Problem sheet 4 for Rigid Analytic Geometry Winter Term 2026/27

Let $\mathbb{K}$ be complete and non-discrete with respect to the ultrametric absolute value $|\cdot|$. In the previous term it was shown on sheet 2 that two norms on a finite dimensional $\mathbb{K}$-vector space V are equivalent and that V, equipped with the topology defined any of these norms, is isomorphic to $\mathbb{K}^d$ equipped with the product topology by a continuous isomorphism with a continuous inverse, where $d = \dim V$. The following is a straightforward consequence of this result. Since it was used in the lecture it may nevertheless be appropriate to work out a careful proof.

Problem 1 (2 points)**.** *Let V is a finite dimensional $\mathbb{K}$-vector space and W a dense subspace of V. Show that $V = W$!*

For a prime number p and a ring R with $p \cdot R = 0$, let $\mathfrak{F}_R$ be the endomorphism of R given by $\mathfrak{F}_R(r) = r^p$. The subscript $_R$ will be dropped when R is obvious from the context, and for ideals I and J such that J contains both I^p and p the ring homomorphism $R/I \to R/J$ sending $x \bmod I$ to $x^p \bmod J$ will also be denoted $\mathfrak{F}$ if this has no potential of causing confusion.

Keeping our previous assumptions about $\mathbb{K}$ we also assume that the residue field $\mathfrak{k} = \mathbb{K}^\circ/\mathbb{K}^{\circ\circ}$ has characteristic p.

Problem 2 (5 points)**.** *Show that the following conditions are equivalent:*

- *The subgroup $|\mathbb{K}^\times|$ of $(0,\infty)_\mathbb{R}$ is dense and the endomorphism $\mathfrak{F}$ of $\mathbb{K}^\circ/p\mathbb{K}^\circ$ surjective.*
- *There is $s \in \mathbb{K}^{\circ\circ} \setminus \{0\}$ such that $p \in s^p\mathbb{K}^\circ$ and for every s with this property*

$$\mathbb{K}^\circ/s\mathbb{K}^\circ \xrightarrow{\mathfrak{F}} \mathbb{K}^\circ/s^p\mathbb{K}^\circ$$

  is an isomorphism.
- *There is $s \in \mathbb{K}^{\circ\circ} \setminus \{0\}$ such that $p \in s^p\mathbb{K}^\circ$ and such that*

$$\mathbb{K}^\circ/s\mathbb{K}^\circ \xrightarrow{\mathfrak{F}} \mathbb{K}^\circ/s^p\mathbb{K}^\circ$$

  is an isomorphism.

Definition 1. *We say that $\mathbb{K}$ is* perfectoid *if it satisfies these equivalent conditions, besides the other assumptions.*

It is clear from the third point of the previous problem that in the equal characteristic case $\mathbb{K}$ is perfectoid if and only if it is perfect.

Problem 3 (2 points)**.** *For perfectoid* $\mathbb{K}$*, show that* $|\mathbb{K}^\times|$ *is p-divisible!*

This notion of being perfectoid has a generalization which applies to arbitrary Huber rings. We will discuss this on one of the next sheets after the basic result which allows us to show sheafiness in this situation has been shown.

On this sheet and the following ones, the notion of a Tate ring A will be applied in the sens of Huber. This means that A has a pair of definition $(A^\sharp, s)$ formed by an open bounded subring and a topologically nilpotent unit $s \in A^{\circ\circ} \cap A^\times$. Then the $s^j A^\sharp$ form a neigbourhood base of 0 and if A is not complete yet then its completion is given by

$$\lim A \,/\, s^j A^\sharp \tag{1}$$

over natural numbers j. This means that the underlying additive group is (1) equipped with the topology of an inverse limit of discrete groups and that $(x_j)_{j=0}^\infty \cdot (y_j)_{j=0}^\infty = (z_j)_{j=0}^\infty$ where $z_j = x_j y_j$ when j is so large that $s^j x_j$ and $s^j y_j$ are both $\in A^\sharp$. If j is smaller, z_j is the image in $A \,/\, s^j A^\sharp$ of z_i with i sufficiently large.

For the following problems we assume A to be complete.

Problem 4 (4 points)**.** *For* $g \in A$*, let* $B \subseteq A$ *be the subring of* A *generated by* $A^\sharp$ *and* g*. Show that there is a unique isomorphism of nat-A-algebras between* $A\langle g\rangle$ *and the completion of* A *with respect to the toplogy for which the* $s^j B$ *form a neighbourhood base of* 0*!*

We denote A equipped with the topology for which the $s^j B$ form a neighbourhood base of 0 by $\mathcal{B}$.

Let A_g be the localization of A with respect to g.

Problem 5 (4 points)**.** *In the previous situation let* $C \subseteq A_j$ *be the subring of* A_g *generated by* $A^\sharp$ *and* g^{-1}*. Show that there is a unique isomorphism of nat-A-algebras between* $A\langle 1/g\rangle$ *and the completion of* A_g *with respect to the toplogy for which the* $s^j C$ *form a neighbourhood base of* 0*!*

Let $\mathcal{C}$ denote A_g equipped with the toplogy for which the $s^j C$ form a neighbourhood base of 0.

By combining the previous problems it can also be shown that $A\langle g, g^{-1}\rangle$ is the completion of $\mathcal{D}$, the ring A_g equipped with the topology for which a neighbourhood base of 0 is given by the $s^j D$, where D is the subring of A_g generated by $A^\sharp$, g and g^{-1}.

Problem 6 (2 points)**.** *Show that the sequence*

$$0 \to A \xrightarrow{i} \mathcal{B} \oplus \mathcal{C} \xrightarrow{p} \mathcal{D} \tag{2}$$

formed in a similar way to the sequence

$$0 \to A \to A\langle g\rangle \oplus A\langle g^{-1}\rangle \to A\langle g, g^{-1}\rangle \to 0 \tag{3}$$

considered in sheafiness and acycliciy proofs is exact!

The problem is thus to show that the exactness of (2) survives completion.

Problem 7 (4 points)**.** *Assume that*

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \tag{4}$$

is a short exact sequence of abelian groups. Moreover, assume that X, Y and Z are equipped with group topologies for which countable neighbourhood bases of 0 *exist, and equip* Ker$p$ *with the topology induced from Y. If p and the bijection $X \xrightarrow{i}$* Ker$p$ *are open, show that* (4) *stays exact after completion.*

Problem 8 (2 points)**.** *Show that p in* (2) *is open!*

Five of the 25 points from this sheet are bonus points which do not count in the calculation of the 50%-threshold for passing this exercise module.

Solutions should be e-mailed to your tutor s45qhou at uni hyphen bonn dot de before Friday November 13. Good luck!