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1. Introduction

The following text gives an outline of Deligne's proof of the Weil conjectures for readers with some basic knowledge of étale cohomology. While I hope it is useful for other readers as well, it is intended as a surrogate for the sequel of an étale cohomology lecture given in the winter term 2019/20 in Bonn. For this sequel I intended to present a brief exposition without proofs of material about étale cohomology which did not fit into the winter term, followed by an exposition of Deligne's Weil I proof. If this left any time at the end of the term, this was to be followed by giving some proofs omitted in the first part of the lecture. Because I never gave a Weil I lecture before I could not predict how much time would be left at the end. For the same reason I will try to err on the side safety in the selection of material for an exam based on the current text or on a textbook.

When the usual form of lecture was canceled because of the COVID-19 pandemic, I had trouble to use zoom for various reasons, including the initial lack of necessary hardware and my own problems of getting acquainted with a web service I had never used before. In addition, the lecture would have made ample use of the blackboard space in the Großer Hörsaal, for which there seems to be no easy replacement using zoom. To aid anyone who nevertheless wants to take an exam on Weil I, my original intention was to give a brief introduction to the exposition in [**FK88**]. Since this exposition is much too long to present all material in it in a single term, this was to be followed by an explanation of which parts of [**FK88**] would be relevant to the exam, and by a list of mistakes I found in [**FK88**].

However, I found [**FK88**] hard to read if one wants to read only part of it. For this reason, I instead wrote a complete exposition which is self-contained in the sense that most material used is actually formulated in the text. Of course, proofs often are quoted from other sources because they would not have fitted into the lecture. I am offering exams based on this text (of course without my list of errata to [**FK88**]). Proofs omitted in the current text may also be omitted in the exam. Also, I am using the machinery of ℓ-adic sheaves after only a partial exposition of it, relying on the reader's familarity with ordinary cohomology and his willingness to believe that things are similar after he has seen how things work in the torsion case. This too can be done in the exam.

I am also willing to offer exams based on [**Del74**]. This is a very nicely written paper with an exposition of part of the needed prerequisites, making the paper understandable for readers with just a little bit of knowledge of étale cohomology. I actually read it when I was twenty and had only a rough idea of what étale cohomology is. However, it is written in French, of which not all students interested in this may have sufficient knowledge. Also, the exposition of background material in [**Del74**] is shorter than this text, so an exam based on [**Del74**] would have to include some other material as well, for the sake of justice. I leave this open to negotiation, if anyone is interested in this sort of exam.

In the next few weeks I may add a few more references to other sources and correct errors, but promiss not to add anything which would make an exam harder. Of course this may change after fall, when this text is no longer needed for its original purpose.

2. Prerequisites

This section is an exposition of prerequisites of the Weil 1 proof not covered in the basic étale cohomology lecture. Only material of a "basic" nature is in this section. The necessary exposition of étale Picard-Lefschetz theory will be given in the next section.

2.1. Cohomological dimension in the affine case.

Let $\mathfrak{k}$ be an algebraically closed field. In the previous term, we have seen that the étale cohomology of affine curves over $\mathfrak{k}$ with coefficients in a torsion sheaf vanishes in degrees > 1. A related result holds for varieties of arbitrary dimension.

THEOREM 1. *Let X be an affine scheme of finite type over $\mathfrak{k}$ and $d = \dim X$. Then $H^p(X_{ét}, \mathcal{F}) = 0$ when $p > d$ and $\mathcal{F}$ is a torsion sheaf on $X_{ét}$.*

For a proof, see [**FK88**, Chapter I.8], [**Mil80**, Chapter VI.7] or [**SGA4.3**, Expose XIV].

The importance of this result for proving the Riemann Hypothesis for the congruence zeta function comes from the fact that the upper vanishing bound for ordinary cohomology becomes a lower vanishing bound for cohomology with compact support by Poincaré duality. This, together with a cohomology sequence, implies an injectivity result Corollary .2.5.3 for the restriction of cohomology to hyperplane sections essentially reducing the proof of the needed result for Frobenius eigenvalues to the case of the middle degree étale cohomology group of X.

The following corollary is also worthwile mentioning. Let ζ_{X*} denote the direct image functor from $X_{\text{ét}}$ to X_{Zar}, which restricts an étale sheaf to the Zariski open subsets of X.

COROLLARY 1. If X is a prescheme of locally finite type over $\mathfrak{k}$, $x \in X$ and $d = \dim \mathcal{O}_{X,x}$, then $(R^p\zeta_{X*}\mathcal{F})_x = 0$ when $p > d$ and $\mathcal{F}$ is torsion.

In the case of curves, we had established this in the previous term.

2.2. Comparison with classical cohomology.

If X is a prescheme of finite type over Spec$\mathbb{C}$, $X(\mathbb{C})$ carries a "classical" topology X^{an} with underlying set $X(\mathbb{C})$, the coarsest one such that for $U \subseteq X$ Zariski-open and $f \in \mathcal{O}_X(U)$, $U(\mathbb{C}) \subseteq X(\mathbb{C})$ is open in X^{an} and $U(\mathbb{C}) \xrightarrow{f} \mathbb{C}$ continuous for the classical topology on $\mathbb{C}$ and the induced topology from X^{an} on $U(\mathbb{C})$.

The topological space X^{an} also carries the structure of a complex analytic space in the sense of Grauert. For morphisms of complex analytic spaces, the property of étaleness can simply defined as being a local isomorphism. One has an analytic étale topology $X^{\text{an}}_{\text{ét}}$ similar to the étale topology of a scheme. In the case where X is smooth $X(\mathbb{C})$ is simply a complex analytic manifold and étale morphisms are local isomorphisms. It thus possible to continue reading this subsection without having a knowledge of complex analytic spaces.

Since étale morphisms in the complex case are just local isomorphisms, every covering sieve for $X^{\text{an}}_{\text{ét}}$ has a covering subsieve generated by morphisms $U \to X$ with are isomorphisms onto open subsets. Because of this,

$$\underline{\text{Sh}}(X^{\text{an}}_{\text{ét}}) \cong \underline{\text{Sh}}(X^{\text{an}}), \tag{1}$$

the category of sheaves on the ordinary topological space X^{an}. Since an étale morphism $U \to X$ of schemes of finite type over $\mathbb{C}$ defines an étale morphism $U^{\mathrm{an}} \to X^{\mathrm{an}}$, one has a direct image functor

$$\underline{\mathrm{Sh}}(X^{\mathrm{an}}_{\text{ét}}) \xrightarrow{\alpha_{X*}} \underline{\mathrm{Sh}}(X_{\text{ét}})$$

with a left adjoint α_X^* similar to our considerations for $X_{\text{ét}} \to X_{\mathrm{Zar}}$ in the previous term. For an object $\mathcal{F}$ on $\underline{\mathrm{Sh}}(X_{\text{ét}})$, let $\mathcal{F}^{\mathrm{an}}$ be the sheaf on the ordinary topological space X^{an} which is the image of $\alpha_X^*\mathcal{F}$ under the equivalence of categories (1).

We have the following comparison theorem:

THEOREM 2. *Let $X \xrightarrow{f} S$ be a separated morphism between preschemes of finite type over $\mathbb{C}$ and $\mathcal{F}$ a torsion sheaf on $X_{\text{ét}}$. Then one has a canonical isomorphism*

(2) $$R^p f^{\mathrm{an}}_! \mathcal{F}^{\mathrm{an}} \cong (R^p f_! \mathcal{F})^{\mathrm{an}}.$$

In particular ($S = \mathrm{Spec}\mathbb{C}$)

(3) $$H^p_c(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) \cong H^p_c(X_{\text{ét}}, \mathcal{F})$$

when X is separated.

This is [**Arcata**, Théorème IV.6.3], where the definition of $\mathcal{F}^{\mathrm{an}}$ has been omitted and a brief sketch of proof added.

2.3. Smooth base change. In the previous term we introduced a base-change homomorphism

(1) $$f^* R^l p_* \mathcal{F} \to R^l \tilde{p}_* f_X^* \mathcal{F}$$

for Cartesian squares

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f_X} & X \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{S} & \xrightarrow{f} & S \end{array}$$

and shown that it is an isomorphism when p is proper and $\mathcal{F}$ a torsion sheaf.

THEOREM 3. *Assume that f is a smooth morphism and that $\mathcal{F}$ is a sheaf of torsion, such that the order of torsion of every element $\mathcal{F}_\xi$ any geometric stalk is invertible in $\mathcal{O}_{X,[\xi]}$, where $[\xi]$ is the ordinary point underlying ξ. Then (1) is an isomorphism.*

This must be shown along with a result called the local acyclicity of smooth morphisms. Formulating this would be essential for presenting the proof because both assertions must be shown together. For the purposes of this exposition we take the result for granted, although the proof is a bit more elementary than for proper base change. It may be found in [**SGA4.3**, Exposé XV and XVI], [**Arcata**, V], [**Mil80**, VI.4] and [**FK88**, I.7].

One important consequence of smooth base change is a type of "homotopy invariance" result for smooth families of proper varieties.

THEOREM 4. *Let $X \xrightarrow{p} S$ be a proper smooth morphism and $\mathcal{F}$ a locally constant constructible torsion sheaf on $X_{ét}$, with the torsion satisfying the assumption from the previous theorem. Then the sheaves $R^k p_* \mathcal{F}$ on $S_{ét}$ are locally constant constructible.*

This may be combined with (.2.2.3) to give the following corollary, which can be used to derive an equality between étale ℓ-adic and ordinary Betti numbers.

COROLLARY 1. Let $X \xrightarrow{p} \mathrm{Spec}\mathbb{Z}_f$ be a proper smooth morphism, G a finite abelian group and s a geometric point of $\mathrm{Spec}\mathbb{Z}_f$ located over an ordinary point corresponding to a prime number not diving f or the order of G. Then there is an isomorphism

$$H^p\Big((X_s)_{\text{ét}}, G\Big) \cong H^p\Big(X(\mathbb{C})^{\text{an}}, G\Big)$$

which is functorial in G.

One important step in deriving Theorem 4 from the theorems of smooth and proper base change is the proof of the fact that a certain cospecialization homomorphism is an isomorphism. It will be necessary to explain this because it is necessary to understand the Picard-Lefschetz formulas needed for the proof of the Weil conjecture. Let $S = \mathrm{Spec}R$ be the spectrum of a strictly Henselian local domain, s the closed and η the generic point of S. Let $\overline{\eta} \xrightarrow{k_S} S$ be the spectrum of a separable closure of the field of quotients of S. Let $X \xrightarrow{p} S$ be a proper morphism and let $X_{\overline{\eta}} \xrightarrow{k_X} X$ be the base-change of k_S. If $\mathcal{F}$ is a sheaf of torsion on $X_{\text{ét}}$, we have a canonical morphism

$$\mathcal{F} \to k_{X*} k_X^* \mathcal{F} \tag{2}$$

giving rise to

$$H^p\Big((X_s)_{\text{ét}}, \mathcal{F}\Big) \cong H^p(X_{\text{ét}}, \mathcal{F}) \to H^p(X_{\text{ét}}, k_{X*} k_X^* \mathcal{F}) \to H^p(X_{\overline{\eta}}, k_X^* \mathcal{F}) \tag{3}$$

where the first isomorphism is by proper base change and the the last morphism comes from the Leray spectral sequence

$$E_2^{p,q} = H^p\big(X_{\text{ét}}, R^q k_{X*} k_X^* \mathcal{F}\big) \Rightarrow H^{p+q}\Big((X_{\overline{\eta}})_{\text{ét}}, k_X^* \mathcal{F}\Big). \tag{4}$$

In the case of Theorem 4, we show (3) to be an isomorphism. See also [**FK88**, Lemma I.8.13] or [**Mil80**, Proof of corollary VI.4.2]:

PROPOSITION 1.
- If in the previous situation $\mathcal{F}$ is locally constant constructible and the torsion of $\mathcal{F}$ invertible in R, then (3) is an isomorphism.
- More generally, without the assumption of $X \to S$ being proper, (2) is an isomorphism and $R^q k_{X*} k_X^* \mathcal{F} = 0$ for $q > 0$.

PROOF. By the Leray spectral sequence (4) and the application of proper base change mentioned after (3), it is sufficient to show the second assertion. The assertion is local on $X_{\text{ét}}$, hence we may assume that $\mathcal{F} = F_X$ is the constant sheaf given by a finite abelian group F of order invertible in R. Since $F_X \cong p^* F_S$ smooth base change reduces the assertion to the case

$X = S$. This case easily follows from the fact that S is connected and $\overline{\eta}$ the spectrum of a separably closed field. □

2.4. Purity and fundamental classes. For a Zariski-closed subset $Z \subseteq X$ with complement U, let $H^0_Z(X_{\text{ét}}, \mathcal{F})$ denote the kernel of the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ and let $H^p_Z(X_{\text{ét}}, \mathcal{F})$ be its p-th derived functor.

Lemma 1. *When $\mathcal{F}$ is an injective object of $\underline{\mathrm{Sh}}(X_{\acute{e}t})$, then the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.*

Proof. This surjectivity holds for the skyscraper sheaves $\xi_* G$ at geometric points ξ of X and is injerited by direct products from their factors and also by direct summands. Since the morphism from $\mathcal{F}$ to the product $\mathcal{P}$ of $\xi_*\mathcal{F}_\xi$ taken over a sufficiently large set of geometric points is a monomorphism and $\mathcal{F}$ is injective, $\mathcal{F}$ is a direct summand of $\mathcal{P}$ hence inherits the surjectivity from $\mathcal{P}$. □

By the machinery of derived functors, we get a long exact cohomology sequence

$$\ldots \to H^{p-1}(U_{\text{ét}}, \mathcal{F}) \xrightarrow{d} H^p_Z(X_{\text{ét}}, \mathcal{F}) \to H^p(X_{\text{ét}}, \mathcal{F}) \to H^p(U_{\text{ét}}, \mathcal{F}) \xrightarrow{d} \ldots \tag{1}$$

If V is an étale X-prescheme, let Z_V denote the preimage of Z in V. There is an étale sheaf $\mathcal{H}^0_Z\mathcal{F}$ defined by

$$\mathcal{H}^0_Z\mathcal{F}(V) = H^0_{Z_V}(V, \mathcal{F}).$$

The sheafification of $V \to H^*_{Z_V}(V_{\text{ét}}, \mathcal{F})$ satisfies the universal property of the derived functor $\mathcal{H}^*_Z\mathcal{F}$ of $\mathcal{H}^0_Z\mathcal{F}$, and if $U \xrightarrow{j} X$ denotes the open embedding of U we have a cohomology sequence

$$\ldots \to \mathcal{H}^{p-1}\mathcal{F} \xrightarrow{d} \mathcal{H}^p_Z\mathcal{F} \to \mathcal{H}^p\mathcal{F} \to \mathcal{H}^p\mathcal{F} \xrightarrow{d} \mathcal{H}^{p+1}_Z\mathcal{F} \to \ldots \tag{2}$$

The machinery of Grothendieck spectral sequences gives

$$E_2^{p,q} = H^p(X_{\text{ét}}, \mathcal{H}^q_Z\mathcal{F}) \Rightarrow H^{p+q}_Z(X_{\text{ét}}, \mathcal{F}) \tag{3}$$

as well as, for $Z \subseteq Y \subseteq X$,

$$E_2^{p,q} = \mathcal{H}^p_Z\mathcal{H}^q_Y\mathcal{F} \Rightarrow \mathcal{H}^{p+q}_Z\mathcal{F} \tag{4}$$

or

$$E_2^{p,q} = i^*_{Z\to Y}\mathcal{H}^p_Z i^*_{Y\to X}\mathcal{H}^q_Y\mathcal{F} \Rightarrow i^*_{Z\to X}\mathcal{H}^{p+q}_Z\mathcal{F}. \tag{5}$$

The derived functor $\mathcal{H}^p_Z$ is taken on $\underline{\mathrm{Sh}}(X_{\text{ét}})$ in (4) while in (5) it is taken on $\underline{\mathrm{Sh}}(X_{\text{ét}})$. The verification of the assumptions for this machinery is not carried out here. It is an amusing exercise for the reader or can be looked up in the various text books. In any case, it will not be part of any exam.

Remark 1. If $\tilde{X} \xrightarrow{\xi} X$ is a morphism, $\tilde{Z} \subseteq \tilde{X}$ a closed subset containing the preimage of Z, by the universal property of derived functors there is a unique morphism

$$H^*_Z(X_{\text{ét}}, \mathcal{F}) \xrightarrow{\xi^*} H^*(\tilde{X}_{\text{ét}}, \mathcal{F})$$

of cohomological functors on $\underline{\mathrm{Sh}}(X_{\text{ét}})$ which in degree zero is the obvious pull-back.

The basic purity result is the following

THEOREM 5. *Let $Z \xrightarrow{i} X$ be a (automatically regular) closed embedding of smooth S-preschemes. Assume that i is of constant codimension d and that $\mathcal{F}$ is a locally constant constructible torsion sheaf on $X_{\text{ét}}$, where the order of torion is invertible on S. Then $\mathcal{H}^p_Z\mathcal{F}$ vanishes unless $p = 2d$, in which case $i^*\mathcal{H}^{2d}_Z\mathcal{F}$ is locally isomorphic to $i^*\mathcal{F}$ and thus locally constant constructible. The formation of these sheaves commutes with base change $S' \to S$.*

The proof works by reduction to the case of a section of the affine line over X, using induction on the codimension and smooth base change. See [**FK88**, I.10], [**Mil80**, VI.5] or [**SGA4.3**, XVI.3].

REMARK 2. Because of the vanishing assertion and the easily seen isomorphism

$$\mathcal{H}^p_Z\mathcal{F} \cong i_*i^*\mathcal{H}^p_Z\mathcal{F},$$

(3) degenerates to

$$H^p_Z(X_{\text{ét}}, \mathcal{F}) \cong H^{p-2d}(X_{\text{ét}}, \mathcal{H}^{2d}_Z\mathcal{F}) \cong H^{p-2d}(Z_{\text{ét}}, i^*\mathcal{H}^{2d}_Z\mathcal{F}). \tag{6}$$

REMARK 3. Assume that i factors as a composition $Z \xrightarrow{j} Y \xrightarrow{k} X$ of closed embeddings also satisfying the assumptions of the theorem. If $d_{Y/X}$ and friends denote the respective codimensions, then $d_{Z/X} = d_{Z/Y} + d_{Y/X}$. In view of the vanishing assertion, the spectral sequence (5) degenerates to

$$i^*\mathcal{H}^{2d_{Z/X}}_Z\mathcal{F} \cong j^*\mathcal{H}^{2d_{Z/Y}}_Z k^*\mathcal{H}^{2d_{Y/X}}_Y\mathcal{F} \tag{7}$$

It will be necessary to give a more canonical description of $\mathcal{H}^{2d}_Z\mathcal{F}$ in the situation of the above theorem. I will give an exposition based on the relation between H^1 and torseurs studied in the previous term. Recall that an $\mathcal{F}$-torseur is a sheaf of sets $\mathcal{T}$ with a morphism $\mathcal{F}\times\mathcal{F} \to \mathcal{T}$ satisfying the axioms for a principal homogenuous space for $\mathcal{F}$. There is an obvious category of $\mathcal{F}$-torseurs which is a groupoid. A torseur $\mathcal{T}$ splits if it has a global section or, equivalently, is isomorphism to the trivial torseur given by $\mathcal{F}$ acting on itself in the obvious way. Let $\mathfrak{T}_X\mathcal{F}$ denote the set of isomorphism classes of $\mathcal{F}$-torseurs on $X_{\text{ét}}$. Of course these definitions already work when $\mathcal{F}$ is a non-abelian sheaf of groups, but we will only need them in the abelian case.

In the previous term we have constructed a bijection $\mathfrak{T}_X\mathcal{F} \cong H^1(X_{\text{ét}}, \mathcal{F})$ when $\mathcal{F}$ is abelian. In particular, in the case where $\mathcal{F}$ is an injective object of $\underline{\mathrm{Sh}}(X_{\text{ét}})$ this implies that every torseur splits. This splitting was, however, shown first and the bijection derived from it. The bijection is such that whenever

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0 \tag{+}$$

is a short exact sequence of sheaves of abelian groups giving rise to $H^0(X_{\text{ét}}, \mathcal{Q}) \xrightarrow{d} H^1(X_{\text{ét}}, \mathcal{F})$, the $\mathcal{F}$-torseur $\mathcal{T}_q$ of preimages of $q \in \mathcal{Q}(X)$ is assigned the cohomology class dq.

There is an obvious candidate for a version of this bijection which applies to $H^1_Z(X_{\text{ét}}, \mathcal{F})$. In the discussion of this, the restrictions in Theorem 5 may be dropped, and arbitrary closed subsets Z of arbitrary preschemes X allowed. Consider a groupoid $\mathfrak{G}_Z\mathcal{F}$ with objects $(\mathcal{T}, t)$ where $\mathcal{T}$ is an $\mathcal{F}$-torseur on $X_{\text{ét}}$ and $t \in \mathcal{T}(U)$. Morphisms $(\mathcal{T}, t) \to (\mathcal{T}', t')$ are given by morphisms $\mathcal{T} \to \mathcal{T}'$ of $\mathcal{F}$-torseurs sending t to t'. Let $\mathfrak{T}_Z\mathcal{F}$ denote the set of isomorphism classes of objects of that category.

Proposition 1. There is a unique bijection $\mathfrak{T}_Z\mathcal{F} \cong H^1_Z(X_{\text{ét}}, F)$ with the following property: For every short exact sequence (+) and $q \in H^0_Z(X_{\text{ét}}, \mathcal{Q})$, the bijection associates to the isomorphism class of the pair $(\mathcal{T}_q, 0)$ the image of q under

(*) $$H^0_Z(X_{\text{ét}}, Q) \xrightarrow{d} H^1_Z(X_{\text{ét}}, \mathcal{F}).$$

Proof. The uniqueness of the bijection is easily seen by applying the condition characterizing it to a single sequence (+) with injective $\mathcal{G}$. To construct the bijection, we fix such a sequence, again with injective $\mathcal{G} = \mathcal{G}_o$. Let $\mathcal{F} \xrightarrow{\iota} \mathcal{G}$ denote the first morphism in that sequence.

There is a direct image $\iota_*\mathcal{T}$ of the $\mathcal{F}$-torseur $\mathcal{T}$ defined as sheafifying the presheaf associating to the object V of $X_{\text{ét}}$ the quotient $\mathcal{T}(V) \times \mathcal{G}(V)$ by the diagonal $\mathcal{F}(V)$-action. There is a morphism $\mathcal{T} \to \iota_*\mathcal{T}$ of sheaves of sets compatible with the $\mathcal{F}$-actions on both sides. By the aforementioned triviality of $\mathfrak{T}_{\mathcal{G}}$ there is an isomorphism $\iota_*\mathcal{T} \cong \mathcal{G}$, hence a morphism $\mathcal{T} \xrightarrow{\gamma} \mathcal{G}$ of sheaves of sets compatible with the $\mathcal{F}$-actions on both sides. By Lemma 1 there is $g \in \mathcal{G}(X)$ restricting to $\gamma(t) \in \mathcal{G}(U)$. Replacing γ by $\tilde{\gamma}(\tau) = \gamma(\tau) - g$ we may assume that γ sends t to 0. If $V \in \mathrm{Ob}X_{\text{ét}}$ such that $\mathcal{T}$ splits on V let $\tau \in \mathcal{T}(V)$, and let $q_V \in \mathcal{Q}(V)$ denote the image of $\gamma(\tau)$ in $\mathcal{Q}(V)$. It is easy to see that this is indeed independent of the choice of τ, and that $\upsilon^*q_V = q_W$ for morphisms $W \to V$ in $X_{\text{ét}}$. There is thus a unique $q \in \mathcal{Q}(X)$ such that each q_V is the image of q in $\mathcal{Q}(V)$. If $V = U$ we may take $\tau = t$ in the definition of q_V. Thus, $q_U = 0$ and $q \in H^0_Z(X_{\text{ét}}, Q)$. We want to associate $d(q) \in H^1_Z(X_{\text{ét}}, \mathcal{F})$ to the isomorphism class of $(\mathcal{T}, t)$. Obviously, γ defines an isomorphism $\mathcal{T} \to \mathcal{T}_q$ sending t to 0, hence our condition will then be satsified in the special case of the initially chosen short exact sequence

(%) $$0 \to \mathcal{F} \to \mathcal{G}_o \to \mathcal{Q}_o \to 0,$$

and surjectivity of $\mathfrak{T}_Z\mathcal{F} \to H^1_Z(X_{\text{ét}}, F)$ follows from this and the surjectivity of (*). Because $\gamma : (\mathcal{T}, t) \to (\mathcal{T}_q, 0)$, the isomorphism class of $(\mathcal{T}, t)$ depends only on that of $(\mathcal{T}_q, 0)$. If q and $\tilde{q}$ define the same element of $H^1_Z(X_{\text{ét}}, \mathcal{F})$, there is $g \in H^0_Z(X_{\text{ét}}, \mathcal{G})$ mapping to $\tilde{q} - q$, and addition of g defines an isomorphism $(\mathcal{T}_q, 0) \to (\mathcal{T}_{\tilde{q}}, 0)$, showing the injectivity of our map $\mathfrak{T}_Z\mathcal{F} \to H^1_Z(X_{\text{ét}}, F)$ provided that it is well-defined.

The only potential ambiguity comes from the choice of γ. As a different γ' must also be compatible with the $\mathcal{F}$-action and $\mathcal{T}$ is locally trivial, $\gamma'(\tau) = \gamma(\tau) + g$ for some $g \in \mathcal{G}(X)$. As $\gamma'(t)$ must also be zero we have $g \in H^0_Z(X_{\text{ét}}, \mathcal{Q})$. As

$$q' = q + \Big(\text{image of } g \text{ under } \mathcal{G}(V) \to \mathcal{Q}(V)\Big)$$

the images of q and q' in $H^1_Z(X_{\text{ét}}, \mathcal{F})$ coincide, and there is no ambiguity.

To confirm the condition of the proposition for sequences (+) different from the initially chosen (%), we use the injectivity of $\mathcal{G}_o$ to extend $\mathcal{F} \to \mathcal{G}_o$ to a morphism $\mathcal{G} \xrightarrow{G} \mathcal{G}_o$ compatible with the morphisms from $\mathcal{F}$, defining $\mathcal{Q} \xrightarrow{Q} \mathcal{Q}_o$ on cokernels. As G defines an isomorphism

$\mathcal{T}_q \cong \mathcal{T}_{Q(q)}$ and (1) is functorial in the short exact sequence, the constructed bijection indeed associates $d(q)$ to $\mathcal{T}_q$. □

In the situation of Theorem 5 with $d = 1$, $Z \to X$ is a regular embedding of codimension one, hence the seaf of ideals $\mathcal{I}$ defining it is a line bundle. Let $\mathcal{L} = \mathcal{I}^{-1}$ be its inverse. These Zariski sheaves of modules define étale sheaves as explained in the previous term, and the subsheaf of sets $\mathcal{L}^* \subseteq \mathcal{L}$ of nowhere vanishing sections of the line bundle is an $\mathcal{O}^*_{X_{\text{ét}}}$-torseur. We have $\mathbf{1} \in \mathcal{L}(X)$ given by the embedding $\mathcal{I} \to \mathcal{O}_X$ which is an isomorphism outside Z, hence $\mathbf{1} \in \mathcal{L}^*(U)$.

Definition 1. In the situation of Theorem 5 with $d = 1$, let

$$[Z]_\infty \in H^1_Z(X_{\text{ét}}, \mathcal{O}^*_{X_{\text{ét}}})$$

be the cohomology class associated to the pair $(\mathcal{L}^*, \mathbf{1})$ by Proposition 1 applied to $\mathcal{F} = \mathcal{O}^*_{X_{\text{ét}}}$. For a natural number ℓ invertible on S, let

$$[Z]_\infty \in H^2_Z(X_{\text{ét}}, \boldsymbol{\mu}_\ell)$$

be the image of $[Z]_\infty$ under d in the instance

$$\ldots \to H^1_Z(X_{\text{ét}}, \mathcal{O}^*_{X_{\text{ét}}}) \xrightarrow{d} H^2_Z(X_{\text{ét}}, \boldsymbol{\mu}_\ell) \to H^2_Z(X_{\text{ét}}, \mathcal{O}^*_{X_{\text{ét}}}) \xrightarrow{\ell} H^2_Z(X_{\text{ét}}, \mathcal{O}^*_{X_{\text{ét}}}) \to \ldots$$

of (1).

The fundamental classes coincide with the ones constructed at the beginning of [**Mil80**, VI.6] for divisors on smooth varieties over a field. This restriction is unnecessary in the previous definition, but Milne shows the canonical version of Theorem 5 only in this case. Since maximal generality is not needed here, we will also eventually impose this condition.

Since the canonical fundamental classes from Definition 1 are in the cohomology with coefficients in $\boldsymbol{\mu}_\ell$, which is only locally and non-canonically isomorphic to $\mathbf{\Lambda}_X = (\mathbb{Z}/\ell\mathbb{Z})_{X_{\text{ét}}}$, the important issue of Tate twists now comes up for the first time. Let $\mathbf{\Lambda}_X(1) = \boldsymbol{\mu}_\ell$ and let, for positive integers k, $\mathbf{\Lambda}_X(k)$ be the k-th tensor power over $\mathbf{\Lambda}$ of this sheaf of $\mathbf{\Lambda}$-modules. Moreover, let $\mathbf{\Lambda}(-k)$ be the k-th tensor power of $\mathbf{\Lambda}(-1)$, the sheaf of homomorphisms from $\mathbf{\Lambda}(1)$ to $\mathbf{\Lambda}$. For a $\mathbf{\Lambda}$-module $\mathcal{F}$, let $\mathcal{F}(k) = \mathcal{F} \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}(k)$. These twists are easily seen to be compatible with derived direct and with inverse images of sheaves of modules and satisfy $\mathcal{F}(k+l) \cong \big(\mathcal{F}(k)\big)(l)$ canonically.

Of course, when working over an algebrically closed field or a strictly Henselian local ring where ℓ is invertible, it is possible to chose a primitive ℓ-th root of 1 to identify $\mathcal{F}$ with all $\mathcal{F}(k)$, but this identification depends on the choice of the $\sqrt[\ell]{1}$. This is similar to ordinary Poincaré duality in the complex analytic case where the choice of a $\sqrt{-1}$ determines an $\mathbb{R}$-orientation for all complex analytic manifolds which however depends on the choice made.

Theorem 6. *In the situation of Theorem 5 with S the spectrum of a field, let i denote the morphism $Z \to X$ and $d = d_{Z/X}$ the codimension of Z in X. For locally free Λ_X-modules $\mathcal{F}$, there is a unique functorial in $\mathcal{F}$ isomorphism*

$$i^*\mathcal{F} \xrightarrow{\iota_{X,Z}} i^*\mathcal{H}^{2d}_Z\mathcal{F}(d)$$

with the following properties:

- $\iota_{X,X} = \mathrm{Id}_{i^*\mathcal{F}}$.
- *If i factors as $Z \xrightarrow{j} Y \xrightarrow{k} X$ where $Y \xrightarrow{k} X$ is also a smooth over $\mathfrak{k}$ closed subprescheme of X, then the diagram*

$$\begin{array}{ccccc} j^*k^*\mathcal{F} & & \xrightarrow{\quad\cong\quad} & & i^*\mathcal{F} \\ \downarrow {\scriptstyle j^*\iota_{X,Y}} & & & & \downarrow {\scriptstyle \iota_{X,Z}} \\ j^*k^*\mathcal{H}_Y^{2d_{Y/X}}\mathcal{F}(d_{Y/X}) & \xrightarrow{\iota_{Y,Z}} & j^*\mathcal{H}_Z^{2d_{Z/Y}}k^*\mathcal{H}_Y^{2d_{Y/X}}\mathcal{F}(d_{Z/X}) & \xrightarrow{(7)} & i^*\mathcal{H}_Z^{2d_{Z/X}}\mathcal{F}(d_{Z/X}) \end{array}$$

  commutes.
- *Let $[Z]_\ell \in H_Z^{2d}\big(X_{ét}, \mathbf{\Lambda}_X(d)\big)$ denote the inverse image of*

$$\mathbf{1} \in H^0(Z, \mathbf{\Lambda}_Z) \cong H^0(Z, i^*\mathbf{\Lambda}_X) \xrightarrow{\iota_{X,Z}} H^0(Z, i^*\mathcal{H}_Z^{2d}\mathbf{\Lambda}_X)$$

  under (6). If $d = 1$, this coincides with the fundamental class defined in Definition 1. Moreover, if $\tilde{X} \xrightarrow{\xi} X$ is a morphism such that $\tilde{Z} = Z \times_X \tilde{X}$ is smooth over $\mathfrak{k}$ and $d_{\tilde{Z}/\tilde{X}} = d_{Z/X}$, then $\xi^*[Z]_\ell = [\tilde{Z}]_\ell$, where the pull-back is defined by Remark 1.

The theorem follows from [**Mil80**, Theorem VI.6.1]. Different treatments of the fundamental class are in [**SGA4$\frac{1}{2}$**] and [**FK88**].

2.5. Poincaré Duality. Let X smooth connected of finite type and separated over an algebraically closed field $\mathfrak{k}$ and let $X \xrightarrow{j} \overline{X}$ a Nagata compactification of X. Moreover, let $Z \subseteq X$ be a Zariski-closed subset which stays closed in $\overline{X}$. We have an obvious homomorphism of extension by zero

$$H_Z^0(X_{\text{ét}}, \mathcal{F}) \to (j_!\mathcal{F})(\overline{X})$$

which gives rise to

$$H_Z^*(X_{\text{ét}}, \mathcal{F}) \to H_c^*(X_{\text{ét}}, \mathcal{F}) \tag{1}$$

by the universal property of derived functors. In the case where Z is smooth of codimension e, we use the same notation $[Z]_\ell$ for the image under (1)

$$[Z]_\ell \in H_c^{2e}\big(X_{\text{ét}}, \mathbb{Z} \,/\, \ell\mathbb{Z}(e)\big) \tag{2}$$

of the fundamental class $[Z]_\ell$ from Theorem 6. In particular, when $x \in X$ is a closed point we have a canonical fundamental class

$$[\{x\}] \in H_{\{x\}}^{2d}\big(X_{\text{ét}}, \mathbb{Z} \,/\, \ell\mathbb{Z}(d)\big) \to H_c^{2d}\big(X_{\text{ét}}, \mathbb{Z} \,/\, \ell\mathbb{Z}(d)\big).$$

where ℓ is assumed to be invertible in $\mathfrak{k}$. We abbreviate $\mathbb{Z}/\ell\mathbb{Z} = \Lambda$ as before. The easiest case of Poincaré duality may now be formulated as follows:

THEOREM 7. *Under the above assumptions, $H_c^{2d}\big(X_{\acute{e}t}, \mathbf{\Lambda}_X(d)\big)$ is a free $\big(\mathbf{\Lambda}_X(d)\big)$-module of rank 1 and for arbitrary $x \in X$ $[\{x\}]$ is a generator of this module. This generator does not depend on the choice of x, giving rise to a canonical isomorphism*

$$H_c^{2d}\big(X_{\acute{e}t}, \mathbf{\Lambda}_X(d)\big) \xrightarrow{\mathrm{Tr}_{X/\mathfrak{k}}} \Lambda.$$

The resulting pairing

$$H_c^k(X_{\acute{e}t}, \mathcal{F}) \times \mathrm{Ext}^{2d-k}_{\mathbf{\Lambda}_X}(\mathcal{F}, \mathbf{\Lambda}_X(d)) \to \Lambda$$

for constructible $\mathbf{\Lambda}_X$-modules $\mathcal{F}$ is a non-degenerate pairing of finitely generated Λ-modules.

This is shown in [**Mil80**, VI.11], [**FK88**, II.1] and [**SGA4.3**, XVIII], where the last two sources provide more general results and have a different exposition. In particular, in [**FK88**] the relation with the fundamental class is only made later on in the book. An introduction to Poincaré duality over fields is also given in [**Arcata**, Chapter VI].

REMARK 1. The pairing considered in the theorem is a special case of a general pairing for arbitrary objects of $\underline{\mathrm{Sh}}(Y_{\text{ét}})$

$$(+) \qquad H^p(Y_{\text{ét}}, \mathcal{A}) \times \mathrm{Ext}^q_{\underline{\mathrm{Sh}}(Y_{\text{ét}})}(\mathcal{A}, \mathcal{B}) \to H^{p+q}(Y_{\text{ét}}, \mathcal{B}).$$

An element of $H^p(Y_{\text{ét}}, \mathcal{A})$ gives rise to a morphism $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \to H^p(Y_{\text{ét}}, \mathcal{B})$ which is functorial in $\mathcal{B}$, and the universal property of derived functors gives (+). This may be applied with $Y = \overline{X}$, $\mathcal{A} = j_!\mathcal{F}$ to give

$$H_c^p(X_{\text{ét}}, \mathcal{F}) \times \mathrm{Ext}^q_{\underline{\mathrm{Sh}}(Y_{\text{ét}})}(j_!\mathcal{F}, \mathcal{B}) \to H^{p+q}(Y_{\text{ét}}, \mathcal{B}).$$

When $\mathcal{B} = j_!\mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ are Λ_X-modules this can be composed with a morphism

$$\mathrm{Ext}^q_{\mathbf{\Lambda}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^q_{\underline{\mathrm{Sh}}(Y_{\text{ét}})}(j_!\mathcal{F}, j_!\mathcal{G})$$

given by the functoriality of $j_!$ for $q = 0$ and the universal property of the derived functor Ext in general. The result is

$$H_c^p(X_{\text{ét}}, \mathcal{F}) \times \mathrm{Ext}^q_{\mathbf{\Lambda}_X}(\mathcal{F}, \mathcal{G}) \to H_c^{p+q}(X_{\text{ét}}, \mathcal{G})$$

which of course must be shown to be independent of the choice of compactification. Applying this with $\mathcal{G} = \mathbf{\Lambda}_X(d)$ gives the pairing considered in the theorem.

REMARK 2. A bilinear form $X \times Y \to \Lambda$ on finitely generated Λ-modules is called non-degenerate if it defines an isomorphism

$$X \to Y^* = \mathrm{Hom}_\Lambda(Y, \lambda)$$

or, equivalently, an isomorphism $Y \xrightarrow{\cong} X^*$. Note that Λ is easily seen to be self-injective (eg, using Baer's criterion) and $X \to X^*$ is an anti-equivalence on the category of finitely generated Λ-modules. The fact that $X \xrightarrow{\cong} X^{**}$ can be verified on cyclic modules which is also easy.

Note that the finiteness assertion made in the theorem will only follow from the result of the previous term directly if X is proper.

When $\mathcal{F}$ is locally free $\mathbf{\Lambda}_X$-module we have

$$\operatorname{Hom}_{\mathbf{\Lambda}_X}(\mathcal{F},\mathcal{G}) \cong \left(\mathcal{F}^* \otimes_{\mathbf{\Lambda}_X} \mathcal{G}\right)$$

for $\mathbf{\Lambda}_X$-modules $\mathcal{G}$, where the sheaf $\mathcal{F}^*$ of homomorphisms from $\mathcal{F}$ to $\mathbf{\Lambda}_X$ is a dual of X. In the category of $\mathbf{\Lambda}_X$-modules sufficiently many injective objects may be constructed using products of skyscraper sheaves, and the functor from $\mathbf{\Lambda}_X$-modules to $\underline{\mathrm{Sh}}(X_{\text{ét}})$ sends them to products of skyscraper sheaves which are acyclic for $H^*(X_{\text{ét}},\cdot)$. Thus,

$$\operatorname{Ext}^q_{\mathbf{\Lambda}_X}(\mathcal{F},\mathcal{G}) \cong H^q(X_{\text{ét}},\mathcal{F}^* \otimes_{\mathbf{\Lambda}_X} \mathcal{G})$$

when $\mathcal{F}$ is locally free.

Corollary 1. If, in the situation of Theorem 7, $\mathcal{F}$ is a locally free $\mathbf{\Lambda}_X$-module, we have

$$H^p_c(X_{\text{ét}},\mathcal{F}) \cong H^{2d-p}(X_{\text{ét}},\mathcal{F}^*(d))^*.$$

Remark 3. In particular, if X is also proper and $p=d$, one gets a non-degenerate bilinear form on $H^d(X_{\text{ét}},\mathbf{\Lambda}_X)$ after chosing a primitive $\sqrt[\ell]{1}$. It can be shown that this bilinear form is symmetric if d is even and symplectic when d is odd.

If X is affine, the previous corollary may be combined with Theorem 1.

Corollary 2. If, in the situation of Theorem 7, X is affine and $\mathcal{F}$ a locally free $\mathbf{\Lambda}_X$-module, we have $H^p_c(X_{\text{ét}},\mathcal{F}) = 0$ when $p < d$.

Combining this with a long exact cohomology sequence

$$\ldots \to H^p_c(X_{\text{ét}},\mathcal{F}|_X) \to H^p_c(Y_{\text{ét}},\mathcal{F}) \to H^p(Z_{\text{ét}},i^*\mathcal{F}) \to$$

obtained using

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

where $X \xrightarrow{j} Y \xleftarrow{i} Z$ are the inclusions, one obtains the following result (usually called *weak Lefschetz*) about restriction of cohomology classes to hyperplane sections.

Corollary 3. Let $Y \subseteq \mathbb{P}^N_{\mathfrak{k}}$ be an irreducible projective $\mathfrak{k}$-scheme and $Z \subseteq Y$ a hyperplane section such that $X = Y \setminus Z$ is smooth. If $\mathcal{F}$ is a locally free $\mathbf{\Lambda}_Y$-module then the restriction

$$H^p(Y_{\text{ét}},\mathcal{F}) \to H^p(Z_{\text{ét}},i^*\mathcal{F})$$

is bijective when $p < d-1$ and injective when $p < d$.

We already mentioned this after Theorem 1. A discussion of related results which are also sometimes called "weak Lefschetz" is in [**Mil80**, VI.7].

It is perhaps already known to the reader that the hard part of giving a full proof of the original Weil conjectures which was still left after the work of Grothendieck on étale cohomology and which was solved by Deligne boils down to the investigation of Frobenius eigenvalues on $\mathbb{Q}_l$-vector spaces $H^p(X,\mathbb{Q}_\ell)$ defined in terms of the $H^p(X_{\text{ét}},\mathbb{Z}/\ell^k\mathbb{Z})$. If $d = \dim X$ then Poincaré duality relates the cohomological degrees p and $2d-p$ to each other. It will thus be sufficient to consider $p \le d$, and weak Lefschetz plus an easy argument of induction on $\dim X$ shows that it is in fact sufficient to consider $p = d$. Since only appropriate estimates on the absolute value of the Frobenius eigenvalues will be obtained directly, it is necessary to also consider arbitrary powers of X which for other reasons are chosen to be even-dimsional. One then considers morphisms $X^k \xrightarrow{\pi} C$ where C is a curve, which are smooth over an open dense subset U of C and degenerate in a controlled way outside U. This is done using the technique of Lefschetz pencils. It will be necessary to know that the symplectic form on $H^{dk-1}\big((X^k)_u, \mathbb{Q}_\ell)$ obtained by Poincaré duality (cf. Remark 3) depends in a controllable way on u, defining a non-degenerate pairing of ℓ-adic sheaves on U. For this, the above simplified discussion of Poincaré duality is insufficient but the parts of [**FK88**] or [**SGA4.3**] cited above actually provide what is needed. Obviously, the only hard point besides what we formulated as Theorem 7 is to define the trace morphism in sufficient generality.

There is another consequence of Poincaré duality will will be needed later on. In [**FK88**] this is presented as a remark after Proposition III.1.10.

Corollary 4. Let $\tilde{X} \xrightarrow{\pi} X$ be a birational map between proper smooth connected schemes of finite type over the algebraically closed field $\mathfrak{k}$ in which the natural number ℓ is invertible. Then

$$H^p(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\pi^*} H^p(\tilde{X}_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$$

is injective.

Proof. Let $U \subseteq X$ be an open dense set over which π is an isomorphism, and let $x \in U$. For the sake of simplicity, we use notations as if U was a subset of $\tilde{X}$ as well, and $\pi\,|_U = \mathrm{Id}_U$. Then the fundamental classes of x in X and $\tilde{X}$ are the images of

$$[\{x\}]_\ell \in H^{2d}_{\{x\}}\big(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big) \cong H^{2d}_{\{x\}}\big(U_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big) \cong H^{2d}_{\{x\}}\big(\tilde{X}_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big)$$

in $H^{2d}\big(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big)$ and $H^{2d}\big(\tilde{X}_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big)$. This implies

$$\pi^*\big([x]_X\big) = [x]_{\tilde{X}}$$

and in view of the way in which the trace isomorphism is introduced in Theorem 7 we have a commutative diagram

$$\begin{array}{ccc} H^{2d}\big(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big) & & \\ {\scriptstyle \pi^*}\downarrow & \searrow^{\mathrm{Tr}_{X/\mathfrak{k}}} & \\ H^{2d}\big(\tilde{X}_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big) & \xrightarrow[\mathrm{Tr}_{\tilde{X}/\mathfrak{k}}]{} & \mathbb{Z}/\ell\mathbb{Z} \end{array}$$

hence

$$\langle h, \eta\rangle_X = \langle \pi^* h, \pi^*\eta\rangle_{\tilde{X}}$$

for the Poincaré duality pairings between $h \in H^p(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$ and $\eta \in H^p\big(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big)$. If $\pi^* h = 0$ this implies that h is in the kernel of $\langle\cdot,\cdot\rangle_X$ hence vanishes by Theorem 7. ☐

REMARK 4. Let X and $\tilde{X}$ be connected, of dimension d, and smooth over the algebraically closed field $\mathfrak{k}$. Let $\tilde{X} \xrightarrow{p} X$ be a finite morphism of degree δ which is not totally inseparable. Then there is a closed point $x \in X$ over which p is étale. It follows from the compatibility with pull-back in Theorem 6 that $p^*([x])$ is the sum of the fundamental classes of the preimages of x. As these have trace one and $H_c^{2d}\big(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big)$ is generated by $[x]$ as a $\mathbb{Z}/\ell\mathbb{Z}$-module, it follows that

(3) $$\mathrm{Tr}_{\tilde{X}/\mathfrak{k}}\Big(p^*\eta) = \delta \mathrm{Tr}_{X/\mathfrak{k}}(\eta)$$

for $\eta \in H_c^{2d}\big(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}(d)\big)$.

2.6. ℓ-adic cohomology. Let now ℓ be a prime number. We will always assume this to be invertible on X. As was explained in the previous term, cohomology groups like $H^*(X_{\text{ét}}, \mathbb{Z})$ or $H^*(X_{\text{ét}}, \mathbb{Z}_\ell)$ are well-defined but pathologicial in the sense that they fail to have the properties one would expect by comparison with classical cohomology. It is thus necessary to define ℓ-adic cohomology by using étale cohomology with finite coefficients.

A quite natural definition for ℓ-adic cohomology with constant coefficients is

$$H^p(X, \mathbb{Z}_\ell) = \lim_k H^p(X_{\text{ét}}, \mathbb{Z}/\ell^k\mathbb{Z}).$$

Similarly,

$$H^p\big(X, \mathbb{Z}_\ell(1)\big) = \lim_k H^p(X_{\text{ét}}, \mu_{\ell^k})$$

where in the latter case the transition is made using

$$\begin{aligned} \mu_{\ell^{k+1}} &\to \mu_{\ell^k} \\ \zeta &\to \zeta^l, \end{aligned}$$

and other Tate twists $\mathbb{Z}_\ell(m)$ of constant coefficients are obtained using the m-th tensor power of these transition morphisms. Cohomology with compact support is introduced in the same way, and cohomology with $\mathbb{Q}_\ell$-coefficients is defined by taking $\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$. One can verify that the fundamental classes $[Z]_{\ell^m}$ from (.2.5.2) assemble to an ℓ-adic fundamental class

(1) $$[Z]_{\mathbb{Q}_\ell} \in H_c^{2e}\big(X, \mathbb{Z}_\ell(e)\big)$$

Since the projective systems occuring in the above construction are made of finite abelian groups, they have the Mittag-Leffler property, and one gets long exact sequences of the type

(2) $$\dots H^{p-1}(Y, \mathbb{Z}_\ell) \to H_c^p(U, \mathbb{Z}_\ell) \to H^p(X, \mathbb{Z}_\ell) \to H^p(Y, \mathbb{Z}_\ell) \to H_c^{p+1}(U, \mathbb{Z}_\ell) \to \dots$$

(X proper, $U = X \setminus Y$) by taking the inverse limit of the similar sequences with finite cyclic coefficients. Very importantly, the projective systems $(A_n)_{n=1}^\infty$ occuring satisfy the stronger Artin-Rees Mittag-Leffler property (ARML): There is a natural number k such that the images of $A_n \to A_m$ and $A_{m+k} \to A_m$ coincide when $n \geq m + k$. For cohomology with compact support this is a special case of [**FK88**, Proposition 12.15]. The proof given there still works for ordinary cohomology because the crucial finiteness result is still available for ordinary direct images when the base is the spectrum of a separably closed field. One can use this to show that the cohomology groups considered above are finitely generated $\mathbb{Z}_\ell$-modules.

The ARML property has the consequence that up to torsion Poincaré duality survives taking the limit. To explain this let $\langle\cdot,\cdot\rangle_n$ denote the Poincaré duality pairing between $A_n = H^p(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ and $B_n = H_c^{2d-p}(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}(d))$, let A and B denote the limits, and let $C_n \xrightarrow{\pi_m^n} C_m$ denote the transition homomorphisms, where C may be A or B. Obviously one gets a pairing $A \times B \to \mathbb{Z}_\ell$ between the limits, but this is not necessarily non-degenerate. Let $a = (a_n)_{n=1}^\infty \in A$ be in the kernel of the pairing. Then $\langle a_m, b\rangle = 0$ whenever

$$b \in \bigcap_{n=m}^{\infty} \pi_m^n(B_m) = \pi_m^{m+k}(B_{m+k}),$$

where k is from the ARML property. Thus, for $b \in B_{m+k}$ we have $\left\langle a_m, \pi_m^{m+k} b\right\rangle_m = 0$. But the last expression is the image in $\mathbb{Z}/\ell^m\mathbb{Z}$ of $\langle a_{m+k}, b\rangle_{m+k}$, hence $\ell^k \langle a_{m+k}, b\rangle_{m+k} = 0$ in $\mathbb{Z}/\ell^{m+k}\mathbb{Z}$ for all $b \in B_{m+k}$, or $\ell^k a = 0$ in A. The same argument works with the roles of A and B interchanged. It follows that we have a Poincaré type duality

$$H^p(X, \mathbb{Q}_\ell(m)) \cong H_c^{2d-p}\big(X, \mathbb{Q}_\ell(d-m)\big)^* \tag{3}$$

between finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

We also record the following consequence of Corollary .2.5.4.

Corollary 1. Let $\tilde{X} \xrightarrow{\pi} X$ be a birational map between proper smooth connected schemes of finite type over the algebraically closed field $\mathfrak{k}$ in which the natural number ℓ is invertible. Then

$$H^p(X, \mathbb{Z}_\ell) \xrightarrow{\pi^*} H^p(\tilde{X}_{\text{ét}}, \mathbb{Z}_\ell)$$

is injective.

In the following we will also use the Künneth decomposition

$$H_c^n(X \times Y, \mathbb{Q}_\ell) \cong \oplus_{n=a+b} H_c^a(X, \mathbb{Q}_\ell) \otimes H_c^b(Y, \mathbb{Q}_\ell). \tag{4}$$

A proof is, for instance, in [**Mil80**, VI.8]. For separably closed $\mathfrak{k}$, we get $H_c^2(\mathbb{A}^1_{\mathfrak{k}}, \mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell$ (canonically) by Poincaré duality or the calculation of the cohomology of curves, while all other $H_c^p(\mathbb{A}^1_{\mathfrak{k}}, \mathbb{Q}_\ell(1))$ vanish by the calculation of the cohomology of curves given in the previous term. By the Künneth formula, $H_c^p(\mathbb{A}^d_{\mathfrak{k}}, \mathbb{Q}_\ell(d)) \cong \mathbb{Q}_\ell$ (canonically) when $p = 2d$ while the same cohomology group vanishes when $p \neq 2d$. Using (2) one obtains, canonically

$$H^p(\mathbb{P}^n_{\mathfrak{k}}, \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-p/2) & p \text{ even and } 0 \le p \le 2n \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

canoncially. Moreover, restriction to $\mathbb{P}^{n-1}_{\mathfrak{k}}$ is an isomorphism in cohomological degrees $< 2n$.

If one is only interested in showing the rationality of the congruence zeta function, then the previous remarks about ℓ-adic cohomology with coefficients that are constant up to Tate twists are sufficient. For the more complicated considerations leading to a complete proof of the Weil conjectures, it is necessary to develop the machinery of ℓ-adic sheaves.

The obvious idea is to consider projective systems $(\mathcal{F}_n)_{n=1}^{\infty}$ of constructible sheaves satisfying

$$\ell^n \mathcal{F}_n = 0. \tag{6}$$

with the following canonical morphism derived from this and the universal property of Coker being an isomorphism:

$$\mathrm{Coker}\big(\mathcal{F}_n \xrightarrow{\ell^{n+1}} \mathcal{F}_n\big) \xrightarrow{\cong} \mathcal{F}_n. \tag{7}$$

By an ℓ-adic sheaf I will understand a system with these properties.

Unfortunately (7) is not preserved under important operations like the taking of kernels or direct images. One must relax it to an ARML-like condition, the existence of a natural number m such that for all $n > m$ and $k > 0$, the images of $\mathcal{F}_{k+m} \to \mathcal{F}_k$ and $\mathcal{F}_{k+n} \to \mathcal{F}_k$ coincide, where now arbitrary systems of constructible sheaves of ℓ-power torsion are considered. Let $\mathcal{F}[k]_n = \mathcal{F}_{k+n}$. Homomorphisms in the category of ARML systems are defined as

$$\mathrm{Hom}_\ell(\mathcal{F}, \mathcal{G}) = \mathrm{colimHom}(\mathcal{F}[k], \mathcal{G}),$$

which coincides with homomorphisms of projective systems when both $\mathcal{F}$ and $\mathcal{G}$ are ℓ-adic sheaves. An ARML ℓ-adic sheaf is a projective system of ℓ-power torsion constructible sheaves which in the category of ARML systems is isomorphic to an ordinary ℓ-adic sheaves.

It turns out [**FK88**, I.12] that the ARML condition is preserved under higher direct images with compact support, and that many of the previous results about étale cohomology with torsion coefficients imply similar results about ℓ-adic sheaves. For instance,

$$R^n f_!(\mathcal{F}_n) = (R^n f_! \mathcal{F})_n$$

where this may only be ARML ℓ-adic even if the original system $(\mathcal{F}_n)$ is an ℓ-adic sheaf. Of course, for results depending on smooth base change ℓ must be invertible on the schemes under consideration. In the subsection on the Weil conjectures, we will use this tacitly, applying results about torsion sheaves previously shown or quoted from other sources to ℓ-adic sheaves as well rather than producing a long list of results in this subsection. Finally, the category of $\mathbb{Q}_\ell$-sheaves is obtained from the category of ℓ-adic sheaves by formally inverting ℓ in the homomorphism groups. An ℓ-adic sheaf $\mathcal{F}$ is called locally constant if all $\mathcal{F}_n$ are locally constant constructible. A $\mathbb{Q}_\ell$ sheaf is called locally constant if it can be defined from a locally constant ℓ-adic sheaf.

2.7. Lefschetz type trace formulas. If one is willing to believe a number of rather plausible facts about ℓ-adic cohomology, it is easy to explain why a basic Lefschetz-type fixed point formula holds for this cohomology. Let $X \xrightarrow{\phi} X$ be an endomorphism of a proper smooth connected d-dimensional scheme of finite type over a separably closed field $\mathfrak{k}$. Let $\big(b_i^{(p)}\big)_{i=1}^{\beta_p}$ be a base of $H^p(X, \mathbb{Q}_\ell)$ and $\big(b_i^{*(p)}\big)_{i=1}^{\beta_p}$ be the Poincaré dual base of $H^p\big(X, \mathbb{Q}_\ell(d)\big)$. One has a Künneth type formula

$$H^{2d}\big(X \times X, \mathbb{Q}_\ell(d)\big) \cong \oplus_{p=0}^{2d} H^p(X, \mathbb{Q}_\ell) \otimes H^{2d-p}\big(X, \mathbb{Q}_\ell(d)\big)$$

and it is natural to assume that the ℓ-adic fundamental class $[\Gamma_\phi]_{\mathbb{Q}_\ell}$ (cf. (.2.6.1)) of the graph Γ_ϕ of ϕ is given by

$$(\sharp) \qquad \sum_{p=0}^{2d} \sum_{i=1}^{\beta_p} \phi^*(\beta_i^{(p)}) \otimes \beta_i^{*(2d-p)}$$

under this decomposition, cf. [**Mil80**, Lemma VI.12.2]. In particular, taking $\phi = \mathrm{Id}_X$ one has

$$[\Delta_X]_{\mathbb{Q}_\ell} = \sum_{p=0}^{2d} \sum_{i=1}^{\beta_p} \beta_i^{(p)} \otimes \beta_i^{*(2d-p)}$$

Exchanging the roles of β and β^*, in other words, applying this with $\gamma_i^{(p)} = g^{-d}\beta_i^{*(p)}$ after fixing a generator g of $\mathbb{Q}_\ell(1)$, one has $\gamma_i^{*(p)} = (-1)^p g^d \beta_i^{(p)}$ because of the symmetry property Remark .2.5.3 of the Poincaré duality pairing. Applied to this base, the last formula for the fundamental class of the diagonal is thus

$$(\flat) \qquad [\Delta_X]_{\mathbb{Q}_\ell} = \sum_{p=0}^{2d} (-1)^p \sum_{i=1}^{\beta_p} \beta_i^{*(p)} \otimes \beta_i^{2d-p}.$$

If one believes that the Poincaré duality pairing $\langle \cdot, \cdot \rangle_{X \times X}$ evaluated on fundamental classes calculates the intersection number of algebraic cycles when the intersection is transversal, the duality pairing of ($\flat$) with ($\sharp$) should give the intersection product $\Gamma_\phi \cdot \Delta_X$ or the number of fixed points of ϕ, provided the intersection is transversal. It is easy to carry out this calculation:

$$\langle [\Gamma_\phi]_{\mathbb{Q}_\ell}, [\Delta_X]_{\mathbb{Q}_\ell} \rangle_{X \times X} = \sum_{p=0}^{2d} (-1)^p \sum_{i=1}^{\beta p} \left\langle \phi^*(\beta_i^{(p)}), \beta_i^{*(2d-p)} \right\rangle_X = \sum_{p=0}^{2d} (-1)^p \mathrm{Tr}\big(\phi^* \mid H^p(X, \mathbb{Q}_\ell)\big).$$

Thus, assuming that one is willing to believe the above assumptions, one has (cf. [**Mil80**, Theorem VI.12.3]):

THEOREM 8. *Let X be a smooth connected proper scheme over an algebraically closed field $\mathfrak{k}$ of dimension d, and let ϕ be an endomorphism of X such that the intersection $\Gamma_\phi \cap \Delta_X$ is transversal in $X \times X$. Then the number of fixed points of ϕ is*

$$(1) \qquad \sum_{p=0}^{2d} (-1)^p \mathrm{Tr}\big(\phi^* \mid H^p(X, \mathbb{Q}_\ell)\big).$$

REMARK 1. The condition that the intersection of Γ_ϕ with Δ_X be transversal translates into the condition that for no fixed point $x \in X(\mathfrak{k})$ of ϕ, 1 is an eigenvalue of the endomorphism ϕ^* of the tangent space of X at x.

While this relatively straightforward Lefschetz-type formula is sufficient to show the rationality of the congruence zeta function it is unfortunately insufficient for Deligne's proof the Weil conjecture. Instead of formulating this here we will briefly quote the relevant result when we need its application (.3.4.4).

2.8. Rationality of the zeta function and formulation of the main result. Let q be a power of the prime p, and let X be scheme such that $p = 0$ in $\mathcal{O}_X$. Then one has an endomorphism $\mathfrak{F}_q = \mathfrak{F}_q^{(X)}$ of X, called the absolute Frobenius, given by $\mathfrak{F}_q(x) = x$ on points and $\mathfrak{F}_q^*(\lambda) = \lambda^q$ on sections of the structure sheaf. It is easy to see that this is canonical in the sense that it is a functor-endomorphism of the identity functor on the category of schemes of characteristic p, and that

$$\mathfrak{F}_{q^k} = \mathfrak{F}_q^k. \tag{1}$$

If $X \xrightarrow{\xi} \operatorname{Spec}\mathfrak{k}$ is a scheme over a perfect field of characteristic p and $X^{(q)}$ denotes the fibre product $\operatorname{Spec}\mathfrak{k} \times_{\operatorname{Spec}\mathfrak{k}} X$ taken with respect to $\operatorname{Spec}\mathfrak{k} \xrightarrow{\mathfrak{F}_q} \operatorname{Spec}\mathfrak{k}$, the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\mathfrak{F}_q} & X \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ \operatorname{Spec}\mathfrak{k} & \xrightarrow{\mathfrak{F}_q} & \operatorname{Spec}\mathfrak{k} \end{array}$$

shows that $\operatorname{Spec}\mathfrak{k} \xleftarrow{\xi} X \xrightarrow{\mathfrak{F}_q} X$ define a morphism $X \xrightarrow{\mathfrak{F}_{q,X/\mathfrak{k}}} X^{(q)}$ by the universal property of the fibre product. This morphism $\mathfrak{F}_{q,X/\mathfrak{k}}$ is called the relative Frobenius. From (1) and the functoriality of the absolute Frobenius one derives that $\mathfrak{F}_{q^k,X/\mathfrak{k}}$ coincides with the composition

$$X \xrightarrow{\mathfrak{F}_{q,X/\mathfrak{k}}} X^{(q)} \xrightarrow{\mathfrak{F}_{q,X^{(q)}/\mathfrak{k}}} X^{(q)(q)} \cong X^{(q^2)} \to \dots \xrightarrow{\mathfrak{F}_{q,X^{(q^{k-1})}}} X^{(q^{k-1})(q)} \cong X^{(q^k)}. \tag{2}$$

If $X \subseteq \mathbb{P}^n$, $X^{(q)}$ is the algebraic variety given by the same equations as X but with all coefficients raised to the q-th power, and $\mathfrak{F}_{q,X/\mathfrak{k}}$ is the morphism of projective varieties over $\mathfrak{k}$ given by raising all homogenuous coordinates to their q-th power.

If $X = X_o \times_{\operatorname{Spec}\mathfrak{k}_o} \operatorname{Spec}\mathfrak{k}$, where $\mathfrak{k}_o$ is a field with q elements and $\mathfrak{k}$ an algebraic closure of $\mathfrak{k}_o$, one has an isomorphism $X^{(q)} \cong X$ which is canonical (functorial in X_o) and the relative Frobenius becomes an endomorphism $\mathfrak{F}_{X_o/\mathfrak{k}_o}$ of X. If $\mathfrak{l} \subseteq \mathfrak{k}$ is the unique degree k extension of $\mathfrak{k}_o$ and $X_\mathfrak{l}$ the base-change of X_o to $\mathfrak{l}$ it follows from our remark about (2) that

$$\mathfrak{F}_{X_\mathfrak{l}/\mathfrak{l}} = \mathfrak{F}_{X_o/\mathfrak{k}_o}^k. \tag{3}$$

The number of $\mathfrak{k}_o$-valued points is equal to the number of fixed points of this endomorphism. Since this is a purely inseparable endomorphism it acts as zero on the tangent spaces and by Remark .2.7.1 it is possible to apply Theorem 8:

$$\#\big(X_o(\mathfrak{k}_o)\big) = \sum_{n=0}^{2d} (-1)^n \operatorname{Tr}\big(\mathfrak{F}_{X_o/\mathfrak{k}_o}^* \mid H^n(X, \mathbb{Q}_\ell)\big). \tag{4}$$

The Weil congruence zeta function of X_o is

$$Z_{X_o}(z) = \exp\Big(\sum_{m=1}^{\infty} \frac{z^m}{m} \#\big(X_o(\mathfrak{k}_m)\big)\Big), \tag{5}$$

where $\mathfrak{k}_n \subseteq \mathfrak{k}$ is the unique subfield of $\mathfrak{k}$ with q^n elements (thus, $\mathfrak{k}_o = \mathfrak{k}_1$). This can be viewed as a formal power series, which is however easily seen to converge when $|z|$ is small enough. The relation with the Riemann zeta function $\zeta(s)$ becomes closes when one puts $z = q^{-s}$, which unlike the number field case turns out to be a rational function of q^{-s}, which thus has a period $\frac{2\pi i}{\log q}$ in the imaginary direction. By the well known power series expansion of $\log(1-z)$, one also has an Euler product representation of $Z_{X_o}(z)$

$$Z_{X_o}(z) = \prod_{x \in X \text{ closed}} \big(1 - z^{[\mathfrak{k}(x):\mathfrak{k}_o]}\big)^{-1} \tag{6}$$

converging in the formal power series ring $\mathbb{Z}[\![z]\!]$.

Applying (4) with $\mathfrak{k}_o$ replaced by $\mathfrak{k}_n$ and using (3), (5) becomes

$$\begin{aligned} Z_{X_o}(z) &= \exp\Big(\sum_{m=1}^{\infty} \frac{1}{n} \sum_{n=0}^{2d} (-1)^n \mathrm{Tr}\big(\mathfrak{F}^{*m}_{X_o/\mathfrak{k}_o} \mid H^n(X, \mathbb{Q}_\ell)\big)\Big) \\ &= \prod_{n=0}^{2d} \exp\Big((-1)^n \sum_{m=1}^{\infty} \mathrm{Tr}\big(\mathfrak{F}^{*m}_{X_o/\mathfrak{k}_o} \mid H^n(X, \mathbb{Q}_\ell)\big)\Big) \\ &= \prod_{n=0}^{2d} \det\big(\mathbf{1} - z\mathfrak{F}^{*}_{X_o/\mathfrak{k}_o} \mid H^n(X, \mathbb{Q}_\ell)\big)^{(-1)^{n+1}} \end{aligned}$$

by the well-known formal power series formula

$$\exp\Big(\sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Tr}\big(A^m \mid V\big)\Big) = \det\big(\mathbf{1} - zA \mid V\big)$$

for endomorphisms A of a finite-dimensional vector space V over an arbitrary field. Thus,

Theorem 9. *Let X be a d-dimensional proper smooth geometrically connected scheme of finite type over a field $\mathfrak{k}_o$ with q elements, then the Weil zeta function $Z_{X_o}(z)$ is a rational function of z which moreover for primes ℓ invertible in $\mathfrak{k}_o$ is given in $\mathbb{Q}_l(z)$ by the formula*

$$Z_{X_o}(z) = \prod_{n=0}^{2d} \det\big(\mathbf{1} - z\mathfrak{F}^{*}_{X_o/\mathfrak{k}_o} \mid H^n(X, \mathbb{Q}_\ell)\big)^{(-1)^{n+1}} \tag{7}$$

Here one uses the easy fact that a formal power series (5) with rational coefficients defines a rational function over $\mathbb{Q}$ if and only if it does so over any extionsion field like $\mathbb{Q}_\ell$. The rationality was proved by Dwork about a decade before étale cohomology was mature enough to obtain this result.

Example 1. If $X_o = \mathbb{P}^n_{\mathfrak{k}_o}$ then an easy calculation shows

$$Z_{\mathbb{P}^n_{\mathfrak{k}_o}}(z) = \prod_{k=0}^{n} \frac{1}{1-q^k z}. \tag{8}$$

We would like to identify the k-th factor with the characteristic polynomial of the Frobenius on $H^{2k}(\mathbb{P}^n_{\mathfrak{k}}, \mathbb{Q}_\ell)$. If $n = 0$ this is trivial. In general, we may use induction on n. The induction assumption together with the remark on restriction of cohomology classes after (.2.6.5) shows that for $k < n$ $\mathfrak{F}_{\mathbb{P}^n_{\mathfrak{k}_o}/\mathfrak{k}_o}$ acts on $H^{2k}_c(\mathbb{P}^n_{\mathfrak{k}}, \mathbb{Q}_\ell)$ by multiplication by q^k. This leaves only the factor belonging to $k = n$ in (8) for the contribution of $n = 2d$ to (7) applied to $X_o = \mathbb{P}^n_{\mathfrak{k}_o}$, proving our claim.

Remark 1. We would like to show that in general the Frobenius eigenvalue on $H^{2d}_c(X, \mathbb{Q}_\ell)$ is q^d, when $X_o/\mathfrak{k}_o$ is smooth of dimension d and geometrically connected. If will be convenient to also consider the case where X is not proper over $\mathfrak{k}$, hence cohomology with compact support is used. If $U \subseteq X$ is open and dense, continuation by zero defines an isomorphism $H^{2d}_c(U, \mathbb{Q}_\ell) \cong H^{2d}_c(X, \mathbb{Q}_\ell)$ by our material on Poincaré duality. Thus the assertion for X and for U is equivalent. In particular, it holds for $\mathbb{A}^d_{\mathfrak{k}_o}$ because the previous example shows its validity for $\mathbb{P}^d_{\mathfrak{k}_o}$, and to prove it in general it suffices to treat the affine case. In this case, there is a finite morphism $X_o \xrightarrow{p} \mathbb{A}^d_{\mathfrak{k}_o}$ by Noether normalization. This factors as $X_o \xrightarrow{q} Y_o \xrightarrow{r} \mathbb{A}^d_{\mathfrak{k}_o}$ where q is purely inseparable while r is separable at the generic point. There is a dense open subset $U_{\mathfrak{k}_o} \subseteq \mathbb{A}^d_{\mathfrak{k}_o}$ over which r is étal, and $r^{-1}U_{\mathfrak{k}_o} \to \mathfrak{k}_o$ is still smooth, and the assertion holds for $U_o/\mathfrak{k}_o$ by our previous remarks. By (.2.5.3) and Poincaré duality,

$$H^{2d}_c(U_{\mathfrak{k}}, \mathbb{Q}_\ell) \xrightarrow{r}{}^{*} H^{2d}_c(r^{-1}U, \mathbb{Q}_\ell)$$

is an isomorphism, and the assertion holds for $V_o = r^{-1}U_o/\mathfrak{k}_o$. But

$$H^{2d}_c(V_{\mathfrak{k}}, \mathbb{Q}_\ell) \xrightarrow{q}{}^{*} H^{2d}_c(q^{-1}V, \mathbb{Q}_\ell)$$

is an isomorphism as the purely inseparable q behaves as a homeomorphism for the étale topology. Thus, our assertion holds for the open dense subset $q^{-1}V_o$ of X_o, hence for X_o as well.

Note that the fact that Z_{X_o} is a rational function with coefficients in $\mathbb{Q}$ does not imply that the individual factors in the product (7) have coefficients in $\mathbb{Q}$ because there may be cancellations between linear factors of type $z - \lambda$ with $\lambda \in \overline{\mathbb{Q}_\ell}$ transcendental over $\mathbb{Q}$.

Moreover, the degrees $\dim H^n(X, \mathbb{Q}_\ell)$ of these polynomials, which are the ℓ-adic Betti numbers, might in principle depend on ℓ for varieties without smooth models over Spec$\mathbb{Z}$. In the case where such smooth models exist, the comparison theorem mentioned earlier (Corollary .2.3.1) shows that the n-th polynomial in (7) has degree equal to the n-th classical Betti number of the variety of complex points, as conjectured by Weil. However, this still leaves the problem of whether or not this factor has rational coefficients and is independent of ℓ remaining open, even in this case. Moreover, Weil had a conjecture about the zeroes of these polynomials generalizing the classical Riemann hypothesis for the congruence zeta function. This form of the Riemann hypothesis is easily seen to imply all the other assertions which remained open so far, as we will explain at the end of this subsection.

Deligne was able to show all this:

THEOREM 10. *Let X be a d-dimensional proper smooth geometrically connected scheme of finite type over a field $\mathfrak{k}_o$ with q elements, then the factor*

$$\det\bigl(\mathbf{1} - z\mathfrak{F}^*_{X_o/\mathfrak{k}_o} \bigm| H^n(X,\mathbb{Q}_\ell)\bigr)$$

in (7) is a polynomial with coefficients in $\mathbb{Z}$ which is independent of ℓ. The complex zeroes of this polynomial have absolute value $q^{-n/2}$.

COROLLARY 1. Under the same assumption, the ℓ-adic Betti numbers

$$\dim_{\mathbb{Q}_\ell} H^n(X,\mathbb{Q}_\ell)$$

with ℓ invertible in $\mathfrak{k}$ are independent of ℓ.

Of course one considers the Frobenius eigenvalues as elements of $\overline{\mathbb{Q}_\ell}$, showing that they are in fact integral elements of $\overline{\mathbb{Q}}$ and that all their complex conjugates have the desired absolute values. With this constraint on the zeroes of the polynomials, the representation of a rational function as (7) becomes unique and all other assertions easily follow, as will be explained in the proof of Proposition 1. Thus, one considers the following property:

(9)

$W(X_o/\mathfrak{k}_o,k,\delta,\varepsilon)$:	All eigenvalues $\lambda \in \overline{\mathbb{Q}_\ell}$ of the endomorphism $\mathfrak{F}_{X_o/\mathfrak{k}_o}$ on $H^k(X,\mathbb{Q}_\ell)$ are algebraic over $\mathbb{Q}$, and $q^{k/2-\delta} \le \iota(\lambda) \le q^{k/2+\varepsilon}$ for all field embeddings $\mathbb{Q}(\lambda) \xrightarrow{\iota} \mathbb{C}$.

Here we allow for $\delta = \infty$, in which case only an upper bound on $|\iota(\lambda)|$ is asserted. The summands ε and δ have been introduced because the main step of the proof only yields estimates containing them. The reduction of Theorem 10 to weaker assertions of this form is so easy that it is best to present them here.

PROPOSITION 1. We assume the following: For every prime number ℓ there exist a positive integer D and a real number $\varepsilon \in (0,\infty)$ such that for every finite field $\mathfrak{k}_o$ of characteristic different from ℓ, every smooth projective geometrically connected $\mathfrak{k}_o$-scheme X_o of dimension d divisible by D there exists a positive integer N (possibly depending on $X_o/\mathfrak{k}_o$) such that for all positive integers n divisible by N, $W(X_n/\mathfrak{k}_n,d,\infty,\varepsilon)$ holds. Under this assumption, both Theorem 10 and its corollary hold. Moreover, for fixed n and X the conditions $W(X_n/\mathfrak{k}_n,d,\infty,\varepsilon)$ and $W(X_n/\mathfrak{k}_n,d,\varepsilon,\varepsilon)$ are actually equivalent.

It is actually possible to apply this with $D = 2$ (thus, the initial estimate is only shown for manifolds of even dimension) and $\varepsilon = 1/2$. The number N depends on $X_o/\mathfrak{k}_o$ in a way which is difficult to describe. It occurs because one wants certain morphisms (Lefschetz pencils) and their (isolated) singularities to be defined over the finite ground field one is working with. One could even allow for N to depend on ℓ, but this is not going to happen in the proof.

Proof. We proceed in several steps, starting with the following strengthening of the initial assumption:

For every prime number ℓ there exist a positive integer D and a real number $\varepsilon \in (0,\infty)$ such that for every finite field $\mathfrak{k}_o$ of characteristic different from ℓ, every smooth projective geometrically connected $\mathfrak{k}_o$-scheme X_o of dimension d divisible by D, $W(X_o/\mathfrak{k}_o, d, \infty, \varepsilon)$ holds.

Indeed, let λ be an eigenvalue of $\mathfrak{F}_{X_o/\mathfrak{k}_o}$ on $H^d(X,\mathbb{Q}_\ell)$. By (3), λ^N is an eigenvalue of $\mathfrak{F}_{X_N/\mathfrak{k}_N}$ on $H^d(X,\mathbb{Q}_\ell)$. It is thus algebraic over $\mathbb{Q}$ and $\left|\iota\lambda^N\right| \le q^{N(d/2+\varepsilon)}$, and the assertion about λ follows.

For every prime number ℓ and every finite field $\mathfrak{k}_o$ of characteristic different from ℓ and every smooth projective geometrically connected $\mathfrak{k}_o$-scheme the condition X_o, $W(X_o/\mathfrak{k}_o, d, 0, 0)$ holds, where $d = \dim(X)$.

Indeed, let λ be an eigenvalue of $\mathfrak{F}_{X_o/\mathfrak{k}_o}$ on $H^d(X,\mathbb{Q}_\ell)$. Let n be a positive integer and let X^n be the n-th Cartesian power of X. By the Künneth decomposition

$$H^{nd}(X^n,\mathbb{Q}_\ell) = \bigoplus_{nd=d_1+\ldots+d_n} \bigotimes_{i=1}^{n} H^{d_i}(X,\mathbb{Q}_\ell),$$

λ^n is an eigenvalue of $\mathfrak{F}_{X_o/\mathfrak{k}_o}$ on $H^{nd}(X,\mathbb{Q}_\ell)$. If n is chosen such that D divides nd, it follows that λ^n is algebraic and $|\iota\lambda^n| \le q^{nd/2+\varepsilon}$. Thus, λ is algebraic and $|\lambda| \le q^{d/2+\varepsilon/n}$. Since n can be made arbitrarily large, the upper bound follows, and the lower bound can be derived by Poincaré duality and Remark 1.

For every prime number ℓ and every finite field $\mathfrak{k}_o$ of characteristic different from ℓ and every smooth projective geometrically connected $\mathfrak{k}_o$-scheme X_o, the condition $W(X_o/\mathfrak{k}_o, p, 0, 0)$ holds for all natural numbers p.

We use induction on the dimension d of X, the case $d = 0$ being trivial. Let $d > 0$ and the assertion be known for manifolds of dimension $< d$. We first deal with the case $p < d$. By Bertini's theorem, there are a positive integer N and a smooth hyperplane section Y_N of X_N. By Corollary .2.5.3, the restriction morphism

$$H^p(X,\mathbb{Q}_\ell) \to H^p(Y,\mathbb{Q}_\ell)$$

is injective. Because Y is defined over $\mathfrak{k}_N$, this morphism is compatible with $\mathfrak{F}_{X_n/\mathfrak{k}_n}$ when N divides n. Thus, every Frobenius eigenvalue of $\mathfrak{F}_{X_n/\mathfrak{k}_n}$ on $H^p(X,\mathbb{Q}_\ell)$ is also an eigenvalue on $H^p(Y,\mathbb{Q}_\ell)$, and by the induction assumption we get $W(X_n/\mathfrak{k}_n, p, 0, 0)$ when N divides n. From this $W(X_o/\mathfrak{k}_o, p, 0, 0)$ can be derived in the same way as in the first reduction step.

Finally, the case $p = d$ follows from our previous intermediate result while the case $p > d$ follows by Poincaré duality and Remark 1.

If ℓ is invertible in $\mathfrak{k}_o$, the polynomial

$$P_{\ell,k} \det\Big(\mathrm{Id} - z\mathfrak{F}^p_{X_o/\mathfrak{k}_o} \;\Big|\; H^k(X,\mathbb{Q}_\ell)\Big)$$

has rational coefficients which are independent of ℓ.

For $k \neq l$ the polynomials $P_{\ell,k}$ and $P_{\ell,l}$ are coprime in $\mathbb{Q}_\ell[T]$ as the properties of their zeroes shown in the previous step are mutually exclusive. The products P_ℓ (resp. Q_ℓ) of $P_{\ell,k}$ over odd (resp. even) k are thus also coprime. As $Z = Z_{X_o/\mathfrak{k}_o}$ satisfies $Z(1) = 1$ we have, for every field K of characteristic 0, a unique representation

$$Z(z) = \frac{P_K(z)}{Q_K(z)}$$

where $P, Q \in K[T]$ are coprime and $P(0) = Q(0) = 1$, and because of the uniqueness P_K (resp. Q_K) is in fact the image of $P = P_\mathbb{Q}$ (resp. $Q_\mathbb{Q}$) in $K[T]$. Also by the uniqueness of the representation,

$$P_\ell = P_{\mathbb{Q}_\ell} = P$$

and

$$Q_\ell = Q_{\mathbb{Q}_\ell} = Q$$

Thus, P_ℓ and Q_ℓ have rational coefficients which are independent of ℓ. It follows from the property of the zeroes shown previously that when $\lambda \in \overline{\mathbb{Q}}$ is a zero of PQ and $\overline{\mathbb{Q}} \xrightarrow{\iota} \mathbb{C}$ a complex embedding that $|\iota(\lambda)| = q^{-k/2}$ with $k = k_\lambda$ independent of ι. Thus, if R is an irreducible factor of PQ in $\mathbb{Q}[T]$ there is a unique natural number k_R such that $k_\lambda = k_R$ for all zeroes λ of R in $\overline{\mathbb{Q}}$. It also follows that

$$P_{\ell,k} = \prod_{R_k = k} R.$$

The product is over all irreducible factors of PQ in $\mathbb{Q}[T]$ subject to the indicated condition and with the same multiplicity with which R occurs in PQ. As this is a rational polynomial independent of ℓ, the claim follows.

The coefficients of $P_{\ell,k}$ are integers.

Because $P_{\ell,k}(0) = 1$, this is equivalent to showing that λ^{-1} is an integer in $\overline{\mathbb{Q}}$, for every zero λ of P or Q. If λ violates this condition, there are a prime number p and an embedding $\overline{\mathbb{Q}} \xrightarrow{\iota} \overline{\mathbb{Q}_p}$ such that $|\iota(\lambda)|_p < 1$ holds for the unique extension of the p-adic absolute value to $\overline{\mathbb{Q}_p}$. If λ is a zero of Q then

$$P(\lambda) = Z(\lambda)Q(\lambda) = 0$$

as the power series $Z(\lambda)$ converges by (6). This is a contradiction to our previous observation that P and Q are coprime. As (6) also shows that Z^{-1} is a formal power series with integer coefficients, the possibility of such a λ being a zero of P can likewise be excluded.

The derivation of the Weil conjectures from the assumption of the Proposition is now complete. The assertion about the equivalence of $W(X_n/\mathfrak{k}_n, d, \infty, \varepsilon)$ and $W(X_n/\mathfrak{k}_n, d, \varepsilon, \varepsilon)$ also follows Poincaré duality as in the second step. □

3. Overview over the proof

By Proposition .2.8.1, the proof of the Weil conjectures is reduced to the proof of a weaker assertion about Frobenius eigenvalues on $H^d(X, \mathbb{Q}_\ell)$, where $d = \dim(X)$ may be assumed to be even. The proof of this proceeds by induction on the even number d and considers morphisms $\tilde{X}_o \xrightarrow{f} S_o = \mathbb{P}^1_{\mathfrak{k}_o}$ with the property that f is smooth over $S \setminus F$, where F is finite. Moreover, the fibres over the elements of F have precisely one singularity of a specific simple type. Here $\tilde{X}_o$ is a suitable blow-up of X_o. By Corollary .2.6.1 it is possible to replace X_o by this blow-up. Let X and $\tilde{X}$ denote the base-changes from Spec$\mathfrak{k}_o$ to Spec$\mathfrak{k}$. We will consider the Frobenius eigenvalues on the terms $E_2^{0,d}$, $E_2^{1,d-1}$ and $E_2^{2,d-2}$ of the spectral sequence

$$E_2^{p,q} = H^p(S, R^q f_* \mathbb{Q}_{\ell,\tilde{X}}) \Rightarrow H^{p+q}(\tilde{X}, \mathbb{Q}_\ell).$$

If f was smooth, Theorem 4 would imply that all $R^q f_* \mathbb{Q}_{\ell,\tilde{X}}$ are locally constant. In our case, the assumption about the singularities of f implies that this still holds with the exception of the cases $p = d - 1$ and $p = d$, in which cases Proposition .2.3.1 fails in a way which is relatively easy to describe. This is done in the subsection on Picard-Lefschetz formulas where the situation near a single $f \in F$ is investigated in terms of vanishing cycles.

The "global" situation where all $f \in F$ are considered is investigated in another subsection. From these results it will then be easy to give a proof of the Weil conjectures.

3.1. Existence of Lefschetz pencils.

Definition 1. Let X be a prescheme of finite type over a field $\mathfrak{k}$ and $x \in X$ a closed point. If $\mathfrak{k}$ is algebraically closed, x is called an ordinary double point if the completion of the local ring admits a representation as

$$\widehat{\mathcal{O}_{X,x}} \cong \mathfrak{k}[\![X_0, \ldots, X_d]\!] \Big/ f \cdot \mathfrak{k}[\![X_0, \ldots, X_d]\!]$$

where f is a formal power series

$$f = \sum_{l=2}^{\infty} f_l$$

such that f_l is a homogenuous polynomial of degree l and the bilinear form

$$B(x, y) = f_2(x + y) - f_2(x) - f_2(y)$$

is non-degenerate. If $\mathfrak{k}$ is arbitrary and $\overline{\mathfrak{k}}$ an algebraic closure, the condition is instead that all preimages of x in the base-change $\overline{X}$ of X to $\overline{\mathfrak{k}}$ are ordinary double points of $\overline{X}$.

Let $\check{V}$ be the dual $\mathfrak{k}$-vector space to $V = \mathfrak{k}^{n+1}$ and $\check{\mathbb{P}}^n_{\mathbb{K}} = \mathbb{P}(\check{V})$ be the dual projective space of hyperplanes in $\mathbb{P}^n$.

Definition 2. Let X be a smooth projective connected variety over an algebraically closed field $\mathfrak{k}$. A closed embedding $X \xrightarrow{i} \mathbb{P}^n_{\mathbb{K}}$ is called a Lefschetz embedding if there is a closed subset $A \subseteq \check{\mathbb{P}}^n_{\mathfrak{k}}$ of codimension > 1 such that no $H \in U = \check{\mathbb{P}}^n \setminus A$ contains an irreducible component of X and such that for $H \in U$, the intersection scheme $H \cap X$ has at most one singular point, which is an ordinary double point.

It can be shown that every X satisfying the assumptions of the definition has a Lefschetz embedding. In fact, if $X \xrightarrow{i_o} \mathbb{P}^n$ is any closed embedding then its composition with the closed embedding $\mathbb{P}^n \to \mathbb{P}^N$ defined by the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ is a Lefschetz embedding for sufficiently large d. In particular, if X comes from a projective scheme X_o over a finite field $\mathfrak{k}_o$ of which $\mathbb{K}$ is an algebraic closure, then on can chose a Lefschetz embedding defined over $\mathfrak{k}_o$.

By a line in $\check{\mathbb{P}}^n_{\mathbb{K}}$ we understand the image D of a morphism $\mathbb{P}(W) \to \mathbb{P}(\check{V})$ where $\dim W = 2$. The lines D are in canonical bijection with the linear projective subspaces $\check{D} \subseteq \mathbb{P}^n$ where $\check{D}$ is the image of $\mathbb{P}(W^\perp) \to \mathbb{P}(V) = \mathbb{P}^n_{\mathfrak{k}}$. If $x \notin \check{D}$ then there is a unique hyperplane $H \in D$ containing x.

For our algebraically closed $\mathfrak{k}$, the generic line D does not intersect the codimension ≥ 2 subset A from Definition 2. Also, the generic $\check{D}$ has transversal intersection with X. We assume that D has been chosen in this way. As the generic hyperplane has transversal intersection with X we can also chose D such that it contains at least one such hyperplane H. We call such D a Lefschetz pencil.

Let $\tilde{X}(\mathfrak{k})$ be the set of pairs (x, H) with $x \in X$, $H \in D$ and $x \in H$. This is the set of closed points of a unique reduced closed subscheme $\tilde{X} \subseteq X \times D$. Let $X \xrightarrow{f} D$ denote the projection to the second factor. The fibres of this projection are the intersections of X with the hyperplanes in D. As the generic element of D intersects X transversally, all but finitely many fibres are regular. As D does not intersect A, the exceptional fibres have but one singularity which is an ordinary double point. It is easy to see that $\tilde{X}$ is the blow-up of the regular codimension two subscheme $X \cap \check{D}$ of X. In particular, $\tilde{X}$ is regular and we have a birational morphism $\tilde{X} \to X$.

If X comes from X_o over a finite field $\mathfrak{k}_o$, our argument for the selection of D may fail over $\mathfrak{k}_o$, but it is still possible to chose a suitable D which is defined over a finite extension field $\mathfrak{k}_n$ of $\mathfrak{k}_o$. The morphism $X \xrightarrow{f} D$ is then defined over $\mathfrak{k}_n$.

The local behaviour of f at a closed point $x \in X$ where it fails to be smooth can be described as follows:

Definition 3. Let $X \xrightarrow{f} S$ be a morphism of finite type where S is one-dimensional and regular. We assume that f is smooth over an open dense subset of S. Let $x \in X$ be a closed point of the geometric fibre of f at a geometric point σ of S, and let A be the strict Henselization $\mathcal{O}_{S_{\text{ét}},\sigma}$. We say that f has an ordinary double point at x if the geometric point of $X \times_S \operatorname{Spec} A$ defined by σ and X has an étale neighbourhood of the form

$$A[X_0, \ldots, X_d] \,/\, (Q + u)A[X_0, \ldots, X_d]$$

where u is an element of the maximal ideal $\mathfrak{m}$ of A and Q is a homogenuous polynomial of degree two such that $Q_o = Q \bmod \mathfrak{m}$ is non-degenerate as in Definition 1, the bilinear form $Q_o(x+y) - Q_o(x) - Q_o(y)$ being non-degenerate.

In particular, the condition implies that f is flat at x.

The considerations in [**FK88**, III.2], in particular Proposition III.2.8, show that all singularities of a morphism $\tilde{X} \xrightarrow{f} D$ defined by a Lefschetz pencil are ordinary double points.

3.2. The Picard-Lefschetz formula. Similar to Proposition .2.3.1 we consider a situation of a proper morphism $X \xrightarrow{f} S$ where X is regular and S the spectrum of a strictly Henselian discrete valuation ring A with field of quotients K. These assumptions, which are a bit more restrictive than for Proposition .2.3.1, are made for the sake of convenience. We relax the condition of f being smooth to the condition that there is a closed point x of the fibre X_s of f over the closed point $s \in S$ such that f is smooth outside x. Then the second part of Proposition .2.3.1 still applies to $X \setminus \{x\} \to S$. Thus, if k_X is as in Proposition .2.3.1, $R^p k_{X*} k_X^* \mathcal{F}$ for $p > 0$ is a skyscraper sheaf supported at x and given by an abelian group Φ^p, and $\mathcal{F} \xrightarrow{\kappa} k_{X*} k_X^* \mathcal{F}$ is an isomorphism outside x. Since X is regular of positive dimension this implies that κ is a monomorphism whose cokernel is again a skyscraper sheaf at x given by an abelian group Φ^0.

Because of this, all terms $E_2^{p,q}$ with $pq > 0$ in the Leray spectral sequence (.2.3.4) vanish. The only non-vanishing differentials are

$$\Phi^{r-1} = E_2^{0,r-1} = E_r^{0,r-1} \to E_r^{r,0} = E_2^{r,0} = H^r(X_{\text{ét}}, k_{X*} k_X^* \mathcal{F})$$

with $r > 1$. Taking appropriate care of the slightly different situation in cohomological degree 0, the assertion of Proposition .2.3.1 about the cospecialisations being isomorphism gets replaced by a long exact sequence involving the groups Φ^p of vanishing cycles:

$$(1) \quad 0 \to H^0\Big((X_s)_{\text{ét}}, \mathcal{F}\Big) \to H^0\Big((X_{\overline{\eta}})_{\text{ét}}, k_X^* \mathcal{F}\Big) \to \Phi^0 \to H^1\Big((X_s)_{\text{ét}}, \mathcal{F}\Big) \to \dots$$
$$\to \Phi^{p-1} \to H^p\Big((X_s)_{\text{ét}}, \mathcal{F}\Big) \to H^p\Big((X_{\overline{\eta}})_{\text{ét}}, k_X^* \mathcal{F}\Big) \to \Phi^p \to \dots$$

We are only interested in applying this to cohomology with coefficients in $\mathbb{Q}_\ell$. Applying the direct limit to (1) is possible without problems as the abelian groups involved are finite and a Mittag-Leffler condition is thus satisfied. In the following, we will thus apply (1) with $\mathfrak{F} = \mathbb{Q}_{\ell,X}$. The Φ^p are then finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

It is necessary to calculate them in the case where the singularity of f at x is an ordinary double point. Because the definition of Φ^p local with respect to $X_{\text{ét}}$, it is possible to replace X by an appropriate quadric with the correct degeneracy. If the dimension d of X over S is 1, one has a $\mathbb{P}^1$ over $\overline{\eta}$ degenerating to X_s, a union of two copies of $\mathbb{P}^1_{\mathfrak{k}(s)}$ joined along ∞. Applying (.2.6.2) with $X = X_s$, $Y = \{\infty\}$ gives

$$\dim H^p(X_s, \mathbb{Q}_\ell) = \begin{cases} 1 & p = 0 \\ 2 & p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this with (1) shows that, at least when $d = 1$, $\Phi^p = 0$ when $p \neq d$ while Φ^d is one-dimensional. It turns out that this is the case for arbitrary d. The considerations for showing this in general are relatively straightforward but lengthy, and would probably not fit into a lecture of four hours per week for one-term in which the proof of the Weil conjecture is also given and the material from our previous section formulated. For instance, in [**FK88**] the relevant chapter III is about three times as long as the sections devoted to the actual proof of the Weil conjecture, which is relatively short but not at all straightforward. Even so, they formulate some of the relevant facts for the complex-analytic situation without proof.

We thus only quote the final result, and we do so only in the case of odd d, which is the only case relevant for our purposes. This case is [**FK88**, Theorem III.4.3]. In [**Del74**, (4.3)] the general case is recalled from SGA7-II. The case of even d requires odd characteristic and has a slightly different description of the monodromy action. In our case, this involves the homomorpism

$$\mathrm{Gal}(\overline{K}/K) \xrightarrow{\chi_{\mathbb{Z}/n\mathbb{Z}}} \mu_n$$

$$\chi_{\mathbb{Z}/n\mathbb{Z}}(\sigma) = \frac{\sigma\big(\sqrt[n]{u}\big)}{\sqrt[n]{u}} \tag{2}$$

where $u \in \mathfrak{m}_A$ is as in Definition .3.1.3 and the same root must be used in the enumerator and denominator. This is well-defined because $\mu_n \subseteq A$ when n is invertible in A. Passing to the limit over $n = \ell^k$ also gives

$$\mathrm{Gal}(\overline{K}/K) \xrightarrow{\chi_{\mathbb{Q}_\ell}} \mathbb{Q}_\ell(1)$$

THEOREM 11. *Let S be the spectrum of a strictly Henselian discrete valuation ring A. Let $\Lambda = \mathbb{Z}/r\mathbb{Z}$ or $\Lambda = \mathbb{Q}_\ell$ with r (resp. ℓ) invertible in A. Let $X \xrightarrow{f} S$ be a proper morphism which is smooth of odd relative dimension d outside a closed point x of the closed fibre, where f has an ordinary double point. Let $\langle\cdot,\cdot\rangle$ denote the canonical alternating Poincaré duality pairing on $H^d(X_{\overline{\eta}}, \Lambda)$ with values in $\Lambda(-d)$. The sheaves $R^p f_* \Lambda_X$ are then locally constant on S unless $p = d$ or $p = d+1$. In these cases, their stalks at the closed and generic geometric points are linked by an exact sequence*

$$0 \to H^d(X_s, \Lambda) \xrightarrow{c} H^d(X_{\overline{\eta}}, \Lambda) \xrightarrow{\alpha} \Lambda\big(-\frac{d+1}{2}\big) \xrightarrow{\beta} H^{d+1}(X_s, \Lambda) \xrightarrow{c} H^{d+1}(X_{\overline{\eta}}, \Lambda) \to 0 \tag{3}$$

involving the cospecializations c and with

$$\alpha(\eta) = \langle \eta, \delta\rangle \qquad \text{with } \delta \in H^d\Big(X_{\overline{\eta}}, \Lambda\big(\frac{d-1}{2}\big)\Big)$$

$$\beta(t) = t \cdot \delta^* \qquad \text{with } \delta^* \in H^p\Big(X_s, \Lambda\big(\frac{d+1}{2}\big)\Big).$$

The action of $\Gamma = \mathrm{Gal}\big(\overline{K} \ / \ K\big)$ on $H^{d+1}(X_{\overline{\eta}}, \Lambda)$ is trivial, and the choice of δ can be made such that the action of $\sigma \in \mathrm{Gal}(\overline{K}/K)$ on $H^d(X_{\overline{\eta}}, \Lambda)$ is given by

$$\sigma(x) = x + (-1)^{\frac{d+1}{2}} \chi_\Lambda(\sigma) \cdot \langle x, \delta\rangle \cdot \delta, \tag{4}$$

with χ_Λ defined before the formulation of the theorem.

REMARK 1. For the investigation of Lefschetz pencils the theorem will be applied with A the strict Henselezation of $\mathcal{O}_{\mathbb{P}^1_{\mathfrak{k}}, f}$. In this application, the Galois group Γ in the Theorem is called the local monodromy group at f.

Remark 2. • It is clear from the exact sequence that precisely one of δ and δ^* vanishes. Our conditions characterize the *vanishing cycle* δ only up to sign. In the case where it does not vanish the dual vanishing cycle δ^* is only characterized uniquely up to multiplication by a unit in Λ. There is a refinement [**FK88**, Proposition III.4.8] of this which however is not needed for our purposes.

- Note that χ_Λ depends on the singularity of f at x, namely $\chi_\Lambda = \vartheta^e$ where e is the valuation exponent of u and ϑ is defined like χ, replacing u by a uniformizer $\pi \in A$. Note that this formula for χ writes its target as a multiplicative group while in (4) it is written as an additive group. The exponent e can be thought of as some order of degeneracy of f near x.

3.3. Global Lefschetz theory. We now apply this result to the case of a morphism $\tilde{X} \xrightarrow{f} \mathbb{P}^1$ defined by a Lefschetz pencil, again limiting considerations to the case of odd fibre dimension d. While Theorem 11 only makes assumptions about the singularities of f most results in this subsection are limited to f obtained from a sufficiently generic Lefschetz pencil, as will become clear from the proof of Theorem 12.

Let $A \subseteq \mathbb{P}^1$ be the finite subset over which f fails to be smooth. Then $R^d f_* \mathbb{Q}_{\ell,X}$ is locally constant over $U = \mathbb{P}^1 \setminus A$, by Theorem 4. If $\mathfrak{k} = \mathbb{C}$ one may pass to a complex-analytic local system (ie., a locally constant sheaf of $\mathbb{Q}_\ell$-vector spaces for the ordinary topology). If $u \in U$ is a closed point this is given by a representation of $\pi_1(U, u)$ on the $\mathbb{Q}_\ell$ vector space $\mathcal{V} = H^d(X_u, \mathbb{Q}_\ell)$, the cohomology of the fibre at u. For $f \in A$ it is possible to construct an element γ_p of $\pi_1(U, u)$ from a path

$$I = [0,1] \xrightarrow{p} \mathbb{P}^1 \setminus (A \setminus \{f\})$$

from u to f as follows. Let U be a contractible neighourhood of f containing no other element of A. Let $\lambda \in I$ be such that $\lambda(t) \neq f$ $p(t) \in U$ for $t \in [\lambda, 1]$. Define γ_p by going from u to $p(\lambda)$ along p, then turning counterclockwise once around f in U, the going back from $p(\lambda)$ to $u = p(0)$ along p. This is easily seen to be independent of the choice of λ, then of the choice of U. It depends on p only by conjugation with an element of $\pi_1(U, u)$. One often supresses this dependence and writes $\gamma_f = \gamma_p$, but then only conjugacy class of γ_f is well-defined. Note that regardless of the choices for p made, the γ_f generate the free group $\pi_1(U, u)$.

We would like to generalize this to arbitrary ground fields. Unfortunately there is a problem in the case where $\mathfrak{k}$ is of characteristic $p > 0$, which is the case we are interested in. In this case, the étale fundamental group $\pi_1^{\text{ét}}(U, u)$ is more complicated than expected from the complex analytic situation. For instance, if $U = \mathbb{A}^1_{\mathfrak{k}}$ it will fail to vanish because we have the finite étale Artin-Schreier morphism

$$\mathbb{A}^1 \to \mathbb{A}^1 \qquad\qquad t \to t^p - t$$

with wild ramification at ∞. Fortunately Theorem 11 implies the tameness of the ramification of $R^d f_* \mathbb{Q}_{\ell,X}$ at the elements of A. Recall that $\pi_1^{\text{ét}}(U, u)$ is the automorphism group of the fibre functor

$$\left(V \xrightarrow{v} U\right) \to v^{-1}(u)$$

on the category of finite étale morphisms $V \to U$. Replacing this by the full subcategory of all $V \to U$ for which the ramification at the elements of A is at most tame we get a quotient of $\pi_1^{\text{ét}}(U, u)$ which for the purposes of this exposition will be denoted $\pi_1^t(U, u)$.

If A is a strictly Henselian discrete valuation ring and $\Omega = \text{Spec}A \setminus \{\mathfrak{m}_A\}$ and $\overline{\eta}_A$ the spectrum of a separable closure of the field of quotients of A we have a similar quotient $\pi_1^t(\Omega, \overline{\eta}_A)$ of $\pi_1^{\text{ét}}(\Omega, \overline{\eta}_A)$ by the subgroup of all elements acting trival on étale covers of Ω with tame ramification at $\mathfrak{m}_A$. Applying (.3.2.2) with u a uniformizing element of A gives us an isomorphism

$$\pi_1^t(\Omega, \overline{\eta}_A) \to \prod_{\ell \neq c} \mathbb{Z}_\ell(1), \tag{1}$$

where c is the characteristic of the residue field. This isomorphism is independent of the choice of uniformizer. We apply this to the strict Henselization $A = \mathcal{O}_{\mathbb{P}^1_{\text{ét}}, u}$ at u. The inverse of (1) composed with the morphism on fundamental groups defined by $\Omega \to U$ defines a morphism

$$\prod_{\ell \neq c} \mathbb{Z}_\ell(1) \to \pi_1^t(U, \overline{\eta}_A),$$

where now c is the characteristic of the ground field. Conjugating with a morphism from $\overline{\eta}_A$ to u in the étale fundamental groupoid $\Pi_1^{\text{ét}}(U)$ we get a homomorphism

$$\prod_{\ell \neq c} \mathbb{Z}_\ell(1) \xrightarrow{\gamma_f} \pi_1^t(U, u)$$

which is unique up to conjugacy by elements of $\pi_1^t(U, u)$. Again it can be shown that, regardless of the choices made, the images of the $\gamma_{f,\ell}$ generate $\pi_1^t(U, u)$ topologically.

If A is as above, Theorem 11 describes the relation between the fibres of $R^d f_* \mathbb{Q}_\ell$ at f and η_A. Using the morphism from u to η_A in $\Pi_1^{\text{ét}}(U)$ chosen for the definition of γ_f, we may identify the fibres at η_A and u, defining a vanishing cycle $\delta_f \in H^d\big(X_u, \mathbb{Q}_\ell(\frac{d-1}{2})\big)$.

$$\gamma_f(\lambda)h = h \pm \lambda_\ell \langle h, \delta_f \rangle \cdot \delta_f \tag{2}$$

where λ_ℓ is the ℓ-component of λ. Let $E \subseteq V$ be the $\pi_1^{\text{ét}}(U, u)$-invariant subspace generated by the $\mathbb{Q}_\ell(frac1-d2)$-multiples of all vanishing cycles δ_f.

THEOREM 12. *Let $X \to \mathbb{P}^N$ be a Lefschetz embedding. Recall that the set V of all lines $D \subseteq \check{\mathbb{P}}^N$ defining a Lefschetz pencil for f is an open dense subset of the Grassmanian of all lines in $\check{\mathbb{P}}^N$. We also assume the dimension of X to be even such that the fibre dimension d of f is odd. For D in an open dense subset W of V, the vanishing cycle γ_f is up to sign and up to conjugacy by an element of $\pi_1^t(U, u)$ independent of f. Moreover, the representation of $\pi_1^t(U, u)$ on $M = E/E \cap E^\perp$ is absolutely irreducible in the sense that it is irreducible and stays so after $\otimes_{\mathbb{Q}_\ell} L$, for any field extension L of $\mathbb{Q}_\ell$.*

PROOF. The proof of the first assertion will only be sketched, referring to [**FK88**, III.7] for details. Let $X \to \mathbb{P}^N$ and $D \subseteq \check{\mathbb{P}}^N$ be the Lefschetz embedding and pencil used to define f. Let Z be the "set" of (x, H) with $H \in \check{\mathbb{P}}^N$ and $x \in X \cap H$. We have a projection $Z \xrightarrow{g} \check{\mathbb{P}}^N$

to the second factor. We also have $\tilde{X} \subseteq Z$. In fact, $\tilde{X} \xrightarrow{f} D$ is derived from $Z \xrightarrow{g} \check{\mathbb{P}}^N$ by base-change with respect to the embedding $D \subseteq \check{\mathbb{P}}^n$. By proper base-change it follows that $R^d f_* \mathbb{Q}_{\ell,\tilde{X}}$ is isomorphic to the pull-back of $R^d f_* \mathbb{Q}_{\ell,Z}$ to D. Let $F \subseteq \check{\mathbb{P}}^n$ be the set of all hyperplanes touching X at least one point. This is easily seen to be irreducible, being the image of a certain projective fibration $\mathbb{P}(\mathcal{E})$ over X. Moreover, g is smooth over $\Omega = \check{\mathbb{P}}^N \setminus Z$, and $A = D \cap F$. It is shown in [**FK88**, Proposition A I.16] that

$$(+) \qquad \pi_1^t(U,u) \to \pi_1^t(\Omega,u)$$

is surjective.

If F has codimension > 1 we define W by excluding all lines intersecting it from V. In this case we are done because A is then empty. Otherwise, the subset $F' \subseteq F$ of all hyperplanes H touching X in more than one point or with an intersection which fails to be an ordinary double point is a proper subset because $\operatorname{codim}(F', \check{\mathbb{P}}^N) \geq 2$ by our definition of Lefschetz embeddings. Applying Theorem 11 to the strict localization of $\check{\mathbb{P}}^N$ at the generic point of F one obtains that the ramification of $R^d g_* \mathbb{Q}_{\ell,Z}$ at F is tame. It follows that the action of $\pi_1^t(U,u)$ on $\mathcal{V}$ factors over (+). As (+) is surjective it is sufficient to show that the different vanishing cycles are $\pi_1^t(\Omega,u)$. However, (+) maps all γ_f to a single conjugacy class of morphisms

$$\prod_{\ell \neq c} \mathbb{Z}_\ell(1) \xrightarrow{\gamma} \pi_1^t(\Omega,u)$$

which is essentially the analog of γ_f with U replaced by Ω and f by the generic point of F. Since δ_f is uniquely determined up to sign by (2), our claim about all $\{\pm\gamma_f\}$ being conjugate follows. We may actually take $W = V$ in this case.

For the other assertion, let $N \subseteq M \otimes_{\mathbb{Q}_\ell} L$, with be a non-vaninshing $\pi_1^t(U,u)$-invariant subspace. As $N \neq 0$ there are $m \in N$ and $f \in A$ such that $\langle m, \delta_f \rangle \neq 0$ for an appropriate choice of δ_f. By (2) and its $\pi_1^t(U,u)$-invariance, N contains the image of δ_f in M. By the previous assertion it also contains the images of all vanishing cycles in M, hence coincides with $M \otimes_{\mathbb{Q}_\ell} L$. ☐

Theorem 13 (Kazhdan-Margulis). *With $M = E/(E \cap E^\perp)$ as before and for Lefschetz pencils as in the previous theorem, the image of $\pi_1^t(U,u)$ in the symplectic group $\mathrm{Sp}(M)$ for the symplectic form defined by the Poincaré duality pairing is open with respect to the ℓ-adic topology.*

Proof. Again, the following is intended to be a sketch of proof. Let $\mathfrak{sp}(M)$ be the set of all endomorphisms A of M such that the dual A^* with respect to the symplectic form coincides with $-A$. This is a Lie algebra with Lie bracket $[A,B] = AB - BA$. For $t \in \mathbb{Q}_\ell$ sufficiently close to zero, the exponential series for $\exp(\varepsilon A)$ converges to an element of $\mathrm{Sp}(M)$. Let $\mathfrak{g} \subseteq \mathfrak{sp}(M)$ be the subset of all A such that $\exp(tA)$ is in the image of $\pi_1^t(U,u)$, for all sufficiently small $t \in \mathbb{Q}_\ell$. Obviously this is $\pi_1^t(U,u)$-invariant, hence also invariant under $\exp(tA)$ with $A \in \mathfrak{g}$ and t sufficiently small. Using the Campbell-Baker-Hausforff formula one can derive that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{sp}(M)$. Let $W \subseteq M$ be the subset of all m such that

$$g_m : x \to \langle x, m \rangle \, m$$

belongs to $\mathfrak{g}$. By (2), W contains all vanishing cycles. Moreover, W is closed and $\mathbb{Q}_\ell \cdot W \subseteq W$. We have

$$\langle m, n\rangle\, g_{m+n} = \langle m, n\rangle\, (g_m + g_n) + [g_m, g_n]$$

and it follows that for $m, n \in W$, $\langle m, n\rangle = 0$ or $m + n \in W$. Let $U \subseteq W$ is a largest subvector space of M contained in W. This does not vanish as W contains the vanishing cycles. If $w \in W$ is not orthogonal to U then $w + \omega \in W$ for $\omega \in U \setminus w^\perp$, and by the closedness of W it follows that $U + \mathbb{Q}_\ell \cdot w \subseteq W$. By the maximality of U it follows that $w \in U$. Thus,

$$W = U \cup U^\perp \cap W.$$

From (2) and because of $\gamma_f \in W$, it follows that U is invariant under the images of γ_f. Since these generate $\pi_1^t(U, u)$ topologically, the previous assertion shows $U = M$. But the g_m generate $\mathfrak{sp}(M)$ as a Lie algebra. Hence $\mathfrak{g} = \mathfrak{sp}(M)$, from which the assertion can be derived. □

3.4. An initial estimate. In this subsection we present the material from [**Del74**, 3. and 6.] or [**FK88**, IV.2 and IV.3]. Let $U = \mathbb{P}^1_{\mathfrak{k}} \setminus A$ be as before. We now assume that $\mathfrak{k}$ is the algebraic closure of a finite field $\mathfrak{k}_o$ with q elements over which U is defined. U is thus the preimage of an open $U_o \subseteq \mathbb{P}^1_{\mathfrak{k}_o}$. Let $\mathfrak{F}_o$ is a locally constant $\mathbb{Q}_\ell$-sheaf on U_o and $\mathcal{F}$ its preimage on U. For every closed point $x \in U$, let

$$\mathrm{Spec}\mathfrak{k} \xrightarrow{\xi} U$$

be a geometric point of U centered at x. We denote the image of ξ under $U \to U_o$ by the same letter ξ. We have a canonical isomorphism $\mathcal{F}_\xi \cong \mathcal{F}_{o,\xi}$. We assume that a choice of ξ has been made for each x and just abbreviate this to $\mathcal{F}_x$. On this space the Galois group $\mathrm{Gal}\big(\mathfrak{k}/\mathfrak{k}(x)\big)$ acts. In particular, we have an action of the Frobenius element $\mathfrak{F}_x$ (sending $\lambda \in \mathfrak{k}$ to $\lambda^{\#\mathfrak{k}(x)}$) on $\mathcal{F}_\xi$. This is a special case of the action of $\pi_1^{\text{ét}}(U_o, \xi)$ on $\mathcal{F}_{o,\xi}$, namely this action applied to the image of $\mathfrak{F}_x$ under $G = \pi_1^{\text{ét}}\big(\mathrm{Spec}\mathfrak{k}(x), \xi\big) \to \pi_1^{\text{ét}}(U_o, \xi)$. For the sake of simplicity, this image will be denote by the same notation $\mathfrak{F}_x \in \pi_1^{\text{ét}}(U_o, \xi)$.

Remark 1. The definition of the Galois group action on geometrical stalks or cohomology groups is by pull-backs. It thus involves a sign change in the sense that the pull-back is taken with respect to the action σ^{-1} rather than σ, as one would get an "anti-action" otherwise. Because of this, there is a similar sign change here in the sense the the "geometric Frobeniuses" we have encountered so far and which are defined in terms of pull-back along the Frobenius morphisms are in fact inverse to the "arithmetic Frobeniuses" defined as the action of the Frobenius element of the Galois group on stalks or cohomology classes. For instance, if ρ denotes the action of $\pi_1^{\text{ét}}(U_o, u)$ on $\mathcal{F}_u$ then the arithmetic Frobenius endomorphism $\mathfrak{F}_{X_o,\mathcal{F}_o}$ of $\mathcal{F}_u$ coming from the definition of X and $\mathcal{F}$ over $\mathfrak{k}_o$ is in fact $\rho(\mathfrak{F}_u^{-1})$ rather than $\rho(\mathfrak{F}_u)$. This must be taken into account when the action of $\pi_1^{\text{ét}}(U_o, u)$ on $\mathcal{F}_u$ is used to obtain information about geometric Frobeniuses in the following considerations.

We say that $\mathcal{F}_o$ is of weight β if all eigenvalues λ of $\mathfrak{F}_x$ on $\mathcal{F}_\xi$ are algebraic numbers satisfying

$$|\iota(\lambda)| = \#\mathfrak{k}(x)^{\beta/2}$$

for all complex embeddings ι of the algebraic number field $\mathbb{Q}(\lambda) \subseteq \overline{\mathbb{Q}_\ell}$.

For the sake of simplicity we also assume that U has a point u defined over $\mathfrak{k}_o$. The geometric point of U_o obtained as $\{u\} \to U \to U_o$ will also be denoted u. If x and ξ are as above then the choice of a morphism $\xi \xrightarrow{p} u$ in $\Pi_1^{\text{ét}}(U_o)$ defines an isomorphism

$$\mathcal{F}_x \cong \mathcal{F}_u \tag{1}$$

identifying the action of $\mathfrak{F}_x \in \pi_1^{\text{ét}}(U,\xi)$ on the former with that of

$$\phi_x = p\mathfrak{F}_x p^{-1} \in \pi_1^{\text{ét}}(U_o, u)$$

on the latter stalk. Of course ϕ_x depends on the choice of the "path" p but its conjugacy class in $\pi_1^{\text{ét}}(U_o, u)$ is unique.

We want to investigate the L-function

$$L_{U_o,\mathcal{F}_o}(t) = \prod_{x \in U_o \text{ closed}} L_{U_o,\mathcal{F}_o,x}(t), \tag{2}$$

where the local factor at $x \in U_o$ is

$$L_{U_o,\mathcal{F}_o,x} = \det\bigl(1 - t^{[\mathfrak{k}(x):\mathfrak{k}]}\mathfrak{F}_x \mid \mathcal{F}_x\bigr)^{-1}. \tag{3}$$

By a suitable generalization of (.2.8.7),

$$L_{U_o,\mathcal{F}_o}(t) = \frac{\det\bigl(1 - t\mathfrak{F}_{X_o,\mathcal{F}_o} \mid H_c^1(U,\mathcal{F})\bigr)}{\det\bigl(1 - t\mathfrak{F}_{X_o,\mathcal{F}_o} \mid H_c^0(U,\mathcal{F})\bigr)\det\bigl(1 - t\mathfrak{F}_{X_o,\mathcal{F}_o} \mid H_c^2(U,\mathcal{F})\bigr)}. \tag{4}$$

This can be derived from an appropriate version of Theorem 8 in the same way in which (.2.8.7) was deduced from Theorem 8. Appropriate versions of Theorem 8 are [**Mil80**, Theorem VI.13.3] or [**FK88**, Corollary II.3.15].

Proposition 1. Assume that $\mathcal{F}_o$ is equipped with a symplectic form

$$\mathcal{F}_o \otimes \mathcal{F}_o \xrightarrow{\psi} \mathbb{Q}_\ell(\beta) \tag{5}$$

for which that the following assumptions hold:

- If ξ and x are as above, the characteristic polynomial of $\mathfrak{F}_x$ on $\mathcal{F}_\xi$ has coefficients in $\mathbb{Q}$.
- The image of

$$G = \pi_1^{\text{ét}}(U,u) \to \mathrm{Sp}(\mathcal{F}_u) \tag{6}$$

  is an open subgroup.

Then $\mathcal{F}$ is of weight β.

Proof. Let $\mathcal{F}^{(2k)}$ be the $2k$-th tensor power of $\mathcal{F}$. Because of the formula $\mathrm{Tr}(A \otimes A) = \mathrm{Tr}(A)$, the formal power series coefficients of the local factor

$$L_{U_o,\mathcal{F}_o,x} = \det\bigl(1 - t^{[\mathfrak{k}(x):\mathfrak{k}]}\mathfrak{F}_x \mid \mathcal{F}_x^{(2k)}\bigr)^{-1} = \exp\Bigl(\sum_{k=1} \mathrm{Tr}\bigl(\mathfrak{F}_x^k \mid \mathcal{F}_x^{(2k)}\bigr) \tag{7}$$

are positive for each $x \in U_o$. The same holds for the coefficients of their product $L_{U_o,\mathcal{F}_o}(t)$ over all closed points $x \in U_o$. The following Lemma 1 applied to $\mathcal{F}^{(2k)}$ shows that this formal power series in $t \in \mathbb{C}$ actually converges when $|t| \leq q^{-k\beta-1}$. By the positivity of the power series coefficients it follows that the radius of convergence of each factor (7) is also $\geq q^{-k\beta-1}$. If λ is an eigenvalue of $\mathfrak{F}_x$ on $\mathcal{F}_x$ then λ^{2k} is an eigenvalue of $\mathfrak{F}_x$ on $\mathcal{F}_x^{(2k)}$ hence

$$\left|\iota(\lambda)^{2k}\right| \leq q^{k\beta+1} \tag{@}$$

as $\iota(\lambda)^{-1}$ is a pole of (7). Because (@) holds for all positive integers k we get $|\iota(\lambda)| \leq q^{\frac{\beta}{2}}$, and the opposite inequality may be derived using the symplectic form ψ. ☐

LEMMA 1. *Under the assumptions of the proposition on $\mathcal{F}$, the product*

$$L_{U_o,\mathcal{F}_o^{(k)}}(t) = \prod_{x\,\in\, U_o \text{ closed}} L_{U_o,\mathcal{F}_o^{(k)},x}(t),$$

with the factors given by (7), is a rational function of t, and all poles have $|t| \geq q^{-1-k\beta/2}$. Moreover, if $\lambda \in \overline{\mathbb{Q}_\ell}$ is an eigenvalue of $\mathfrak{F}_{X_o,\mathcal{F}_o}$ on $H_c^2(U,\mathcal{F}^{(k)})$ then λ is algebraic over $\mathbb{Q}$ and

$$|\iota(\lambda)| = q^{1+k\beta/2} \tag{8}$$

for all embeddings $\mathbb{Q}(\lambda) \xrightarrow{\iota} \mathbb{C}$.

PROOF. This is trivial when $\mathcal{F} = 0$. Thus, let $\mathcal{F} \neq 0$. Then the assumption about (6) is obviously violated for $U = \mathbb{P}^1$ as this is simply connected. Therefore, U is affine, and by (4) and the vanishing of $H_c^0(U,\mathcal{F})$, we have

$$L_{U_o,\mathcal{F}_o^{(k)}}(t) = \frac{\det\left(1 - t\mathfrak{F}_{X_o,\mathcal{F}_o^{(k)}} \mid H_c^1(U,\mathcal{F}^{(k)})\right)}{\det\left(1 - t\mathfrak{F}_{X_o,\mathcal{F}_o^{(k)}} \mid H_c^2(U,\mathcal{F}^{(k)})\right)} \tag{9}$$

and it is sufficient to show the second assertion.

Let $\mathcal{V} = \mathcal{F}_u$ be the stalk at the closed point $u \in U$ selected. By Poincaré duality, the H_c^2 occuring in the denominator is dual to $H^0(U,\mathcal{F}^{(k)*}) \cong \mathcal{V}^{\otimes 2k*G}$, where V^G denotes the space of G-invariants in V. Hence

$$H_c^2(U,\mathcal{F}^{(k)}) \cong \mathcal{V}_G^{\otimes 2k}(-1)$$

as the duality pairing is with values in $H_c^2(U,\mathbb{Q}_\ell(1))$. In this, V_G denotes the space of G-coinvariants of V, the quotient of V by the subspace generated by the $gv - v$ for $g \in G$, $v \in V$. It is thus sufficient to show the algebraicity of λ and

$$|\iota(\lambda)| = q^{k\beta} \tag{+}$$

for every eigenvalue λ of $\mathfrak{F}_u^{-1}$ on

$$\left(\mathcal{V}^{\otimes 2k}\right)_G \tag{@}$$

and every $\mathbb{Q}(\lambda) \xrightarrow{\iota} \mathbb{C}$. [1] It follows from our assumption about (6) that the action of the group $\mathrm{Gp} = \mathrm{Gp}(\mathcal{V},\psi)$ of symplectic similitudes of $\mathcal{V}$ on (@) is trivial on $\mathrm{Sp}(\mathcal{V},\psi)$. Let

$$\mathrm{Gp} \xrightarrow{\chi} \mathbb{Q}_\ell$$

---

[1]The reason for considering eigenvalues of $\mathfrak{F}_u^{-u}$ instead of $\mathfrak{F}_u$ is explained in Remark 1

be the character defined by $\psi(gv,gw)=\chi(g)\psi(v,w)$ for $v,w\in\mathcal{V}$ and $g\in G$. Our previous remark shows that the action of Gp on (@) factors over Gp $\xrightarrow{\chi}\mathbb{Q}_\ell^*$. Because this is commutative the eigenspace of λ in (@) contains a one-dimensional Gp-invariant subspace W defined over $\overline{\mathbb{Q}_\ell}$. On this $g\in$ Gp acts by $\theta(\chi(g))$ for a certain group homomorphism

$$\mathbb{Q}_\ell^* \xrightarrow{\theta} \overline{\mathbb{Q}_\ell}^*$$

Applying this with g given by multiplication by $t\in\mathbb{Q}_\ell^*$, $\chi(g)=t^2$ we find that $\theta(t^2)=t^k$. It follows that

$$\lambda^2=\theta\big(\chi(\mathfrak{F}_u^{-1})\big)^2=\theta(q^\beta)^2=q^{k\beta}$$

and the desired assertion (+) follows. □

Corollary 1. Under the assumptions of the Theorem let $\lambda\in\overline{\mathbb{Q}_\ell}$ be an eigenvalue of $\mathfrak{F}_{U_o,\mathcal{F}_o}$ on $H_c^1(U,\mathcal{F})$. Then λ is algebraic over $\mathbb{Q}$ and

$$|\iota(\lambda)|\le q^{1+\beta/2} \tag{10}$$

holds for all $\mathbb{Q}(\lambda)\xrightarrow{\iota}\mathbb{C}$.

Proof. If $\mathcal{F}=0$ this is trivial and otherwise U is affine. Thus, we may apply (9) with $k=1$. Let P and Q denote the polynomials in the enumerator and denominator. By the Lemma applied with $k=1$ the coefficients of Q are algebraic numbers and all zeroes satisfy

$$|\iota(\lambda)|=q^{1+\frac{\beta}{2}}. \tag{@}$$

Because the formal power series $L_{U_o,\mathcal{F}_o^{(k)}}(t)$ has rational coefficients by our assumption on $\mathcal{F}$, the coefficients of P are also algebraic over $\mathbb{Q}$. Thus, λ must be algebraic over $\mathbb{Q}$. Extending ι to a subfield containing all coefficients of P and Q in addition to λ we have

$$L_{U_o,\mathcal{F}_o^{(k)}}(z)\big(\iota(Q)\big)(z)=\big(\iota(P)\big)(z) \tag{+}$$

for all complex numbers z for which $L_{U_o,\mathcal{F}_o^{(k)}}(z)$ converges absolutely. It follows from the trivial bound $\#U_o(\mathfrak{l})=O(\#(\mathfrak{l}))$ for finite fields $\mathfrak{l}$ and the fact that $\mathcal{F}_o$ has weight β that the Euler product (2) converges when $|t|\le q^{-1-\beta/2}$. Hence (+) holds when $|z|<q^{-1-\beta/2}$, and moreover $L(z)\ne 0$ in this case. By (@) we also have $\big(\iota(Q)\big)(z)\ne 0$. Thus, $\big(\iota(Q)\big)(z)\ne 0$ when $z\in\mathbb{C}$ satisfies $|z|<q^{-1-\beta/2}$. Applying this to $z=\iota(\lambda)$ gives (10). □

The verification of the rationality assumption in Proposition 1 uses the following Proposition, which reduces this assumption to a considerably weaker assumption. In its formulation, the reason for the occurence of $\mathfrak{F}_x^{-1}$ rather than $\mathfrak{F}_x$ is again Remark 1. For a closed point $x\in U$ let $d(x)=[\mathfrak{k}(x):\mathfrak{k}_o]$. We have $d(x)=d(\mathfrak{F}_x)$ where d on the right hand side denotes the group homomorphism

$$\pi_1^{\text{ét}}(U_o,u)\to\pi_1^{\text{ét}}(\mathrm{Spec}\mathfrak{k}_o)\cong\hat{\mathbb{Z}}$$

to the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$, with kernel $\pi_1^{\text{ét}}(U,u)$.

PROPOSITION 2. Let ρ denote an action of $\pi_1^{\text{ét}}(U_o, u)$ on the finite-dimensional symplectic $\mathbb{Q}_\ell$-vector space (V, ψ) by symplectic similitudes, such that the image

$$\rho\bigl(\pi_1^{\text{ét}}(U, u)\bigr) \tag{11}$$

is contained in and an open subgroup of $\mathrm{Sp}(V, \psi)$. Assume that there are sequences $\bigl(\lambda_i\bigr)_{i=1}^m$ and $\bigl(\mu_j\bigr)_{j=1}^n$ of elements of $\overline{\mathbb{Q}_l}$ such that for each $x \in U$, the rational function

$$\frac{\prod_{i=1}^m (1 - \lambda_i^{-d(x)} t)}{\prod_{j=1}^n (1 - \mu_j^{-d(x)} t)} \det\bigl(1 - t\rho(\mathfrak{F}_x^{-1})\bigr) \tag{12}$$

belongs to the subfield $\mathbb{Q}(t) \subset \overline{\mathbb{Q}_\ell}(t)$. Then for every $x \in U$, the polynomial $\det\bigl(1 - t\rho(\mathfrak{F}_x^{-1})\bigr)$ has coefficients in $\mathbb{Q}$.

PROOF. By shortening the lists othwerise we may assume the sets $\bigl\{\lambda_i \bigm| 1 \le i \le m\bigr\}$ and $\bigl\{\mu_j \bigm| 1 \le j \le n\bigr\}$ to be disjoint. Let γ be an automorphism of $\overline{\mathbb{Q}_\ell}$ over $\mathbb{Q}$. By our rationality assumption

$$\begin{aligned} (+) \quad \det\bigl(1 - t\rho(\mathfrak{F}_x^{-1})\bigr) \prod_{i=1}^m (1 - \lambda_i^{-d(x)} t) \prod_{j=1}^n (1 - \gamma(\mu)_j^{-d(x)} t) = \\ = \gamma\Bigl(\det\bigl(1 - t\rho(\mathfrak{F}_x^{-1})\bigr)\Bigr) \prod_{i=1}^m (1 - \gamma(\lambda_i)^{-d(x)} t) \prod_{j=1}^n (1 - \mu_j^{-d(x)} t). \end{aligned}$$

Let H be the set ℓ-adic units $\eta\overline{\mathbb{Q}_\ell}$ occuring among the λ_i or μ_j. In the Lemma 2 following this proof we derive from our opennes assumption about (11) that, for each ℓ-adic unit $\eta \in \overline{\mathbb{Q}_\ell}$, the closed set F_η of all $\sigma \in \pi_1^{\text{ét}}(U_o, u)$ for which $\eta^{-d(\sigma)}$ is an eigenvalue has no internal point. The union F of the F_η with η ranging over the finite set $H \cup \gamma(H) \cup \gamma^{-1}(H)$ likewise has no internal point. Let E be the set of integers $e > 1$ for which one of the λ_i/μ_j, $\gamma(\lambda_i)/\lambda_j$ or $\gamma(\mu_i)/\mu_j$ is a primtive $\sqrt[e]{1}$. As E is finite, $G = \bigcup_{e \in E} e\hat{\mathbb{Z}}$ is a proper close subset of $\hat{\mathbb{Z}}$. Its preimage under the surjection d is a proper closed subset of $\pi_1^{\text{ét}}(U_o, u)$. As F has no internal points,

$$\Omega = \pi_1^{\text{ét}}(U_o, u) \setminus \Bigl(d^{-1}(G) \cup F\Bigr)$$

is a non-empty open subset. By the Chebotarev density theorem for number fields there is $x \in U$ with $\mathfrak{F}_x^{-1} \in \Omega$. If this x is used in (+) then the only way for the equation to hold is by

$$\det\bigl(1 - t\rho(\mathfrak{F}_x^{-1})\bigr) = \gamma\Bigl(\det\bigl(1 - t\rho(\mathfrak{F}_x^{-1})\bigr)\Bigr)$$

$$(\flat) \qquad \prod_{i=1}^m (1 - \lambda_i^{-1} t) = \prod_{i=1}^m (1 - \gamma(\lambda_i)^{-1} t)$$

$$(\sharp) \qquad \prod_{j=1}^n (1 - \gamma(\mu)_j^{-1} t) = \prod_{j=1}^n (1 - \mu_j^{-1} t)$$

because there is no other way to make a bijection between the zeroes of the polynomials on both sides. For instance, if λ is an eigenvalue of $\rho(\mathfrak{F}_x)$, our choice of F has made sure that it is not among the $\mu_j^{d(x)}$ or the $\gamma(\lambda_i)^{d(x)}$. From $\flat$ and $\sharp$ it follows that the coefficients of the polynomials $\prod_{i=1}^m(1-\lambda_i^{-1}t)$ and $\prod_{j=1}^n(1-\mu_j^{-1}t)$ are γ-invariant. As this holds for all $\gamma \in \mathrm{Aut}\big(\overline{\mathbb{Q}_\ell} \;/\; \mathbb{Q}\big)$, these polynomials have rational coefficients. But then the remaining factor in (12) must have rational coefficients as well. $\square$

LEMMA 2. *Under the assumptions of the proposition on ρ, let $\eta \in \overline{\mathbb{Q}_\ell}$ be an ℓ-adic unit. Then the closed set of $\sigma \in \pi_1^{\acute{e}t}(U_o, u)$ for which $\eta^{d(\sigma)}$ is an eigenvalue of $\rho(\sigma)$ has no internal points.*

PROOF. Let $\lambda = \eta^{d(\sigma)}$. If σ is an internal point of the above set, then the set

$$\big\{\theta \in \pi_1^{\text{ét}}(U, u) \bigm| \lambda \text{ is an eigenvalue of } \rho(\theta)\rho(\sigma)\big\}$$

is neighbourhood of the neutral element in $\pi_1^{\text{ét}}(U, u)$. By our assumption about (11) this implies that

$$\big\{g \in \mathrm{Sp}(V, \psi) \bigm| \lambda \text{ is an eigenvalue of } g\rho(\sigma)\big\}$$

is neighbourhood of the neutral element in $\mathrm{Sp}(V, \psi)$. It is easy to confirm that this cannot be the case. $\square$

3.5. Proof of the Weil conjectures. As we announced after Proposition .2.8.1 we are going to prove $W(X_n/\mathfrak{k}_n, d, \infty, 1/2)$ for X of even dimension d and n divisible by a suitable positive integer N depending on $X_o/\mathfrak{k}_o$. Because we are free to modify N it is actually sufficient to show this with $n = 1$ under an additional assumption that $\mathfrak{k}_o$ is as large enough to satisfy certain assumptions depending on $X_o/\mathfrak{k}_o$. We will use induction on the even number d, the case $d = 0$ being trivial. Let $d > 0$ and the assertion be shown when $\dim(X)$ is an even number $< d$. If $\mathfrak{k}_o$ is large enough we have a Lefschetz pencil

$$\tilde{X}_o \xrightarrow{f_o} U_o$$

where $U_o = \mathbb{P}^1_{\mathfrak{k}_o} \setminus A$ to which Theorem 12 and Theorem 13 apply, such that the elements of A are defined over $\mathfrak{k}_o$ and such that U still contains a point u defined over $\mathfrak{k}_o$. In this we often indicate base change to $\mathfrak{k}$ from $\mathfrak{k}_o$ by dropping the subscript $_o$. As was explained before it is possible to replace X by its blow-up $\tilde{X}$ in the proof of $W(X_o/\mathfrak{k}_o, d, \infty, 1/2)$. We will always use q to denote the cardinality of $\mathfrak{k}_o$. The embedding $U \to \mathbb{P}^1$ will be denoted by j.

We consider the Frobenius eigenvalues on the E_2-term of

$$E_2^{k,l} = H^k(\mathbb{P}^1, R^l f_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\tilde{X}, \mathbb{Q}_\ell)$$

with $k + l = d$. The only cases where this holds and $E_2^{p,q}$ may be $\neq 0$ are the following.

3.5.1. *The case $k = 2$, $l = d - 2$.* In this case, $\mathcal{F} = R^q f_* \mathbb{Q}_\ell$ is locally constant on $\mathbb{P}^1$, hence constant as $\mathbb{P}^1$ is simply connected. By the calculation of the cohomology of $\mathbb{P}^1_{\bar{\mathfrak{k}}}$ it follows that

$$H^2(\mathbb{P}^1_{\mathfrak{k}}, \mathcal{F}) \cong \mathbb{Q}_\ell(1) \otimes \mathcal{F}_u \cong \mathbb{Q}_\ell(1) \otimes H^{d-2}(\tilde{X}_u, \mathbb{Q}_\ell)$$

where the last isomorphisms holds by proper base change. The Frobenius eigenvalues on $E_2^{2,d-2}$ are thus $q\lambda$ where λ is a Frobenius eigenvalue on $H^{d-2}(\tilde{X}_u, \mathbb{Q}_\ell)$. Let Y be a smooth hyperplane section of $\tilde{X}_u$. By Corollary .2.5.3, λ is also a Frobenius eigenvalue on $H^{d-2}(Y, \mathbb{Q}_\ell)$. As the induction assumption applies to Y, and by the last assertion of Proposition .2.8.1, we have $W(Y_o/\mathfrak{k}_o, d-2, \frac{1}{2}, \frac{1}{2})$ provided that $\mathfrak{k}_o$ is large enough. Then λ is algebraic and all $\iota(\lambda)$ satisfy

$$q^{\frac{d-3}{2}} \leq |\iota(\lambda)| \leq q^{\frac{d-1}{2}}. \tag{1}$$

Thus, $q\lambda$ is algebraic and $|\iota(\lambda)| \leq q^{\frac{d+1}{2}}$ as desired. The lower bound in (1) is not needed here but has been derived for later use.

3.5.2. *The case $k = 0$, $l = d$.* When applying (.3.2.3) at a point $f \in A$ as explained in Remark .3.2.1, the Γ-invariants in last term in the sequence describe the stalk of $j_* j^* R^d f_* \mathbb{Q}_\ell$ at f. By Theorem 11, Γ actually acts trivially on that term, hence the last term is actually the stalk of $\mathcal{F} = j_* j^* R^d f_* \mathbb{Q}_\ell$ at f. The triviality of the Γ-action also shows that $\mathcal{F}$ is locally constant at f. It is thus locally constant, hence constant on all of $\mathbb{P}^1$. Moreover, in (.3.2.3) the term before the last one is the stalk at f of $R^d f_* \mathbb{Q}_\ell$. Before this there comes the stalk of a skyscraper sheaf. Using these considerations, (.3.2.3) gives us an exact sequence

$$0 \to \mathcal{K} \to R^d f_* \mathbb{Q}_{\ell, \tilde{X}} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a constant sheaf of $\mathbb{Q}_\ell$-vector spaces on $\mathbb{P}^1$ with $\mathcal{F}_u \cong H^d(\tilde{X}_u, \mathbb{Q}_\ell)$ and where $\mathcal{K}$ is a sum of skyscraper sheaves $\mathbb{Q}_\ell(\frac{d}{2})_f$ over $f \in A$ when $E = 0$ (in which case all vanishing cycles are 0) and $\mathcal{K} = 0$ otherwise (in which case no vanishing cycle is 0). In any case the only Frobenius eigenvalue which may occur on $H^0(\mathbb{P}^1, \mathcal{K})$ is $q^{d/2}$. The Frobenius eigenvalues ϑ on $H^0(\mathbb{P}^1, \mathcal{F})$ are the ones on $H^d(\tilde{X}_u, \mathbb{Q}_\ell)$. Since this is in Poincaré duality with $H^{d-2}(\tilde{X}_u, \mathbb{Q}_\ell)$, $\lambda = q^{d-1-\vartheta}$ is a Frobenius eigenvalue in the latter space. We have seen before that λ (and hence ϑ) is algebraic, and applying the lower bound from (1) to λ gives

$$|\iota\vartheta| \leq q^{d-1-\frac{d-3}{2}} = q^{\frac{d+1}{2}}$$

as desired. Because of

$$0 \to H^0(\mathbb{P}^1, \mathcal{F}) \to E_2^{0,d} \to H^0(\mathbb{P}^1, \mathcal{F})$$

it follows that all Frobenius eigenvalues on $E_2^{0,d}$ have the desired property.

Our remark about the vanishing of $\mathcal{F}$ when $E \neq 0$ will be needed again, hence we repeat it:

REMARK 1. If $E \neq 0$, $R^d f_* \mathbb{Q}_{\ell, \tilde{X}}$ is a constant $\mathbb{Q}_\ell$-sheaf on $\mathbb{P}^1_{\mathfrak{k}}$.

3.5.3. *The case $k = 1$, $l = d - 1$.* This is the hardest the three cases. It is this case for which Proposition .3.4.1, its corollary and the proposition needed for its application had to be developed.

As we did in the previous case, we apply Theorem 11 at $f \in F$ as announced in Remark .3.2.1. We denote the local monodromy group by Γ as we did in this remark. The first non-zero term in (.3.2.3) is then canoncially isomorphic to the stalk of $\mathcal{R} = R^{d-1} f_* \mathbb{Q}_\ell$ at f, while Γ-invariants in the second term give the stalk of $j_* j^* \mathcal{R}$ at f. It follows from (.3.2.4) that the space of Γ-invariants on that term coincides with the kernel of the arrow from it to the third in (.3.2.3). By the exactness of the sequence, it follows that $\mathcal{R}_f \to (j_* j^* \mathcal{R})_f$ is an isomorphism. Thus, the canonical morphism

$$\mathcal{R} \xrightarrow{\cong} j_* j^* \mathcal{R} \tag{2}$$

is an isomorphism.

If $E = 0$ all vanishing cycles are zero the local monodromy group at each $f \in F$ acts trivially on the generic stalk of $\mathcal{R}$. This triviality of the monodromy action implies that $j_* j^* \mathcal{R}$ is locally constant near f. As this is the case for all $f \in A$, because of (2) $\mathcal{R} = R^{d-1} f_* \mathbb{Q}_{\ell, \tilde{X}}$ is locally constant, hence constant on $\mathbb{P}^1$. By the calculation of the cohomology of $\mathbb{P}^1$ we have $E_2^{1,d-1} = 0$ in this case.

If $E \neq 0$, we denote by $\mathcal{E} \subseteq j^* \mathcal{R}$ and $\mathcal{E}^\perp \subseteq j^* \mathcal{R}$ the subsheaves defined by the $\pi_1^{\text{ét}}(U, u)$-invariant subspaces E and $E^\perp$ of $H^{d-1}(\tilde{X}_u, \mathbb{Q}_\ell)$. It follows from (.3.2.4) that the local monodromy group at each $f \in A$ trivially acts on generic stalk of $\mathcal{E}^\perp$. Thus, $j_* \mathcal{E}^\perp$ and $j_*\big(\mathcal{E} \cap \mathcal{E}^\perp\big)$ are locally constant, hence constant sheaves of $\mathbb{Q}_\ell$-vector spaces on $\mathbb{P}^1$.

Unfortunately, it is now necessary to split the case $E \neq 0$ into two subcases. Deligne was able to deduce the Hard Lefschetz theorem from the Weil conjectures, and from this one can derive that one of these cases will never actually occur. Since the Weil conjectures are used in that proof, it is nevertheless necessary to go through that case. We will do this first as it is the easier of the two subcases, in which Proposition .3.4.1 is not needed and the Frobenius eigenvalues turn out to be $q^{d/2}$.

Let $E \subseteq E^\perp$. In this case, let $\mathcal{F}$ denote the quotient of $\mathcal{R}$ by the constant $\mathbb{Q}_\ell$-subsheaf $j_* \mathcal{E}^\perp$. We have

$$\mathcal{F}_u \cong H^{d-1}(\tilde{X}_u, \mathbb{Q}_\ell) \;/\; E^\perp.$$

By our assumption $E \subseteq E^\perp$ and by (.3.2.4), the local monodromy group at each $f \in A$ trivially acts on this quotient. Thus, $j_* \mathcal{F}$ is a locally constant, hence constant $\mathbb{Q}_\ell$-sheaf on $\mathbb{P}^1$. From

$$0 \to j_* \mathcal{E} \to \mathcal{R} \to \mathcal{F} \to 0,$$

the fact that $j_* \mathcal{E}$ is constant and the calculation of the cohomology of $\mathbb{P}^1$ we derive $H^1(\mathbb{P}^1_{\mathfrak{k}}, \mathcal{R}) \subseteq H^1(\mathbb{P}^1_{\mathfrak{k}}, \mathcal{R})$ and we must investigate the Frobenius eivenvalues on this space.

It is easy to see that the morphism $\mathcal{F} \to j_* j^* \mathcal{F}$ is a monomorphism. It is an isomorphism over U, and the stalk of its cokernel at $f \in A$ is easily identifed with the third term in (.3.2.3). Thus,

$$0 \to \mathcal{F} \to j_* j^* \mathcal{F} \to \mathcal{S} \to 0$$

where is a sum of skyscraper sheaves $i_{f*}\mathbb{Q}_\ell\big(\frac{d}{2}\big)$ erected at the points $f \in A$. The Frobenius eivenvalues on $H^0(\mathbb{P}^1_{\mathfrak{k}}, \mathcal{S})$ are thus $q^{d/2}$. Finally, we have

$$H^0(\mathbb{P}^1_{\mathfrak{k}}, \mathcal{S}) \to H^1(\mathbb{P}^1, \mathcal{F}) \to 0$$

as $j_*j^*\mathcal{F}$ is constant. This proves our claim that $q^{d/2}$ is the only Frobenius eigenvalue on $E_2^{1,d-1}$ which may occur in this case.

Finally, let E not be contained in $E^\perp$. Because all vanishing cycles are $\pi_1^{\text{ét}}(U, u)$-conjugate to each other, this implies for each $f \in A$ that the vanishing cycle δ_f at f fails to be orthogonal to E. Because of (.3.2.4) this implies that the generic stalk $\mathcal{E}_{\overline{\eta}}$ of the subsheaf $j_*\mathcal{E} \subseteq \mathcal{R}$ surjectively maps to the cokernel of the cospecialization morphism

$$\mathcal{R}_f \to \mathcal{R}_{\overline{\eta}}.$$

If $\mathcal{F}$ denotes the quotient of $\mathcal{R}$ by its subsheaf $j_*\mathcal{E}$,

$$0 \to j_*\mathcal{E} \to \mathcal{R} \to \mathcal{F} \to 0, \tag{3}$$

this implies that $\mathcal{F} \cong j_*j^*\mathcal{F}$. Because the vanishing cycles have vanishing images in $\mathcal{F}_{\overline{\eta}}$ an by (.3.2.4), the local monodromy group at f trivially acts on $\mathcal{F}_{\overline{\eta}}$. Thus $\mathcal{F} \cong j_*j^*\mathcal{F}$ is a locally constant, hence constant $\mathbb{Q}_\ell$ sheaf on $\mathbb{P}^1_{\mathfrak{k}}$. Its first cohomology is thus zero, and by

$$H^1(\mathbb{P}^1, j_*\mathcal{E}) \to H^1(\mathbb{P}^1, \mathcal{R}) \to 0$$

it is sufficient to show that the Frobenius Eigenvalues on $H^1(\mathbb{P}^1, j_*\mathcal{E})$ have the desired properties.

As was explained before, the subsheaf $j_*\big(\mathcal{E} \cap \mathcal{E}^\perp\big)$ of $j_*\mathcal{E}$ is constant. Because the vanishing cycle at f is not orthogonal to E, the morphism

$$\mathcal{E}_{\overline{\eta}} \to \big(\mathcal{E} \;/\; \mathcal{E} \cap \mathcal{E}^\perp\big)_{\overline{\eta}}$$

induces an isomorphism on cokernels of the cospecialization map from stalks at f. Because of this, the sequence

$$0 \to j_*\big(\mathcal{E} \cap \mathcal{E}^\perp\big) \to j_*\mathcal{E} \to j_*\big(\mathcal{E} \;/\; \mathcal{E} \cap \mathcal{E}^\perp\big) \to 0$$

is exact at f. As the first term is constant we obtain

$$0 \to H^1(\mathbb{P}^1, j_*\mathcal{E}) \to H^1\Big(\mathbb{P}^1, j_*\big(\mathcal{E} \;/\; \mathcal{E} \cap \mathcal{E}^\perp\big)\Big)$$

and it is sufficient to study Frobenius eigenvalues on the latter space. Let $\mathcal{G} = j_*\big(\mathcal{E} \;/\; \mathcal{E} \cap \mathcal{E}^\perp\big)$. We have

$$H^1_c(U, \mathcal{G}) \to H^1(\mathbb{P}^1, j_*\mathcal{G}) \to 0$$

by an application of (.2.6.2) with $X = \mathbb{P}^1$, $Y = A$. By Corollary .3.4.1 it is sufficient to show that the assumptions of Proposition .3.4.1 hold for $\mathcal{G}$. For the symplecticity this holds by definition and the assumption about (.3.4.6) is Theorem 13. The rationality assumption will be verified using Proposition .3.4.2. The assumption about (.3.4.11) is again Theorem 13 and it remains to verify condition (.3.4.12). This can be considered for arbitrary $\mathbb{Q}_\ell$-adic sheaves $\mathcal{H}$ on U_o. Let us denote this property or $\mathcal{H}$ as $\mathrm{Rat}(\mathcal{H})$. Of course this condition Rat is much weaker than the rationality of the coefficients of the Frobenius characteristic polynomial on the stalks of $\mathcal{H}$.

Then $\mathrm{Rat}(\mathcal{H})$ holds when $\mathcal{H}$ extens to a locally constant $\mathbb{Q}_\ell$-sheaf on $\mathbb{P}^1_{\mathfrak{k}}$ because this extension is then constant, implying that $\mathcal{H}$ comes from an ℓ-adic sheaf on $\mathrm{Spec}\mathfrak{k}_o$ and the Frobenius eigenvalues on its stalk at x are just $\lambda^{[\mathfrak{k}(x):\mathfrak{k}]}$ with λ running over a fixed finite set. The characteristic polynomial of the Frobenius on the stalks of $\mathcal{H}$ thus has a form allowed for the denominator in (.3.4.12), which can thus be chosen such that (.3.4.12) is the constant rational function 1.

By Theorem 11, our previous remark implies $\mathrm{Rat}(R^p f_* \mathbb{Q}_{\ell,\tilde{X}})$ when $p \notin \{d, d+1\}$. By Remark 1 this still holds when $p = d+1$. An application of (.2.8.7) to the (smooth) fibre $\left(\tilde{X}_o\right)_x$ at $x \in U$ now shows $\mathrm{Rat}(R^p f_* \mathbb{Q}_{\ell,\tilde{X}})$ when $p = d$ because this is the only remaining factor in (.2.8.7) and the congruence zeta function has rational coefficients. Thus, we have $\mathrm{Rat}(\mathcal{R})$

Finally, we remark that if Rat holds for two terms in a short exact sequence, then it does for the third, by the multiplicativity of determinants in short exact sequences. From (3) and $\mathrm{Rat}(\mathcal{R})$ we may thus decuce $\mathrm{Rat}(\mathcal{E})$ as $\mathcal{F}$ is constant. The fact that $\mathcal{E} \cap \mathcal{E}^\perp$ not implies $\mathrm{Rat}(\mathcal{G})$. The proof of the Weil conjectures is now complete.

4. Errata to [FK88]

Following is a brief list of mistakes I found in [**FK88**]. It is likely to be incomplete and I may extend it in the next few weeks, in particular if I find more errors in the parts relevant to an exam. Most of the errors are typographical and quite easy to guess and correct, although a few them may present stumbling stones for people with a limited knowledge of étale cohomology attempting to force their way through [**FK88**] by reading line by line.

122_9. The direct limit should be taken.

144_6. left side

149^{14}. $H^{2d}_c(X, \mathbb{Q}_\ell(d)) \to \mathbb{Q}_\ell$

149_{13}. I am not fully convinced that “It follows from the previous that the induced mapping . . . has only torsion modules for its kernel and cokernel” unless X is proper. Passing to the limit here, using the ARML property, is indeed so easy that the typical reader of a book like [**FK88**] may be assumed to be able to do it for himself. But it seems to be necessary to use the ARML property on both sides, as we do before (.2.6.3), while [**FK88**, Theorem I.12.15] only gives it for the H^*_c-side. However, I have not read [**FK88**] careful enough to be 100% sure that there is indeed an issue here, and in addition I think that the proof of the aforementioned theorem still works without compact support when S is a field. A carfule reader or editor of a new edition of [**FK88**] may want to have a look at this, however.

150_{17}. The (after $\oplus$ is not needed and is never closed.

150_6. $\mathrm{Hom}_\Lambda(H^{2(d-s)}_c(Y, \mu_n^{d-s}))$. The $\to$ after the opening (in the text of the book was probably inserted by mistake.

180_2. The text is misleading here. The proof started on 180_2 and continued on the next page is for the remark made on 180_8, not proposition 1.10.

181^6. It is the aforementioned remark which allows one to replace X by $\tilde{X}$ in the Weil 1 proof, not proposition 1.10.

251^2. $\nu \neq n, n+1$

266_{18}. $\psi(\sigma x, \sigma y) = \chi(sig)\psi(x, y)$.

271^{17}. Read $H^1(D, \ldots)$ for $H^1(d, \ldots)$.

Literaturverzeichnis

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