Problem sheet 3 for Rigid Analytic Geometry Winter Term 2026/27

Before starting the formulation of problems I would like to recall some properties of topological rings considered in the previous term when studying the properties of the space $\mathrm{Cont}(A)$ of continuous valuations and of its subspaces $\mathrm{Spa}(A^{\triangleleft}, A^{+})$.

(oisq): If I is an open ideal of A then so is I^2.
(oifin): Every open ideal of A contains a finitely generated open ideal.
(tn0): The ideal of A generated by the set $A^{\circ\circ}$ of topologically nilpotent elements of A is open.
(tn1): $A^{\circ\circ}$ is a neighbourhood of 0.

It is easy to see that (oisq) implies the openness of any finite product of open ideals and that (tn1) implies (tn0). On the current sheet only (tn1) will be used. On other sheets however, as well as in the lecture, all of these conditions, may from now on be used without further explanation. Note also that for sequentially complete nat-rings (tn1) implies (by using the geometric series) that $A^{\times}$ is a neighbourhood of 1 in A. As a consequence, there is an open subgroup U such every finite square matrix M with $m_{ij} \in \delta_{ij} + U$ is invertible. All of this may be used without proof in the solutions to all problems in this exercise module.

The notions of topological Noetherianness and of topological finite generation of modules have the same meaning as in the lecture.

The following is a fairly straightforward consequence of the open mapping theorem shown at the end of the previous lecture.

Problem 1 (4 points)**.** *Let A be a complete nat-ring with a countable neighbourhood base of 0 and a topologically nilpotent unit. Show that every finitely generated closed ideal of A is topologically finitely generated!*

Problem 2 (4 points)**.** *Let A be a complete nat-ring with (tn1) and I an ideal in A such that the closure $\overline{I}$ of I in A is topologically finitely generated. Show that $I = \overline{I}$ (in other words, I is closed)!*

As a fruit of our efforts since the last problem of the first sheet we obtain the following result, confirming the claim I made in Remark 4.2.1 in the lecture of Monday October 19.

Problem 3 (3 points)**.** *Let A be a complete nat-ring with (tn1), a topologically nilpotent unit and a countable neighbourhood base of* 0.[1] *Show that the following conditions are equivalent:*

- *A is Noetherian.*
- *A is topologically Noetherian.*
- *Every ideal of A is closed in A.*

In particular, all assertions of Proposition 3.6.2 apply to finitely generated modules over such rings.

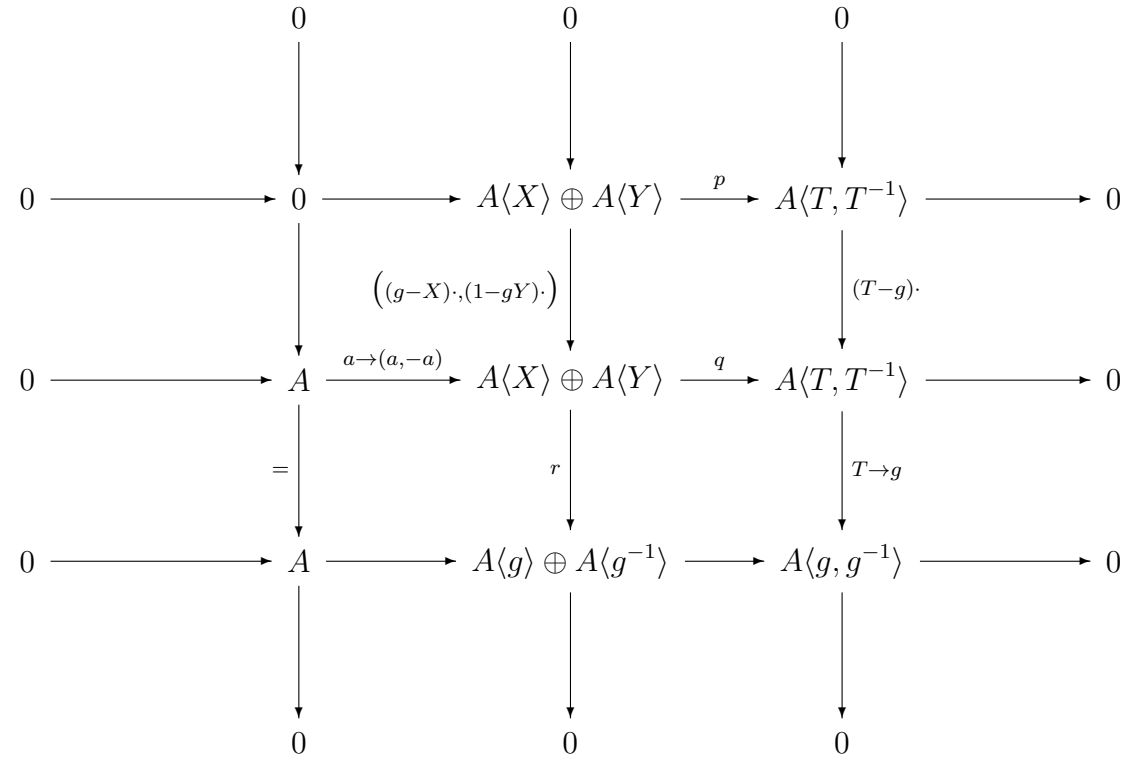
With Proposition 4.2.2 we thus get sheafiness in the complete Tate strongly Noetherian case, where for a property P of topological rings (like Noetheriannes, topological Noetherianness or rigid Noetherianness) "A is strongly P" means that all $A\langle X_1, \ldots, X_m\rangle$ have property P. It may be worthwile to point out that strong Noetherianness can be replaced by a suitable condition of strong topological coherence.

This kind of sheafiness result is based upon considerations similar to the proof of Propositions 4.2.1 and 4.2.2 which are in fact somewhat similar to, but easier than, the considerations I presented in the lectures on October 15 and 22.[2] The considerations are actually entirely

[1] Eg., a Tate ring in the sense of Huber which is complete.

[2] With the hindsight of the preparation of this exercise sheet, I now think that this is actualy what should have been done in these lectures.

straightforward, using the diagram



(1)

Problem 4 (6 points)**.** *Let A be a complete nat-ring and assume that all principal ideals of $A\langle T\rangle$ and of $A\langle T, T^{-1}\rangle$ are closed. Assume also that $g \in A$ is such that $1-gT$ is not a zero divisor in $A\langle T, T^{-1}\rangle$. Show that the sequence*

$$0 \to A^{\triangleleft} \to A^{\triangleleft}\langle g\rangle \oplus A^{\triangleleft}\langle g^{-1}\rangle \to A^{\triangleleft}\langle g, g^{-1}\rangle \to 0$$

is exact!

The following is useful to derive the injectivity condition of the previous problem from a suitable condition of topological coherence.

Problem 5 (4 points)**.** *Let A be a sequentially complete nat-ring with (tn1) and $g \in A^{\circ} \cap A^{\times}$. Assume that M is a topologically finitely generated A-module and that*

$$\lim_{j\to\infty} g^{-j} m = 0$$

holds for all $m \in M$. Show that $M = 0$!

One of the 21 points from this sheet is a bonus point which does not count in the calculation of the 50%-threshold for passing this exercise module.

Solutions should be e-mailed to your tutor s45qhou at uni hyphen bonn dot de before Friday November 6. Good luck!