Fourth exercise sheet Advanced Algebra II. We consider the following two conditions on a class $\mathcal{X}$ of objects of an abelian category $\mathcal{A}$:

A: Every direct summand of an element of $\mathcal{X}$ belongs to $\mathcal{X}$.

In particular, every object isomorphic to an element of $\mathcal{X}$ belongs to $\mathcal{X}$.

B: For every object A of $\mathcal{A}$ there is a monomorphism $A \to X$ with $X \in \mathcal{X}$.

Problem 1 (3 points)**.** *If assumptions A and B hold, show that every injective object of $\mathcal{A}$ belongs to $\mathcal{X}$.*

If $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a right exact functor to an abelian category $\mathcal{B}$, we consider the following assumption:

C: If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence in $\mathcal{A}$ with $X' \in \mathcal{X}$ and $X \in \mathcal{X}$, then $X'' \in \mathcal{X}$ and

$$0 \to FX' \to FX \to FX'' \to 0$$

is exact.

Problem 2 (4 points)**.** *Assume that F and $\mathcal{X}$ are as above, that assumptions A, B and C hold and in addition that $\mathcal{A}$ has sufficiently many injective objects. Show that the elements $X \in \mathcal{X}$ are F-acyclic in the sense that $R^pFX = 0$ when $p > 0$.*

Problem 3 (5 points)**.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on the topological space X, let $\mathcal{X}$ be the class of objects $\mathcal{F}$ of $\mathcal{A}$ such that the restriction map*

$$\mathcal{F}(X) \to \mathcal{F}(U)$$

is surjective for arbitrary open $U \subseteq X$, and let F be the functor of sections on an open subset U. Verify the above conditions A, B and C!

Problem 4 (3 points)**.** *Use the previous results for alternative proofs of Remark 1.4.1 (that $\mathcal{F} \to H^*(U, \mathcal{F}|_U)$ is the derived functor of $\mathcal{F} \to \mathcal{F}(U)$) and Remark 1.4.3 (the Leray spectral sequences).*

Problem 5 (5 points)**.** *Let $X \xrightarrow{f} Y$ be continuous and $\mathcal{F}$ a sheaf of abelian groups on X. Derive an exact sequence*

$$0 \to H^1(Y, f_*\mathcal{F}) \to H^1(X, \mathcal{F}) \to H^0(Y, R^1 f_*\mathcal{F}) \to H^2(Y, f_*\mathcal{F}) \to H^2(Y, \mathcal{F})$$

from the Leary spectral sequence.

Solutions should be submitted in the lecture Friday, May 10.