Sixth exercise sheet Advanced Algebra II. Let K be a real closed field and $A = K[T]$. We order $\mathrm{Sper}A$ by $\mathfrak{P} \subseteq \mathfrak{Q}$ if and only if the following two conditions are both satisfied:

- $\{x \in K \mid T - x \in \mathfrak{P}\} \subseteq \{x \in K \mid T - x \in \mathfrak{Q}\}$
- $\{x \in K \mid x - T \in \mathfrak{Q}\} \subseteq \{x \in K \mid x - T \in \mathfrak{P}\}$.

It is easy to see that this is a linear order and that this coincides with the one introduced in the previous term. For $x \in K$ let

$$\mathfrak{P}_x = \{f \in A \mid f(x) \geq 0\} \in X,$$

and we write $\mathfrak{Q} \prec x$ (resp. $x \prec \mathfrak{Q}$) for $\mathfrak{Q} \prec \mathfrak{p}_x$ (resp. $\mathfrak{P}_x \prec \mathfrak{Q}$). If $a < b$ are elements of $K \cup \{\pm\infty\}$ let

$$(a,b)_X = \{x \in X \mid a \prec x \prec b\}.$$

Here ∞ and $-\infty$ are non-elements of X with the convention that $-\infty \prec x \prec \infty$ holds for all $x \in X$.

Problem 1 (5 points). *Let $X = \mathrm{Sper}A$ as above, and let*

$$\mathfrak{B}_o = \{(a,b)_X \mid a < b, a \in K \cup \{-\infty\}, b \in K \cup \{\infty\}$$
$$\mathfrak{B}_c = \{\mathfrak{P}_x \mid x \in K\}.$$

Show that the assumptions of Problem 4 of sheet 5 hold, and show that every constructible closed point of X equals $\mathfrak{P}_x$ for some $x \in K$.

We now consider $S = \mathrm{Sper}R$ where $R = K[X,Y]/HK[X,Y]$ with $H = X^2 + Y^2 - 1)$. For $(x,y) \in K^2$ with $x^2 + y^2 = 1$ let

$$\mathfrak{P}_{x,y} = \{f \bmod H \mid f(x,y) \geq 0\},$$

let $P = \mathfrak{P}_{0,-1}$ and let $U = S \setminus \{P\}$.

Problem 2 (2 points). *Show that $V_{\mathrm{Sper}R}(Y+1) = \{P\}$ and $\mathfrak{P}_{\mathrm{Sper}R}(Y+1) = U$.*

In the following the homeomorphism $U \xrightarrow{\iota} X$ from the lecture can be used. The same holds for the results of Example 2.4.1 and Example 2.4.2. Let $U \xrightarrow{j} S$ be the inclusion, and for a topological space Y let $\mathbb{Z}_Y$ be the constant sheaf. For an abelian group G let G_P be the skyscraper sheaf.

Problem 3 (5 points). *Calculate the stalks of $j_*\mathbb{Z}_U$ and show that $R^1 j_*\mathbb{Z}_u$ vanishes!*

Problem 4 (5 points). *Construct a short exact sequence*

$$0 \to \mathbb{Z}_S \to j_*\mathbb{Z}_U \to (\mathbb{Z} \oplus \mathbb{Z})_P \to 0$$

and use it to calculate $H^*(S, \mathbb{Z})$!

Recall that $U \xrightarrow{\iota} X$ denotes the homeomorphism from the lecture.

Problem 5 (5 points). *Consider the spectral space S with*

$$\begin{aligned}\mathfrak{B}_o = &\left\{\iota^{-1}\big(a,b\big)_X \;\middle|\; -\infty \le a < b \le \infty\right\} \cup \\ &\qquad\left\{\{P\} \cup \iota^{-1}\big(-\infty,a\big)_X \cup \big(b,\infty\big) \;\middle|\; -\infty < a < b < \infty\right\} \\ \mathfrak{B}_c = &\left\{\mathfrak{P}_{x,y} \;\middle|\; (x,y) \in K^2 \text{ and } x^2+y^2 = 1\right\}\end{aligned}$$

where in the definition of $\mathfrak{B}_o$ a and b must be from $K \cup \{\pm\infty\}$. Show that the assumptions of Problem 4 of the previous sheet hold and that every constructible closed point of S has the form $\mathfrak{P}_{x,y}$ with x and y as above.

Remark 1. *It is not hard to show that the elements of $\mathfrak{B}_o$ are the subsets of S which have the form $\mathcal{P}(c - aX + bY)$ with $(a,b) \in K^2$, $a^2 + b^2 = 1$ and $c \in (-1,1)_K$. Intuitively, one can think of this as the "open" arcs of "length" $\in (0, 2\pi)$. Therefore the topology base $\mathfrak{B}_o$ does not really depend on the choice of P and the homeomorphism $S \setminus \{P\} \cong X$.*

Consider the continuous map $S^{\mathrm{con}} \xrightarrow{c} S$.

Problem 6 (2 points). *Show that $c_*\underline{\mathbb{Z}}_{S^{\mathrm{con}}}$ is the set of functions $S \xrightarrow{f} \mathbb{Z}$ such that there elements $a_0 = -\infty < a_1 < \cdots < a_N = \infty$ such that $f\iota$ is constant on $(a_{i-1}, a_i)_X$ for $1 \le i \le N$!*

Problem 7 (2 points). *Calculate the stalks of $c_*\underline{\mathbb{Z}}_{S^{\mathrm{con}}}$!*

Problem 8 (5 points). *Identify the cokernel $\mathcal{C}$ of $\underline{\mathbb{Z}}_S \to c_*\underline{\mathbb{Z}}_{S^{\mathrm{con}}}$ with the coproduct of the skyscraper sheaves $(\mathbb{Z}\oplus\mathbb{Z})_s$ taken over the constructible closed points s of S and use the short exact sequence*

$$0 \to \underline{\mathbb{Z}}_S \to c_*\underline{\mathbb{Z}}_{S^{\mathrm{con}}} \to \mathcal{C} \to 0$$

to calculate $H^*(S, \mathbb{Z})$.

Recall that we call a topological space *compact* if it is quasi-compact and Hausdorff.

Problem 9 (3 points). *Let X be a compact space. Show that every point of X has a neighbourhood base whose elements are closed in X.*

As a consequence, the following conditions for a Hausdorff space X are equivalent:

- Every point has some compact (and thus closed) neighbourhood in X.

- Every point has a neighbourhood base whose elements are comapct (and thus closed).

Such a Hausdorff space will be called *locally compact*.

Problem 10 (6 points)**.** *Let $I = [0,1]_{\mathbb{R}}$, $\mathcal{A}$ the category of sheaves of abelian groups on I and $\mathcal{X}$ the class of all sheaves $\mathcal{F}$ of abelian groups on I such that the restriction map $\mathcal{F}(I) \to \mathcal{F}(U)$ is surjective for every connected open $U \subseteq I$. Let $F(\mathcal{F}) = \mathcal{F}(U)$ where $U \subseteq I$ is connected and open. Show that the assumptions A-C of the fourth exercise sheet hold.*

Remark 2. *The proof can be formulated in such a way that it goes through for any topological space X in which the connected open subsets form a neighbourhood base closed under finite (including empty) intersections in X. In particular, the result also holds for open and open-closed intervals in $\mathbb{R}$.*

Remark 3. *It may be used without proof that the open connected subsets of I are:*

- $\emptyset$ *and* I.
- (a,b) *with* $0 \leq a < b \leq 1$.
- $[0,a)$ *and* $(a,1]$ *with* $0 < a < 1$.

Out of the 42 points available from this exercise sheet, 22 will be considered bonus points which do not count in the calculation of the 50%-limit for admission to the exam.

Solutions should be submitted in the lecture Friday, May 31.