes u_j] + \pi_1),\ j = 1, 2, \cdots$$

Each element of this sequence belongs to $\mathbf{X}^q \times \mathbf{Z}^q$. In fact, since $(u_1, \pi_1) \in \mathbf{X}^q \times \mathbf{Z}^q$ (see Lemma 2.7) one may proceed via an induction argument to prove the claim. Choose $(\sigma, \varLambda)$ such that $\frac{1}{\sigma} = \frac{1}{2} + \frac{1}{q}$, $\frac{1}{\varLambda} = \frac{1}{2(n-1)} + \frac{1}{p}$ and invoke Proposition 2.19, Hölder's inequality in tent spaces simultaneously to have for each j,

$$\begin{aligned}
\|(u_{j+1}, \pi_{j+1})\|_{\mathbf{X}^q \times \mathbf{Z}^q} &= \|\mathscr{G}[F, u_j \otimes u_j] + u_1\|_{\mathbf{X}^q} + \|\Psi[F, u_j \otimes u_j] + \pi_j\|_{\mathbf{Z}^q} \\
&\le C\big(\|u_1\|_{\mathbf{X}^q} + \|\mathscr{G}[F, u_j \otimes u_j]\|_{\mathbf{X}^q} + \|\pi_1\|_{\mathbf{Z}^q} + \|\Psi[F, u_j \otimes u_j]\|_{\mathbf{Z}^q}\big) \\
&\le C\big(\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau_1,\eta_1}} + \|u_j \otimes u_j\|_{\mathbf{Y}^{\varLambda,\sigma}}\big) \\
&\le C\big(\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau_1,\eta_1}} + \|u_j\|_{T^{2(n-1),2}}\|u_j\|_{T^{p,q}} + \\
&\qquad \sup_{x_n > 0} x_n^{\frac{1}{\eta} + \frac{n-1}{\varLambda}} \|u_j \otimes u_j(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\big) \\
&\le C\big(\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau_1,\eta_1}} + \sup_{x_n > 0} x_n^{\frac{1}{q-1}} \|u_j(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \cdot \\
&\qquad \sup_{x_n > 0} x_n \|u_j(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \|u_j\|_{T^{2(n-1),2}}\|u_j\|_{T^{p,q}}\big) \\
&\le C\big(\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau_1,\eta_1}} + \|u_j\|_{\mathbf{X}}\|u_j\|_{\mathbf{X}^q}\big)
\end{aligned}$$

so that if $(u_j, \pi_j) \in \mathbf{X}^q \times \mathbf{Z}^q$, then so is (u_{j+1}, π_{j+1}). Next, we show that the latter sequence is Cauchy in $\mathbf{X}^q \times \mathbf{Z}^q$. We estimate $(w_j, q_j) = (u_{j+1} - u_j, \pi_{j+1} - \pi_j)$, $j = 1, 2, \ldots$

$$\begin{aligned}
\|(w_j, q_j)\|_{\mathbf{X}^q \times \mathbf{Z}^q} &= \big\|\mathscr{G}[0, u_j \otimes u_j - u_{j-1} \otimes u_{j-1}]\big\|_{\mathbf{X}^q} + \big\|\Psi[0, u_j \otimes u_j - u_{j-1} \otimes u_{j-1}]\big\|_{\mathbf{Z}^q} \\
&\le c\|u_j \otimes u_j - u_{j-1} \otimes u_{j-1}\|_{\mathbf{Y}^{\varLambda,\eta}} \\
&\le c\|u_j \otimes w_{j-1} + w_{j-1} \otimes u_{j-1}\|_{\mathbf{Y}^{\varLambda,\eta}} \\
&\le c\|w_{j-1}\|_{\mathbf{X}^q}(\|u_j\|_{\mathbf{X}} + \|u_{j-1}\|_{\mathbf{X}}) \\
&\le c\|(w_{j-1}, q_{j-1})\|_{\mathbf{X}^q \times \mathbf{Z}^q}(\|u_j\|_{\mathbf{X}} + \|u_{j-1}\|_{\mathbf{X}}).
\end{aligned}$$

Let $\varepsilon > 0$ be as in Theorem 1.15 and take $0 < \varepsilon_q < \varepsilon$. If $\|f\|_{\dot{H}^{-\frac{1}{2},2(n-1)}} + \|F\|_{\mathbf{Y}^{\tau,\eta}} \le \varepsilon_q$, then the conclusion of Theorem 1.15 shows that $\|u_j\|_{\mathbf{X}} \le 2\kappa_q$, $\kappa_q = \kappa_q(\varepsilon_q)$. Whence,

$$\|(w_j, q_j)\|_{\mathbf{X}^q \times \mathbf{Z}^q} \le c2^2\kappa_q \|(w_{j-1}, q_{j-1})\|_{\mathbf{X}^q \times \mathbf{Z}^q}.$$

With $\varepsilon_q > 0$ chosen sufficiently small with $c2^2\kappa_q < 1$, a simple iteration of the previous inequality yields

$$\|(w_j, q_j)\|_{\mathbf{X}^q \times \mathbf{Z}^q} \le (c2^2\kappa_q)^{j-1} \|(w_1, q_1)\|_{\mathbf{X}^q \times \mathbf{Z}^q}$$

thus implying the convergence of the sequence (w_j, q_j) in $\mathbf{X}^q \times \mathbf{Z}^q$. The limit of this sequence solves (**NS**) and by uniqueness, it is the same as that constructed in Theorem 1.15. □

4. Appendix

Here we sketch the proof of Lemma 1.5. Let $K \subset \mathbb{R}^n_+$ be a compact set. Then by Lemma 2.10 we know that $E(K) = \{x' \in \mathbb{R}^{n-1} : K \cap \Gamma(x') \neq \emptyset\}$ has a finite Lebesgue measure. Let us denote by $\mathbf{1}_K$ the characteristic function of the compact set K. If $p \le q$, then via Hölder's inequality, one obtains

$$\begin{aligned}
\|\mathbf{1}_K f\|_{T^{p,q}} &= \left(\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{p/q} dx' \right)^{1/p} \\
&= \left(\int_{E(K)} \left(\iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n dx' \right)^{p/q} dx' \right)^{1/p} \\
&\le \left(\int_{E(K)} \iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \big|E(K)\big|^{\frac{1}{p}-\frac{1}{q}} \\
&\le \big|E(K)\big|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')\cap K} |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\le \big|E(K)\big|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(K)}.
\end{aligned}$$

Moving on, for $q < p$, applying Minkowski's inequality implies

$$\begin{aligned}
\|\mathbf{1}_K f\|_{T^{p,q}} &= \left(\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n \right)^{p/q} dx' \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty \mathbf{1}_{B_{y_n}(y')}(x') \mathbf{1}_K(y', y_n) |f|^q y_n^{1-n} dy_n dy' \right)^{p/q} dx' \right)^{1/p} \\
&\le C_K \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty \mathbf{1}_K |f|^q dy' dy_n dx' \right)^{1/q} \\
&\le C_K \|f\|_{L^q(K)}.
\end{aligned}$$

Assuming that $p \le q$, we use Lemma 2.9 and Minkowski's inequality simultaneously to get

$$\begin{aligned}
\|f\|_{L^q(K)} &= \left(\int_{\mathbb{R}^n_+} \mathbf{1}_K |f|^q dy' dy_n \right)^{1/q} \\
&\le C \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\le C_K \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} y_n^{\frac{(n-1)p}{q}} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\le C_K \left(\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n \right)^{p/q} dx' \right)^{1/p} \\
&\le C_K \|f\|_{T^{p,q}}.
\end{aligned}$$

When $p > q$, the desired bound follows from Hölder's inequality. Indeed, we have

$$\begin{aligned}
\|f\|_{L^q(K)} &= \left(\int_{\mathbb{R}^n_+} \mathbf{1}_K |f|^q dy' dy_n \right)^{1/q} \\
&\le C \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\Big(\int_{E(K)}\iint_{\Gamma(x')}|f|^q y_n^{1-n}dy'dy_n dx'\Big)^{1/q}\\
&\leq C\Big(\int_{\mathbb{R}^{n-1}}\Big(\iint_{\Gamma(x')}|f|^q y_n^{1-n}dy'dy_n\Big)^{p/q}dx'\Big)^{1/p}\big|E(K)\big|^{\frac{1}{q}-\frac{1}{p}}\\
&\leq C\big|E(K)\big|^{\frac{1}{q}-\frac{1}{p}}\|f\|_{T^{p,q}}.
\end{aligned}
$$

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