

Parametrization and reduction to depth zero of $\overline{\mathbb{Z}}[\frac{1}{p}]$ -blocks of tame p -adic groups

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Abstract

Let \mathbf{G} be a reductive group over a non-archimedean local field F of residue characteristic p . We consider pairs (ϕ, I) consisting of a “wild inertia” Langlands parameter $\phi : P_F \rightarrow \hat{\mathbf{G}}$ whose centralizer $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$, and a cohomological invariant I whose definition is inspired by the theory of endoscopy. Assuming that p is odd and not a torsion prime of \mathbf{G} nor of $\hat{\mathbf{G}}$, we associate to each such pair (ϕ, I) a Serre subcategory $\text{Rep}^{\phi, I}(\mathbf{G}(F))$ of the category of smooth $\overline{\mathbb{Z}}[\frac{1}{p}]$ -representations of $\mathbf{G}(F)$. Then we construct an equivalence between this Serre subcategory and the category of depth-zero $\overline{\mathbb{Z}}[\frac{1}{p}]$ -representations of a twisted Levi subgroup $\mathbf{G}_{\phi, I}$ of \mathbf{G} , which is dual to $C_{\hat{\mathbf{G}}}(\phi)$. This pattern for reduction to depth zero fits well with the conjectural (categorical) local Langlands correspondence.

When \mathbf{G} is tamely ramified and p does not divide the order of its Weyl group, then the above Serre subcategories provide the block decomposition of the category of all smooth $\overline{\mathbb{Z}}[\frac{1}{p}]$ -representations of $\mathbf{G}(F)$. In this case, we thus obtain a reduction-to-depth-zero process for smooth representations of $\mathbf{G}(F)$ valued in any algebraically closed field of characteristic different from p . When that field has characteristic 0, this recovers some of the recent results of Adler–Fintzen–Mishra–Ohara. When that field is $\overline{\mathbb{F}}_\ell$, we use our results together with Zhu’s unipotent categorical correspondence to produce a fully faithful embedding of $D\text{Rep}_{\overline{\mathbb{F}}_\ell}(\text{GL}_n(F))$ into a suitable category of coherent sheaves on the moduli space of n -dimensional $\overline{\mathbb{F}}_\ell$ -representations of the Weil group.

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1 Introduction

Let \mathbf{G} be a connected reductive group defined over a non-archimedean local field F and set $G := \mathbf{G}(F)$. For any commutative ring R , we denote by $\text{Rep}_R(G)$ the category of smooth R -representations of G . Here is a coarse representation-theoretic takeaway of what is done in this paper.

Main result. (Take 1) — *Assume that \mathbf{G} is tamely ramified and that p does not divide the order of the absolute Weyl group of \mathbf{G} . We construct the block decomposition of $\text{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)$ and show that each block is equivalent to the depth-0 block of an explicitly associated tamely ramified reductive group (which turns out to be a twisted Levi subgroup of \mathbf{G}). Moreover, each block is defined over $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ and the equivalences can be defined over $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$.*

Note that this implies a reduction to depth 0 for blocks over any $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra R , and in particular over any algebraically closed field of characteristic different from p .

We like to remind readers mostly acquainted with complex representation theory of p -adic groups that there is in general no Bernstein decomposition nor type theoretic construction of blocks of $\text{Rep}_R(G)$ whenever the pro-order of G has a divisor that is not invertible in R . Even when such a decomposition exists, e.g., in the settings of [HKSS24], [Lan23], [MS14], [Vig98], there is in general no equivalence between a block and a category of modules over a nice Hecke algebra. Therefore, our techniques have to be quite different from those used in [AFMO24b], where a similar “reduction to depth 0” result is proved for complex representations by comparing suitable Hecke algebras.¹

To give an idea of our techniques, we start with the following description of the depth-0 subcategory (the notation for the latter will be explained below)

$$\text{Rep}_R^1(G) = \left\{ V \in \text{Rep}_R(G), V = \sum_{x \in \mathcal{B}} e_{G_{x,0+}} V \right\}$$

where \mathcal{B} denotes the Bruhat–Tits building of \mathbf{G} , the group $G_{x,0+}$ is the pro- p -radical of the parahoric subgroup attached to x , and $e_{G_{x,0+}}$ is the averaging idempotent along this pro- p -group, which is an element of the ring $\mathcal{H}_R(G)$ of all compactly supported, locally constant R -valued distributions on G , and thus acts on V (as a projector onto the fixed vectors $V^{G_{x,0+}}$).

Our blocks are constructed in a similar way in Section 2, via families of idempotents indexed by points in the building. These families have nice consistency properties that allow us to apply results of [MS10] and [Lan21], which show that our blocks are equivalent to certain categories of coefficient systems on the building. This is the replacement for the missing Hecke algebras bridge in this setting. Our equivalences between blocks are then obtained in Section 3 by constructing equivalences between these categories of coefficient systems, following an overall strategy introduced in [Dat18].

The definition of these families of idempotents ultimately relies on Yu’s notion of generic characters, which is part of the construction of types in [Yu01] and [KY17]. The hypothesis on p in the first take on our main result above ensures that we obtain the whole category, thanks to the exhaustion result of [Fin21]. For the construction of the equivalences, we need more than the generic characters, namely we need consistent families of twisted Heisenberg–Weil representations, building on the twists introduced in [FKS23].

We note that in [HKSS24], the block decomposition of $\text{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ is worked out with minimal hypothesis ($p \neq 2$) when \mathbf{G} is a classical group, using the notion of semisimple characters due to Bushnell–Kutzko and Stevens. It would be interesting to see if our strategy for reduction to depth-0 can be carried out in that setting.

This paper could have been written purely in representation theoretic terms, labeling our blocks by some suitable equivalence classes of open pro- p -subgroups of G and characters thereof, that appear in the construction of types in [KY17]. However, *this is not the path we follow*.

Instead, we will rather start from the spectral side of the Langlands correspondence, and our blocks will be indexed by “extended” wild inertia parameters, to be defined just below. A

¹Note however that our main result, stated below, allows us to reprove that the relevant Hecke algebras of [AFMO24b] are isomorphic, albeit in a less explicit way, see Corollary 4.1.2.

practical reason is that the reductive groups appearing in the targets of our equivalences then have a transparent interpretation as suitable inner forms of the dual of the centralizer of the parameter. A deeper reason is that the pattern for reduction to depth 0 given by this point of view (exposed in [Dat25, §4.5] and anticipated in [Dat17]) should work in greater generality, a hope that is strongly supported by the categorical local Langlands program, as we will explain below. In order to state our main result with more details we first need to introduce some additional notation.

Parameters. We denote by ${}^L\mathbf{G} = \hat{\mathbf{G}} \rtimes W_F$ the Weil form of the L -group of \mathbf{G} over \mathbb{C} and let $\Phi(\mathbf{G}) \subset H^1(W'_F, \hat{\mathbf{G}})$ be the set of relevant Langlands parameters for \mathbf{G} , where W_F denotes the Weil group of F and $W'_F = W_F \times \mathrm{SL}_2(\mathbb{C})$.

Let P_F be the wild inertia subgroup of W_F , and let $\Phi(P_F, \mathbf{G})$ denote the image of $\Phi(\mathbf{G})$ under the restriction map $H^1(W'_F, \hat{\mathbf{G}}) \rightarrow H^1(P_F, \hat{\mathbf{G}})$. We call the elements of $\Phi(P_F, \mathbf{G})$ “wild inertia parameters” for \mathbf{G} . So, an element $\phi \in \Phi(P_F, \mathbf{G})$ is a $\hat{\mathbf{G}}$ -conjugacy class of L -homomorphisms $P_F \rightarrow {}^L\mathbf{G}$ (still denoted ϕ) that can be extended to some $\varphi : W_F \rightarrow {}^L\mathbf{G}$ (which is a restriction of a relevant Langlands parameter). In this setting, $\varphi(W_F)$ normalizes the centralizer $C_{\hat{\mathbf{G}}}(\phi)$ of ϕ in $\hat{\mathbf{G}}$. The action, induced by conjugation, of W_F on the center $Z(C_{\hat{\mathbf{G}}}(\phi))$ and the cocenter $(C_{\hat{\mathbf{G}}}(\phi))_{\mathrm{ab}}$, as well as the induced outer action $W_F \rightarrow \mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi))$, are tamely ramified, and independent of the choice of φ .

When the reductive group $C_{\hat{\mathbf{G}}}(\phi)$ is connected, which is, for example, the case if $p > 2$ when $\hat{\mathbf{G}}$ is a classical group, then we denote by:

- \mathbf{G}_ϕ the quasi-split reductive F -group that is dual to $C_{\hat{\mathbf{G}}}(\phi)$ over a separable closure \bar{F} of F , with F -rational structure given by the above outer action.
- X_ϕ the kernel of the map $X^*(Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F)})_{\mathrm{tors}} \xrightarrow{h_\phi} X^*(Z(\hat{\mathbf{G}})^{W_F})_{\mathrm{tors}}$ induced by the inclusion $Z(\hat{\mathbf{G}})^{W_F} \subset Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F)}$ after taking torsion subgroups of character groups. For any inner form \mathbf{G}'_ϕ of \mathbf{G}_ϕ , composition with Kottwitz’ isomorphisms turns the above map into a map $H^1(F, \mathbf{G}'_\phi) \xrightarrow{h_\phi} H^1(F, \mathbf{G})$ that allows to view X_ϕ as a subset of $H^1(F, \mathbf{G}'_\phi)$.

When $C_{\hat{\mathbf{G}}}(\phi)$ is additionally a Levi subgroup of $\hat{\mathbf{G}}$, which is, for example, the case if p does not divide the order of the Weyl group of $\hat{\mathbf{G}}$, we denote by:

- \mathbf{S}_ϕ the F -torus that is dual to the cocenter of $C_{\hat{\mathbf{G}}}(\phi)$ with the action of W_F described above. In 2.2.3 we explain how \mathbf{S}_ϕ comes with a $\mathbf{G}(\bar{F})$ -conjugacy class of “Levi-center-embeddings” $\iota : (\mathbf{S}_\phi)_{\bar{F}} \hookrightarrow \mathbf{G}_{\bar{F}}$, meaning an embedding whose image is the connected center of its centralizer. For each such ι , the centralizer $C_{\mathbf{G}_{\bar{F}}}(\iota((\mathbf{S}_\phi)_{\bar{F}}))$ is a Levi subgroup of $\mathbf{G}_{\bar{F}}$ isomorphic to $(\mathbf{G}_\phi)_{\bar{F}}$.
- \mathcal{I}_ϕ the set of G -conjugacy classes of F -rational Levi-center-embeddings $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in the above $\mathbf{G}(\bar{F})$ -conjugacy class. For each such ι , the centralizer $\mathbf{G}_\iota := C_{\mathbf{G}}(\iota(\mathbf{S}_\phi))$ is a twisted Levi subgroup of \mathbf{G} and an inner form of \mathbf{G}_ϕ . The group X_ϕ acts simply transitively on \mathcal{I}_ϕ , but in general there is no natural base point.
- $\mathbf{G}_{\phi, I}$, for $I \in \mathcal{I}_\phi$, a reductive F -group in the isomorphism class of all centralizers \mathbf{G}_ι for $\iota \in I$. For any $\alpha \in X_\phi$, the group $\mathbf{G}_{\phi, \alpha, I}$ is the pure inner form of $\mathbf{G}_{\phi, I}$ associated to $\alpha \in X_\phi \subset H^1(F, \mathbf{G}_{\phi, I})$.

We refer to Sections 2.1 and 2.2 for more details on these objects. Here is a more faithful account of what is done in this paper.

Main result. (Take 2) — Assume that \mathbf{G} is tamely ramified and that p is odd, and not a torsion prime of \mathbf{G} , nor of $\hat{\mathbf{G}}$. Let R be a commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra.

i) To any $\phi \in \Phi(P_F, \mathbf{G})$ such that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$, we associate a non-zero Serre subcategory $\text{Rep}_R^\phi(G)$ of $\text{Rep}_R(G)$, and a decomposition with non-zero factors

$$\text{Rep}_R^\phi(G) = \prod_{I \in \mathcal{I}_\phi} \text{Rep}_R^{\phi, I}(G).$$

Moreover, $\text{Rep}_R^\phi(G)$ and $\text{Rep}_R^{\phi'}(G)$ are orthogonal if $\phi \neq \phi'$, and the category $\text{Rep}_R^1(G)$ associated to the trivial parameter is the depth-0 category (note that \mathcal{I}_1 is a singleton).

ii) If R is a $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra, we construct equivalences of categories

$$\mathcal{I}_{\phi, I} : \text{Rep}_R^1(G_{\phi, I}) \xrightarrow{\sim} \text{Rep}_R^{\phi, I}(G).$$

When $R \subseteq \overline{\mathbb{Z}}[\frac{1}{p}]$, it follows from [DL25, Thm 1.0.5] that each $\text{Rep}_R^{\phi, I}(G)$ is a block. Moreover, there is a G -conjugacy class of embeddings of extended Bruhat–Tits buildings $i : \mathcal{B}_{\phi, I} \hookrightarrow \mathcal{B}$ of the building $\mathcal{B}_{\phi, I}$ of $\mathbf{G}_{\phi, I}$ into the building \mathcal{B} of \mathbf{G} such that for any $x \in \mathcal{B}_{\phi, I}$ and any R -representation ρ of the stabilizer $(G_{\phi, I})_x$ of x in $G_{\phi, I}$ that is trivial on $(G_{\phi, I})_{x, 0+}$, we have

$$\mathcal{I}_{\phi, I} \left(\text{ind}_{(G_{\phi, I})_x}^{G_{\phi, I}}(\rho) \right) \simeq \text{ind}_{K_{\phi, I, i(x)}}^G \left(\kappa_{\phi, I, i(x)} \otimes_R \rho \right).$$

Here ind denotes compact induction, $K_{\phi, I, i(x)}$ is an explicit open subgroup of $G_{i(x)}$ that maps surjectively to $(G_{\phi, I})_x / (G_{\phi, I})_{x, 0+}$, and $\kappa_{\phi, I, i(x)}$ is an explicit representation, called a twisted Heisenberg–Weil representation and defined in 3.6.

iii) If p does not divide the order of the absolute Weyl group of \mathbf{G} , then

$$\text{Rep}_R(G) = \prod_{\phi \in \Phi(P_F, \mathbf{G})} \text{Rep}_R^\phi(G).$$

iv) Let \mathbf{P} be a parabolic F -subgroup of \mathbf{G} with Levi subgroup \mathbf{M} , and denote by i_P the associated parabolic induction functor, and by r_P the corresponding Jacquet functor.

(a) If ϕ is the image of $\phi_M \in \Phi(P_F, \mathbf{M})$ under the natural map $\Phi(P_F, \mathbf{M}) \rightarrow \Phi(P_F, \mathbf{G})$, and I is the image of $I_M \in \mathcal{I}_{\phi_M}$ under the natural map $\mathcal{I}_{\phi_M} \rightarrow \mathcal{I}_\phi$, then we have

$$i_P \left(\text{Rep}_R^{\phi_M, I_M}(M) \right) \subseteq \text{Rep}_R^{\phi, I}(G).$$

(b) Assuming that p does not divide the order of the absolute Weyl group of \mathbf{G} , we have

$$r_P(\text{Rep}_R^{\phi, I}(G)) \subseteq \prod_{(\phi_M, I_M) \mapsto (\phi, I)} \text{Rep}_R^{\phi_M, I_M}(M).$$

(c) Assuming further that $C_{\hat{\mathbf{G}}}(\phi) \subseteq \hat{\mathbf{M}}$, then i_P induces an equivalence of categories

$$\mathrm{Rep}_R^{\phi_M, I_M}(M) \xrightarrow{\sim} \mathrm{Rep}_R^{\phi, I}(G)$$

with quasi inverse the composition of r_P and the projection onto the (ϕ_M, I_M) -factor.

Let us comment on the hypotheses, the construction, and where to find proofs in the text.

- We recall that a torsion prime for \mathbf{G} is a prime that either is bad for the root system of \mathbf{G} , or divides the order of $\pi_1(\mathbf{G}_{\mathrm{der}})$. We refer to Section 2.2.9 for a discussion of the effect of the various hypotheses we impose on the prime p in our results. We also note that the assumption that \mathbf{G} is tamely ramified is actually implied by the existence of a ϕ whose centralizer is a Levi subgroup.
- Let us outline the construction of $\mathrm{Rep}_R^{\phi, I}(G)$, which is achieved in Section 2. The full subcategory $\mathrm{Rep}_R^{\phi, I}(G)$ is defined by a system of idempotents $(e_{\phi, I, x})_{x \in \mathcal{B}}$ by letting the objects of $\mathrm{Rep}_R^{\phi, I}(G)$ be $\{V \in \mathrm{Rep}_R(G), V = \sum_{x \in \mathcal{B}} e_{\phi, I, x} V\}$. For $x \in \mathcal{B}$, each idempotent $e_{\phi, I, x}$ is a sum of idempotents $e_{\phi, \iota, x}$ associated to embeddings $\iota \in I$ such that x belongs to the building $\mathcal{B}_\iota \subset \mathcal{B}$ of \mathbf{G}_ι . In turn, $e_{\phi, \iota, x}$ is the averaging idempotent along a character $\check{\phi}_{\iota, x}^+$ of a certain open pro- p -subgroups of $G_{x, 0+}$. This character is obtained by applying a construction of Yu [Yu01] that takes as input a sequence of characters of twisted Levi subgroups of \mathbf{G} . In this case, the smallest member of the twisted Levi subgroup sequence is \mathbf{G}_ι , and the product $\check{\varphi} : G_\iota \rightarrow \mu_{p^\infty} \rightarrow R^\times$ of the desired characters restricted to G_ι is obtained by the Borel–Langlands reciprocity applied to a 1-cocycle $\varphi : W_F \rightarrow Z(\hat{\mathbf{G}}_\phi) = Z(\hat{\mathbf{G}}_\phi)$ of p -power order that extends ϕ (whose existence is proved in Lemma 2.2.5). The intermediate twisted Levi subgroups and (products of their) characters are obtained in a similar way by replacing ϕ by its restriction to suitable higher ramification subgroups of P_F , following a strategy of Kaletha in [Kal19]. This is explained in Section 2.3. Yu’s construction is recalled in Section 2.4. Thanks to the strong intertwining properties of Yu’s characters (Section 2.5), it turns out that for two ι, ι' , the idempotents $e_{\phi, \iota, x}$ and $e_{\phi, \iota', x}$ are either equal or orthogonal for $x \in \mathcal{B}_\iota \cap \mathcal{B}_{\iota'}$. Summing over non-equal such idempotents, we get the desired idempotent $e_{\phi, I, x}$ (which can be zero) for any $x \in \mathcal{B}$. Their behavior when x is moving is studied in Section 2.6, and applied in Section 2.7 to the proof that $\mathrm{Rep}_R^{\phi, I}(G)$ is indeed a Serre subcategory (Theorem 2.7.1).
- The main properties of $\mathrm{Rep}_R^{\phi, I}(G)$ are proved in Section 2.8. The compatibility with parabolic induction, iv), is Theorem 2.8.7 and the decomposition of iii) is Theorem 2.8.4.
- The construction of the equivalences of categories in ii) is the subject of Section 3. The strategy is the following. In Section 3.2, we show how the categories $\mathrm{Rep}_R^{\phi, I}(G)$ and $\mathrm{Rep}_R^1(G_{\phi, I})$ are equivalent to certain categories of equivariant coefficient systems on the respective Bruhat–Tits buildings of \mathbf{G} and $\mathbf{G}_{\phi, I}$. Then in Section 3.3 we introduce the notion of a “Heisenberg–Weil” coefficient system on the Bruhat–Tits building of $\mathbf{G}_{\phi, I}$ and state an existence claim, Theorem 3.3.7. These gadgets will serve as a bridge between coefficient systems on both buildings, pretty much like a bimodule is used to produce Morita equivalences. We refer to the appendix of [Dat18] for more details on the analogy with

Morita equivalences. In this paper, our approach will be more akin to the familiar “pull-push with kernel” functors from sheaf theory, as explained in 3.4.1. The construction of these “Morita-type” or “pull-push” equivalences is achieved in Section 3.4, admitting Theorem 3.3.7. The proof of this theorem, i.e., the construction of a Heisenberg–Weil coefficient system, is obtained in Section 3.7, after preliminary work on Heisenberg representations in Section 3.5, and on Weil representations in Section 3.6, where we adapt classical and more recent results from coefficients \mathbb{C} to our coefficient ring R , which might be of independent interest.

Let us now comment on the connections with the existing literature on reduction to depth 0. Since our functor $\mathcal{I}_{\phi,I}$ is an equivalence, it induces an isomorphism between the endomorphism algebra of an object and that of its image. Therefore, the explicit description in ii) above of the evaluation of $\mathcal{I}_{\phi,I}$ on certain compactly induced representations implies the existence of isomorphisms between certain Hecke algebras, some of them being the key to previous reduction to depth 0 results.

- In the case $R = \mathbb{C}$ (or more generally when R is an algebraically closed field), we recover in 4.1.2 the existence of an isomorphism between the Hecke algebra attached to the types constructed by Kim and Yu and that of its “twisted depth-0 component”, as first proved in [AFMO24b]. Note that we do not compare explicitly both constructions. In particular, we do not verify here whether our isomorphism is “support-preserving”.
- In the case $R = \overline{\mathbb{Z}}[\frac{1}{p}]$ and G an inner form of GL_n , we recover in 4.2.2 the existence of an isomorphism between two relevant Hecke algebras arising in Bushnell–Kutzko’s type theory that were previously shown to be isomorphic by explicit computation of generators and relations by Chinello in [Chi18]. Again, we do not compare both isomorphisms. Note however that Chinello’s results do not need any tameness assumption.

Connections, and expected connections, with the local Langlands correspondence, its geometrization, and its categorification.

- When the local Langlands correspondence $\mathrm{Irr}_{\mathbb{C}}(G) \longrightarrow \Phi(W'_F, \mathbf{G})$, $\pi \mapsto \varphi_{\pi}$ is available for \mathbf{G} , we conjecture that for an irreducible complex representation π of G , we have $\pi \in \mathrm{Rep}_{\mathbb{C}}^{\phi}(G) \Leftrightarrow (\varphi_{\pi})|_{P_F} \sim \phi$. By design, this is at least true when π is a regular supercuspidal representation and φ_{π} is the parameter defined in [Kal19].

Assume further that \mathbf{G} is quasi-split and that the extended local Langlands correspondence $\pi \mapsto (\varphi_{\pi}, \varepsilon_{\pi})$ is available for \mathbf{G} , where ε_{π} is a finite order character of $C_{\hat{\mathbf{G}}}(\varphi_{\pi})$. Then we conjecture that, for a good choice of base point $I \in \mathcal{I}_{\phi}$, we have, for any irreducible complex representations π of G :

$$\pi \in \mathrm{Rep}_{\mathbb{C}}^{\phi, \alpha \cdot I}(G) \Leftrightarrow \left[(\varphi_{\pi})|_{P_F} \sim \phi \text{ and } (\varepsilon_{\pi})|_{Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi_{\pi}(W_F)}} = \alpha \right],$$

where the restriction is along the inclusion $Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi_{\pi}(W_F)} \subset C_{\hat{\mathbf{G}}}(\varphi_{\pi})$.

- Without any hypothesis on \mathbf{G} nor on ϕ , a construction of a direct factor $\mathrm{Rep}_R^{\phi}(G)$ and a decomposition as in iii) can be deduced from the motivic version of the Fargues–Scholze

spectral action of [Sch25]. Indeed, a consequence of this spectral action is the existence of a ring homomorphism $\mathrm{FS}_{\mathrm{mot}}$

$$\mathcal{O}(Z^1(W_F^0, \hat{\mathbf{G}}))^{\hat{\mathbf{G}}} \xleftarrow{\approx} \mathcal{E}_{\mathrm{xc}}(W_F^0, \hat{\mathbf{G}}) \xrightarrow{\mathrm{FS}_{\mathrm{mot}}} \mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G),$$

where $Z^1(W_F^0, \hat{\mathbf{G}})$ is the space of parameters over $\overline{\mathbb{Z}}[\frac{1}{p}]$ introduced in [DHKM25] (so that here $\hat{\mathbf{G}}$ denotes the dual over $\overline{\mathbb{Z}}[\frac{1}{p}]$ and W_F^0 is the subgroup of W_F defined in [DHKM25, §1.2]), $\mathcal{E}_{\mathrm{xc}}$ denotes an “excursion algebra” and the \approx sign indicates a map of rings that induces a universal homeomorphism on spectra, and $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ denotes the center of $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. According to [DHKM25], the stack $Z^1(W_F^0, \hat{\mathbf{G}})/\hat{\mathbf{G}}$ is the disjoint union over all $\phi \in \Phi(P_F, \mathbf{G})$ of open closed substacks $Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}/C_{\hat{\mathbf{G}}}(\phi)$, where a representative $\phi : P_F \rightarrow \hat{\mathbf{G}}$ has been chosen and $Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}$ is the space of extensions of ϕ to W_F^0 . So each ϕ provides an idempotent e_{ϕ}^{spec} in $\mathcal{O}(Z^1(W_F^0, \hat{\mathbf{G}}))^{\hat{\mathbf{G}}}$, that provides in turn an idempotent e_{ϕ}^{geom} in $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$, and we get a decomposition $\mathrm{Rep}_R(G) = \prod_{\phi} e_{\phi}^{\mathrm{geom}} \mathrm{Rep}_R(G)$. In general, not much is known about this decomposition, but we conjecture that it coincides with ours when applicable. Proving this is equivalent to proving that the Fargues–Scholze semisimple correspondence for the so-called “regular cuspidal” representations of Levi subgroups of G matches the semisimplification of Kaletha’s correspondence in [Kal19].

- In our context, and more generally whenever $C_{\hat{\mathbf{G}}}(\phi)$ is connected, the scheme $Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}$ is connected and the idempotent e_{ϕ}^{spec} is thus primitive. So our decomposition of $\mathrm{Rep}_{\hat{\mathbf{G}}}^{\phi}(G)$ does not seem to be explained by the spectral action. However, it would be explained by (a motivic form of) the categorical local Langlands correspondence (CLLC below). To explain this, recall that the CLLC predicts an equivalence of categories between some variant $\mathrm{IndCoh}_{\mathrm{nilp}}(Z^1(W_F^0, \hat{\mathbf{G}})/\hat{\mathbf{G}})$ of the category of coherent sheaves on $Z^1(W_F^0, \hat{\mathbf{G}})/\hat{\mathbf{G}}$ and a category of sheaves Shv_G on either the v -stack Bun_G of [FS24], or the perfect stack Isoc_G of [Zhu25]. In each case, $\mathrm{Rep}(G)$ identifies with the subcategory of sheaves supported on a certain stratum “ $b = 1$ ”, which is isomorphic to the classifying space of G . The connected components of both these stacks are labeled by $X^*(Z(\hat{\mathbf{G}})^{W_F})$, and the resulting decomposition of Shv_G corresponds, on the parameters side, to the grading coming from the triviality of the action of $Z(\hat{\mathbf{G}})^{W_F}$ on $Z^1(W_F^0, \hat{\mathbf{G}})$. The stratum “ $b = 1$ ” lies in the component corresponding to the trivial character 1 of $Z(\hat{\mathbf{G}})^{W_F}$. Now recall further that the spectral action is defined over Shv_G , yielding a direct summand $e_{\phi}^{\mathrm{geom}} \mathrm{Shv}_G$, so that the CLLC should restrict to an equivalence between $\mathrm{IndCoh}_{\mathrm{nilp}}(Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}/C_{\hat{\mathbf{G}}}(\phi))$ and $e_{\phi}^{\mathrm{geom}} \mathrm{Shv}_G$. Since $Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F)}$ acts trivially on $Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}$, the category $\mathrm{IndCoh}_{\mathrm{nilp}}(Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}/C_{\hat{\mathbf{G}}}(\phi))$ is graded over $X^*(Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F)})$. We then infer a splitting of $e_{\phi}^{\mathrm{geom}} \mathrm{Shv}_G$ indexed by $X^*(Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F)})$ that refines the one indexed by $X^*(Z(\hat{\mathbf{G}})^{W_F})$. When we restrict to the stratum $b = 1$, only the summands labeled by X_{ϕ} can contribute, yielding a decomposition of $e_{\phi}^{\mathrm{geom}} \mathrm{Rep}(G)$ indexed by X_{ϕ} , as in our results above.
- Our equivalences also fit in the framework of the CLLC. Namely, in our setting there exists $\varphi \in Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}(\overline{\mathbb{Z}}[\frac{1}{p}])$ such that $\varphi(W_F)$ normalizes a pinning of $C_{\hat{\mathbf{G}}}(\phi)$. Multiplication by φ then induces an isomorphism $Z^1(W_F^0, \hat{\mathbf{G}})_{\phi,1}/\hat{\mathbf{G}}_{\phi} \xrightarrow{\sim} Z^1(W_F^0, \hat{\mathbf{G}})_{\phi}/C_{\hat{\mathbf{G}}}(\phi)$. Applying

the CLLC for both \mathbf{G} and \mathbf{G}_ϕ , we would get an equivalence $e_1^{\text{geom}} \text{Shv}_{G_\phi} \xrightarrow{\sim} e_\phi^{\text{geom}} \text{Shv}_G$. Of course, it would need further work to see whether this equivalence restricts well to the $b = 1$ strata, and is t -exact there, for the standard t -structure. Our result suggests that it actually does.

- Some important particular cases of the categorical Langlands correspondence have been recently proved on Isoc_G by X. Zhu. They are concerned with the tame part over $\overline{\mathbb{Q}}_\ell$ -coefficients and the unipotent part over $\overline{\mathbb{F}}_\ell$ -coefficients. In particular, [Zhu25, Thm. 5.4] constructs a fully faithful embedding $D\text{Rep}_{\overline{\mathbb{F}}_\ell}(G)_{\text{unip}} \hookrightarrow \text{IndCoh}(Z^1(W_F^0, \hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell})_{\text{unip}}/\hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell})$ when \mathbf{G} is unramified, and where $\hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell}$ denotes a Langlands dual group over $\overline{\mathbb{F}}_\ell$, $D\text{Rep}_{\overline{\mathbb{F}}_\ell}(G)$ denotes the derived (∞)-category of $\text{Rep}_{\overline{\mathbb{F}}_\ell}(G)$, the subscript unip denotes the unipotent summand on the left hand side, resp., the unipotent connected component (i.e. the one that contains the trivial parameter) on the right hand side.

In the case $\mathbf{G} = \text{GL}_n$, one can combine our “reduction to depth 0” result here with the “reduction to unipotent” result of [Dat18]. Recall that, in this case, Vignéras decomposed $\text{Rep}_{\overline{\mathbb{F}}_\ell}(G)$ in [Vig98], as a product of blocks indexed by semisimple representations $W_F \longrightarrow \text{GL}_n(\overline{\mathbb{F}}_\ell)$ up to inertial equivalence, while the same indexing set also parametrizes the connected components of $Z^1(W_F^0, \hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell})$, according to [DHKM25, Cor 4.21].

Corollary. — *Let $\mathbf{G} = \text{GL}_n$, and let p and ℓ be greater than n . Then there is a fully faithful embedding $D\text{Rep}_{\overline{\mathbb{F}}_\ell}(G) \hookrightarrow \text{IndCoh}(Z^1(W_F^0, \hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell})/\hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell})$ that sends $D\text{Rep}_{\overline{\mathbb{F}}_\ell}(G)_{[\varphi]}$ into $\text{IndCoh}(Z^1(W_F^0, \hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell})_{[\varphi]}/\hat{\mathbf{G}}_{\overline{\mathbb{F}}_\ell})$ for each inertia class $[\varphi]$ of semi-simple representations $W_F \longrightarrow \text{GL}_n(\overline{\mathbb{F}}_\ell)$.*

Actually, we construct a whole class of embeddings as in this corollary, and we do not quite specify the “correct” one, i.e., the one that is compatible with Vignéras’ correspondence. We refer to 4.3.3 for a discussion of this question. Among the ingredients needed to find the correct one are the compatibility of our equivalences with parabolic induction, as well as the compatibility with Whittaker/Gelfand–Graev representations. We plan to address these points in future work.

1.1 Notation

- F denotes a non-archimedean local field whose residue field characteristic is denoted by p . We fix a separable closure \overline{F} of F and denote by Γ_F Galois group of \overline{F}/F . Inside Γ_F we have the usual subgroups $W_F \supset I_F \supset P_F$, respectively, the Weil group, its inertia subgroup, and the wild inertia subgroup.
- Bold letters \mathbf{G} , \mathbf{B} , \mathbf{T} , etc., denote algebraic groups over F , unless specified otherwise, and we use the corresponding plain letters to denote their groups of F -rational points, e.g., $G = \mathbf{G}(F)$.
- For a connected reductive group \mathbf{G} over some field, we denote by \mathbf{G}_{ad} the adjoint quotient group of \mathbf{G} , and by \mathbf{G}_{sc} the simply-connected cover of the derived subgroup of \mathbf{G} .
- R will denote the commutative coefficient ring of our representations. We always assume $p \in R^\times$, and most often will assume that R is a $\mathbb{Z}[\frac{1}{p}, \mu_{p^\infty}]$ -algebra.

- For a locally pro- p -group H ,
 - $\text{Rep}_R(H)$ denotes the category of smooth R -representations of H .
 - $R[H]$ denotes the group algebra of H .
 - RH denotes the R -algebra of compactly supported R -valued distributions on H . It contains $R[H]$ as the span of Dirac distributions, as well as averaging idempotents e_K for any closed pro- p -subgroup $K \subseteq H$. The action of $R[H]$ on $V \in \text{Rep}_R(H)$ extends canonically to RH , identifying $\text{Rep}_R(H)$ with the category of smooth RH -modules.
 - $\mathcal{H}_R(H)$ denotes the R -subalgebra of RH consisting of locally constant distributions. This non-unital ring contains averaging idempotents e_K for all open pro- p -subgroups $K \subseteq H$. By restricting the action of RH , $\text{Rep}_R(H)$ identifies with the category of non-degenerate $\mathcal{H}_R(H)$ -modules.
- For a connected reductive group \mathbf{G} over F ,
 - $\hat{\mathbf{G}}$ denotes the complex² dual reductive group of \mathbf{G} , which comes with a pinning $(\hat{\mathbf{B}}, \hat{\mathbf{T}}, \hat{X})$ and a pinning-preserving action of Γ_F . We denote by ${}^L\mathbf{G} := \mathbf{G} \rtimes W_F$ the Weil form of Langlands’ dual group.
 - ϕ will denote a *wild inertia parameter* for \mathbf{G} , i.e., a continuous L -homomorphism $P_F \rightarrow {}^L\mathbf{G}$, that admits an extension to a relevant Langlands parameter $\varphi : W'_F \rightarrow {}^L\mathbf{G}$, where W'_F denotes any form the Weil–Deligne group of F .
 - $\Phi(P_F, \mathbf{G})$ denotes the set of $\hat{\mathbf{G}}$ -conjugacy classes of wild inertia parameters for \mathbf{G} .

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2 From parameters to subcategories

Throughout the paper, \mathbf{G} denotes a connected reductive group over F .

2.1 Levi-center-embeddings and duality

From the construction of the dual group $\hat{\mathbf{G}}$ we have a bijection between $\mathbf{G}(\overline{F})$ -conjugacy classes of *maximal* \overline{F} -tori embeddings $\mathbf{S} \hookrightarrow \mathbf{G}_{\overline{F}}$ and $\hat{\mathbf{G}}$ -conjugacy classes of *maximal* tori embeddings $\hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$. We seek a generalization of this for embeddings of tori as connected center of a Levi subgroup.

²For many purposes, including the relation with the geometrization and the categorification of the LLC, it would be more natural to consider the dual group over the coefficient ring R . However, the complex setting will be sufficient in this paper, and saves us additional technical difficulties.

2.1.1 Lemma. — Let \mathbf{S} be a \overline{F} -torus contained in $\mathbf{G}_{\overline{F}}$. The following are equivalent

i) \mathbf{S} is the connected center of a Levi subgroup of $\mathbf{G}_{\overline{F}}$,

ii) $\mathbf{S} = C_{\mathbf{G}_{\overline{F}}}(C_{\mathbf{G}_{\overline{F}}}(\mathbf{S}))^\circ$,

iii) there is a maximal \overline{F} -torus \mathbf{T} of $\mathbf{G}_{\overline{F}}$ containing \mathbf{S} and a Levi subroot system $\Sigma' \subseteq \Sigma(\mathbf{T}, \mathbf{G}_{\overline{F}})$ of the root system of $\mathbf{G}_{\overline{F}}$ with respect to \mathbf{T} such that $\mathbf{S} = \left(\bigcap_{\alpha \in \Phi'} \ker(\alpha)\right)^\circ$.

Proof. Standard, cf., for example, [Bor91, 14.18]. \square

2.1.2 Definition.— A *Levi-center-embedding* in $\mathbf{G}_{\overline{F}}$ is a pair (\mathbf{S}, ι) with \mathbf{S} a \overline{F} -torus and $\iota : \mathbf{S} \hookrightarrow \mathbf{G}_{\overline{F}}$ an embedding such that $\iota(\mathbf{S})$ satisfies the properties of Lemma 2.1.1.

2.1.3 Duality. — Let $\hat{\mathbf{S}}$ be a complex algebraic torus. The algebraic group $\hat{\mathbf{G}}$ acts by conjugation on Levi-center-embeddings $\hat{\iota} : \hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$. Our aim is to attach to a conjugacy class $\{\hat{\iota}\}$ of such embeddings, a “dual” conjugacy class $\{\iota\}$ of Levi-center-embeddings in the \overline{F} -algebraic group $\mathbf{G}_{\overline{F}}$.

The stabilizer $\hat{\mathbf{G}}_{\hat{\iota}}$ of $\hat{\iota}$ for the action induced by the conjugation action of $\hat{\mathbf{G}}$ on itself is the centralizer $C_{\hat{\mathbf{G}}}(\hat{\iota}(\hat{\mathbf{S}}))$ of the torus $\hat{\iota}(\hat{\mathbf{S}})$, which is a Levi subgroup of $\hat{\mathbf{G}}$. The embedding $\hat{\iota}$ induces an isogeny from $\hat{\mathbf{S}}$ to the cocenter $\hat{\mathbf{G}}_{\hat{\iota}, \text{ab}}$ of $\hat{\mathbf{G}}_{\hat{\iota}}$, which is a torus. An important observation is that *the kernel* $H_{\{\hat{\iota}\}} := \ker(\hat{\mathbf{S}} \rightarrow \hat{\mathbf{G}}_{\hat{\iota}, \text{ab}})$ *of this isogeny only depends on the conjugacy class* $\{\hat{\iota}\}$ *of* $\hat{\iota}$. We then set $\hat{\mathbf{S}}_{\{\hat{\iota}\}} := \hat{\mathbf{S}}/H_{\{\hat{\iota}\}}$.

Now let us choose a maximal torus $\hat{\mathbf{T}}$ in $\hat{\mathbf{G}}_{\hat{\iota}}$. The isogeny $\hat{\mathbf{S}} \rightarrow \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ factors as $\hat{\mathbf{S}} \xrightarrow{\hat{\iota}} \hat{\mathbf{T}} \xrightarrow{\hat{\pi}} \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ where $\hat{\pi}$ is the composition of the projection $\hat{\mathbf{T}} \rightarrow \hat{\mathbf{G}}_{\hat{\iota}, \text{ab}}$ and the inverse of the isomorphism $\hat{\mathbf{S}}_{\{\hat{\iota}\}} \xrightarrow{\sim} \hat{\mathbf{G}}_{\hat{\iota}, \text{ab}}$ induced by $\hat{\iota}$. We then have isomorphisms

$$\hat{\iota} : \hat{\mathbf{S}} \xrightarrow{\sim} \left(\bigcap_{\alpha \in \Sigma_{\hat{\iota}}} \ker(\alpha) \right)^\circ \subset \hat{\mathbf{T}} \quad \text{and} \quad \hat{\pi} : \hat{\mathbf{T}} / \left(\sum_{\alpha \in \Sigma_{\hat{\iota}}} \text{im}(\alpha^\vee) \right) \xrightarrow{\sim} \hat{\mathbf{S}}_{\{\hat{\iota}\}},$$

where $\Sigma_{\hat{\iota}}$ is the root system $\Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}}_{\hat{\iota}})$. Hence dualizing $\hat{\mathbf{S}} \xrightarrow{\hat{\iota}} \hat{\mathbf{T}} \xrightarrow{\hat{\pi}} \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ we see that the dual isogeny $\mathbf{S}_{\{\hat{\iota}\}} \rightarrow \mathbf{S}$ of tori over \overline{F} factors through the dual \overline{F} -torus \mathbf{T} of $\hat{\mathbf{T}}$ giving isomorphisms

$$\pi : \mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\sim} \left(\bigcap_{\alpha \in \Sigma_{\hat{\iota}}} \ker(\alpha^\vee) \right)^\circ \subset \mathbf{T} \quad \text{and} \quad \hat{\iota} : \mathbf{T} / \left(\sum_{\alpha \in \Sigma_{\hat{\iota}}} \text{im}(\alpha) \right) \xrightarrow{\sim} \mathbf{S},$$

where α^\vee , resp., α , is seen as a character, resp., a cocharacter, of \mathbf{T} . Now recall that the embedding $\hat{\mathbf{T}} \hookrightarrow \hat{\mathbf{G}}$ gives rise to a canonical $\mathbf{G}(\overline{F})$ -conjugacy class of embeddings $\mathbf{T} \hookrightarrow \mathbf{G}_{\overline{F}}$. Choose such a “dual embedding” j . By construction it identifies $\Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}})$ with $\Sigma(\mathbf{T}, \mathbf{G}_{\overline{F}})^\vee$. In particular, we see from point iii) in Lemma 2.1.1 that the composition $\iota : \mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\pi} \mathbf{T} \hookrightarrow \mathbf{G}_{\overline{F}}$ is a Levi-center embedding such that, by construction, the Levi subgroup $\mathbf{G}_\iota = C_{\mathbf{G}_{\overline{F}}}(\iota(\mathbf{S}_{\{\hat{\iota}\}}))$ is dual to $\hat{\mathbf{G}}_{\hat{\iota}}$.

Lemma. — The $\mathbf{G}(\overline{F})$ -conjugacy class $\{\iota\}$ of ι only depends on the $\hat{\mathbf{G}}$ -conjugacy class $\{\hat{\iota}\}$.

Proof. Let $\hat{\iota}'$ be conjugate to $\hat{\iota}$, let $\hat{\mathbf{T}}'$ be a maximal torus in $\hat{\mathbf{G}}_{\ell'}$ and let $j' : \mathbf{T}' \hookrightarrow \mathbf{G}_{\overline{F}}$ be a choice of dual embedding. Since all maximal tori of $\hat{\mathbf{G}}_{\ell}$ are conjugate, there is an element $\hat{g} \in \hat{\mathbf{G}}$ which conjugates $\hat{\iota}$ to $\hat{\iota}'$ and $\hat{\mathbf{T}}$ to $\hat{\mathbf{T}}'$. Then we have a commutative diagram

$$\begin{array}{ccc} \hat{\mathbf{S}} & \xrightarrow{\hat{\iota}} & \hat{\mathbf{T}} \\ \hat{\iota}' \downarrow & \text{Ad}_{\hat{g}} \swarrow \sim & \downarrow \hat{\pi} \\ \hat{\mathbf{T}}' & \xrightarrow{\hat{\pi}'} & \hat{\mathbf{S}}_{\{\hat{\iota}\}} \end{array} .$$

It follows that on the dual side we get $\pi = \hat{\text{Ad}}_{\hat{g}} \circ \hat{\pi}'$, where $\hat{\text{Ad}}_{\hat{g}}$ is the isomorphism $\mathbf{T}' \xrightarrow{\sim} \mathbf{T}$ dual to $\text{Ad}_{\hat{g}}$. On the other hand, $j \circ \hat{\text{Ad}}_{\hat{g}}$ is a dual embedding of \mathbf{T}' into $\mathbf{G}_{\overline{F}}$, hence there is $g \in \mathbf{G}(\overline{F})$ such that $\text{Ad}_g \circ j' = j \circ \hat{\text{Ad}}_{\hat{g}}$. It follows that g conjugates the embedding $\iota' = j' \circ \hat{\pi}'$ to the embedding $\iota = j \circ \pi$. \square

We refer to $\{\iota\}$ as the $\mathbf{G}(\overline{F})$ -conjugacy class of Levi-center-embeddings that is dual to $\{\hat{\iota}\}$. Although we will not need it in this paper, note that we can play the game in the other direction and get the following result.

Proposition. — *The above construction sets up a bijection between $\mathbf{G}(\overline{F})$ -conjugacy classes of Levi-center-embeddings in $\mathbf{G}_{\overline{F}}$ and \mathbf{G} -conjugacy classes of Levi-center-embeddings in $\hat{\mathbf{G}}$.*

Proof. It follows from the above construction that the dual of the dual of a conjugacy class of Levi-center-embeddings is the initial conjugacy class of Levi-center-embeddings. \square

2.1.4 Rationality. — We now assume that the complex torus $\hat{\mathbf{S}}$ is endowed with a finite action of W_F , thus corresponding to an F -rational structure on the torus \mathbf{S} . Then W_F acts on the set of Levi-center-embeddings $\hat{\iota} : \hat{\mathbf{S}} \rightarrow \hat{\mathbf{G}}$ by the formula ${}^\gamma \hat{\iota} := \gamma_{\hat{\mathbf{G}}} \circ \hat{\iota} \circ \gamma_{\hat{\mathbf{S}}}^{-1}$. We further assume that the conjugacy class $\{\hat{\iota}\}$ is W_F -stable.

In this case, the finite subgroup $\hat{H}_{\{\hat{\iota}\}}$ of $\hat{\mathbf{S}}$ is W_F -stable, its quotient torus $\hat{\mathbf{S}}_{\{\hat{\iota}\}}$ is therefore also equipped with a finite action of W_F , allowing to define an F -structure on the dual torus $\mathbf{S}_{\{\hat{\iota}\}}$. We also define a quasi-split F -group $\mathbf{G}_{\{\hat{\iota}\}}$ as follows. First note that we have an action of the L -group ${}^L \mathbf{G} = \hat{\mathbf{G}} \rtimes W_F$ on Levi-center-embeddings in $\hat{\mathbf{G}}$ given by ${}^{(\hat{g}, \gamma)} \hat{\iota} := \text{Ad}_{\hat{g}} \circ \gamma \hat{\iota}$, and that the $\hat{\mathbf{G}}$ -conjugacy class $\{\hat{\iota}\}$ is W_F -stable if and only if the stabilizer $({}^L \mathbf{G})_{\hat{\iota}}$ surjects onto W_F through the projection ${}^L \mathbf{G} \rightarrow W_F$. Since we assumed that the $\hat{\mathbf{G}}$ -conjugacy class $\{\hat{\iota}\}$ is W_F -stable, we therefore obtain a short exact sequence $\hat{\mathbf{G}}_{\hat{\iota}} \hookrightarrow ({}^L \mathbf{G})_{\hat{\iota}} \twoheadrightarrow W_F$. It follows that the conjugation action $({}^L \mathbf{G})_{\hat{\iota}} \rightarrow \text{Aut}(\hat{\mathbf{G}}_{\hat{\iota}})$ induces an outer action

$$W_F \longrightarrow \text{Out}(\hat{\mathbf{G}}_{\hat{\iota}}) = \text{Aut}(\psi_0(\hat{\mathbf{G}}_{\hat{\iota}})),$$

where ψ_0 denotes the based root datum associated to a reductive group. For any conjugate $\hat{\iota}'$, this outer action is compatible with the canonical isomorphism $\psi_0(\hat{\mathbf{G}}_{\hat{\iota}}) \xrightarrow{\sim} \psi_0(\hat{\mathbf{G}}_{\hat{\iota}'})$ induced by conjugation under any \hat{g} such that $\hat{\iota}' = \text{Ad}_{\hat{g}} \circ \hat{\iota}$. Further, this outer action is *finite* since it induces the given action on the connected center $\hat{\mathbf{S}}$ of $\hat{\mathbf{G}}_{\hat{\iota}}$. Therefore, there is a quasi-split F -group $\mathbf{G}_{\{\hat{\iota}\}}$ endowed with a W_F -equivariant isomorphism $\alpha : \psi_0(\mathbf{G}_{\{\hat{\iota}\}}) \xrightarrow{\sim} \psi_0(\hat{\mathbf{G}}_{\hat{\iota}})^\vee$. This pair is unique up to isomorphism and its automorphism group is $\mathbf{G}_{\{\hat{\iota}\}, \text{ad}}(F)$. Thanks to α , we have F -rational isomorphisms

$$\mathbf{G}_{\{\hat{\iota}\}, \text{ab}} \xrightarrow{\sim} \mathbf{S} \quad \text{and} \quad \mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\sim} Z(\mathbf{G}_{\{\hat{\iota}\}})^\circ.$$

Moreover, and again thanks to α , we also have a map

$$H^1(F, \mathbf{G}_{\{\hat{\iota}\}}) \longrightarrow H^1(F, \mathbf{G}),$$

defined through ‘‘Kottwitz duality’’ [Kot84, Prop. 6.4] by the inclusion $Z(\hat{\mathbf{G}})^{W_F} \subset Z(\hat{\mathbf{G}}_{\hat{\iota}})^{W_F}$ where the action of W_F on $Z(\hat{\mathbf{G}}_{\hat{\iota}})$ is induced by the conjugation action of $({}^L\mathbf{G})_{\hat{\iota}}$.³

2.1.5 Proposition. — *Assume that \mathbf{G} is quasi-split over F . Then the $\mathbf{G}(\overline{F})$ -conjugacy class of Levi-center-embeddings $\{\mathbf{S}_{\{\hat{\iota}\}} \hookrightarrow \mathbf{G}_{\overline{F}}\}$ dual to $\{\hat{\iota}\}$ contains an F -rational embedding ι whose stabilizer $\mathbf{G}_{\iota} := C_{\mathbf{G}}(\iota(\mathbf{S}_{\{\hat{\iota}\}}))$ is naturally isomorphic to $\mathbf{G}_{\{\hat{\iota}\}}$. Moreover, the map $H^1(F, \mathbf{G}_{\iota}) \longrightarrow H^1(F, \mathbf{G})$ induced by the inclusion $\mathbf{G}_{\iota} \subseteq \mathbf{G}$ coincides with the map defined above.*

Here, ‘‘naturally’’ isomorphic means that there is an isomorphism unique up to inner automorphism, or equivalently that there is a W_F -equivariant isomorphism between the associated based root data.

Proof. Let us choose a maximal torus $\hat{\mathbf{T}} \subset \hat{\mathbf{G}}_{\hat{\iota}}$ and a Borel subgroup $\hat{\mathbf{B}}$ of $\hat{\mathbf{G}}_{\hat{\iota}}$ that contains $\hat{\mathbf{T}}$. Let $\mathcal{T}_{\hat{\mathbf{B}}}$ be the normalizer of the Borel pair $(\hat{\mathbf{T}}, \hat{\mathbf{B}})$ in $({}^L\mathbf{G})_{\hat{\iota}}$. Since $\hat{\mathbf{G}}_{\hat{\iota}}$ acts transitively on the set of its Borel pairs, we see that the map $\mathcal{T}_{\hat{\mathbf{B}}} \longrightarrow W_F$ is surjective, and we have a short exact sequence $\hat{\mathbf{T}} \hookrightarrow \mathcal{T}_{\hat{\mathbf{B}}} \twoheadrightarrow W_F$. In particular, the conjugation action of $\mathcal{T}_{\hat{\mathbf{B}}}$ on $\hat{\mathbf{T}}$ factors through an action of W_F on $\hat{\mathbf{T}}$. By construction, this action preserves the based root datum $\psi_{(\hat{\mathbf{T}}, \hat{\mathbf{B}})} = (X^*(\hat{\mathbf{T}}), \Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}}_{\hat{\iota}}), \Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}}), X_*(\hat{\mathbf{T}}), \Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}}_{\hat{\iota}})^\vee, \Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}})^\vee)$ of $\hat{\mathbf{G}}_{\hat{\iota}}$ associated to the Borel pair $(\hat{\mathbf{T}}, \hat{\mathbf{B}})$ and the induced action $W_F \longrightarrow \text{Aut}(\psi_{(\hat{\mathbf{T}}, \hat{\mathbf{B}})})$ coincides with the outer action $W_F \longrightarrow \text{Aut}(\psi_0(\hat{\mathbf{G}}_{\hat{\iota}}))$ defined before the proposition, through the canonical isomorphism $\psi_{(\hat{\mathbf{T}}, \hat{\mathbf{B}})} = \psi_0(\hat{\mathbf{G}}_{\hat{\iota}})$.

In particular the action of W_F on $\hat{\mathbf{T}}$ is finite and induces the given action of W_F on $\hat{\mathbf{S}}$, so that the whole factorization $\hat{\mathbf{S}} \xrightarrow{\hat{\iota}} \hat{\mathbf{T}} \xrightarrow{\hat{\pi}} \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ is W_F -equivariant. Therefore, endowing the dual \overline{F} -torus \mathbf{T} with the F -structure associated with this W_F -action, the dual morphism $\mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\hat{\pi}} \mathbf{T}$ is defined over F . But since $\mathcal{T}_{\hat{\mathbf{B}}}$ surjects onto W_F , any W_F -conjugate of the embedding $\hat{\mathbf{T}} \subset \hat{\mathbf{G}}$ is also $\hat{\mathbf{G}}$ -conjugate to it. In other words, the $\hat{\mathbf{G}}$ -conjugacy class of this embedding is W_F -stable. It follows that the dual $\mathbf{G}(\overline{F})$ -conjugacy class of embeddings $\mathbf{T}_{\overline{F}} \hookrightarrow \mathbf{G}_{\overline{F}}$ is also Galois stable. Since \mathbf{G} is quasisplit, we know by [Rag04] that there is a dual embedding $j : \mathbf{T} \hookrightarrow \mathbf{G}$ defined over F . Then the composite $\iota = j \circ \hat{\pi}$ is also defined over F .

Now, the stabilizer $\mathbf{G}_{\iota} = C_{\mathbf{G}}(\iota(\mathbf{S}_{\{\hat{\iota}\}}))$ is an F -subgroup of \mathbf{G} , with $j(\mathbf{T})$ a maximal F -torus. By construction, the W_F action on $\hat{\mathbf{T}}$ preserves the basis $\Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}})$ of the root system $\Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}}_{\hat{\iota}})$, therefore the W_F -action on $\mathbf{T}_{\overline{F}}$ preserves a basis of the absolute root system $\Sigma(j(\mathbf{T}), \mathbf{G}_{\iota})$, which determines a Borel subgroup \mathbf{B} of \mathbf{G}_{ι} defined over F and containing $j(\mathbf{T})$. The associated based root datum $\psi_{(j(\mathbf{T}), \mathbf{B})}$ of \mathbf{G}_{ι} is then W_F -equivariantly dual to $\psi_{(\hat{\mathbf{T}}, \hat{\mathbf{B}})}$, and this provides a W_F -equivariant isomorphism $\psi_0(\mathbf{G}_{\iota}) \xrightarrow{\sim} \psi_0(\hat{\mathbf{G}}_{\hat{\iota}})$, hence a whole class of F -rational isomorphisms $\mathbf{G}_{\iota} \xrightarrow{\sim} \mathbf{G}_{\{\hat{\iota}\}}$ modulo inner automorphisms.

It remains to check that the map $H^1(F, \mathbf{G}_{\iota}) \longrightarrow H^1(F, \mathbf{G})$ induced by the inclusion $\mathbf{G}_{\iota} \subseteq \mathbf{G}$ coincides with the map $H^1(F, \mathbf{G}_{\{\hat{\iota}\}}) \longrightarrow H^1(F, \mathbf{G})$ defined before the proposition through any such isomorphism. We have Kottwitz’ isomorphisms $\xi_{\mathbf{G}} : H^1(F, \mathbf{G}) \xrightarrow{\sim} \pi_0(Z(\hat{\mathbf{G}})^{W_F})^*$ and $\xi_{\mathbf{G}_{\iota}} :$

³We could also write $Z({}^L\mathbf{G})$ for $Z(\hat{\mathbf{G}})^{W_F}$ and $Z({}^L\mathbf{G})_{\hat{\iota}}$ for $Z(\hat{\mathbf{G}}_{\hat{\iota}})^{W_F}$ since the center of W_F is trivial.

$H^1(F, \mathbf{G}_\iota) \xrightarrow{\sim} \pi_0(Z(\widehat{\mathbf{G}}_\iota)^{W_F})^*$ and a canonical W_F -equivariant isomorphism $Z(\widehat{\mathbf{G}}_\iota) = Z(\widehat{\mathbf{G}}_\iota)$, so the question is a matter of compatibility of Kottwitz' isomorphisms with the inclusion maps $\mathbf{G}_\iota \subseteq \mathbf{G}$ on one side, and $Z(\widehat{\mathbf{G}}) \subseteq Z(\widehat{\mathbf{G}}_\iota)$ on the other side. This compatibility easily follows from Kottwitz' argument in [Kot84, Prop 6.4]. Indeed, assume first that \mathbf{G}_{der} is simply connected, so that also $\mathbf{G}_{\iota, \text{der}}$ is simply connected. Then $\xi_{\mathbf{G}}$ and $\xi_{\mathbf{G}_\iota}$ factor as follows

$$\begin{array}{ccccc} \xi_{\mathbf{G}_\iota} : H^1(F, \mathbf{G}_\iota) & \xrightarrow{\sim} & H^1(F, \mathbf{G}_{\iota, \text{ab}}) & \xrightarrow{\sim} & \pi_0(Z(\widehat{\mathbf{G}}_\iota)^{W_F})^* \\ \downarrow & & \downarrow & & \downarrow \\ \xi_{\mathbf{G}} : H^1(F, \mathbf{G}) & \xrightarrow{\sim} & H^1(F, \mathbf{G}_{\text{ab}}) & \xrightarrow{\sim} & \pi_0(Z(\widehat{\mathbf{G}})^{W_F})^* \end{array}$$

where the first square is obviously commutative (since it is obtained by applying $H^1(F, -)$ to a commutative diagram of algebraic F -groups) and the second square is also commutative since it can be reduced to local duality for tori, which is functorial. Now, to tackle the general case, Kottwitz considers a central extension \mathbf{H} of \mathbf{G} by an anisotropic torus \mathbf{Z} such that \mathbf{H}_{der} is simply connected. Then the fiber product $\mathbf{H}_\iota = \mathbf{G}_\iota \times_{\mathbf{H}} \mathbf{G}$ is a central extension of \mathbf{G}_ι by \mathbf{Z} with simply connected derived subgroup. We have just seen that the diagram

$$\begin{array}{ccc} H^1(F, \mathbf{H}_\iota) & \xrightarrow[\xi_{\mathbf{H}_\iota}]{\sim} & \pi_0(Z(\widehat{\mathbf{H}}_\iota)^{W_F})^* \\ \downarrow & & \downarrow \\ H^1(F, \mathbf{H}) & \xrightarrow[\xi_{\mathbf{H}}]{\sim} & \pi_0(Z(\widehat{\mathbf{H}})^{W_F})^* \end{array}$$

is commutative. It is moreover equivariant for the action of $H^1(F, \mathbf{Z})$ given as usual on the first column and through $\pi_0(\widehat{\mathbf{Z}}^{W_F})^*$ on the right column. But Kottwitz shows that the diagram we are interested in (with \mathbf{G} 's instead of \mathbf{H} 's) is obtained from this one by modding out by this action. Therefore this diagram is commutative too. \square

When \mathbf{G} is not quasi-split, there may be no F -rational Levi-center-embedding $\iota : \mathbf{S}_{\{\iota\}} \hookrightarrow \mathbf{G}$ dual to $\{\iota\}$. We call $\{\iota\}$ *relevant to \mathbf{G}* if there exists such an F -rational embedding ι . We will make a connection with the notion of relevance of [Bor79, 3]. To this aim, consider the centralizer \mathcal{M}_ι in ${}^L\mathbf{G}$ of the torus $Z(({}^L\mathbf{G})_\iota)^\circ$. It contains $({}^L\mathbf{G})_\iota$, hence it surjects to W_F and by [Bor79, Lemma 3.5], it is a Levi subgroup of ${}^L\mathbf{G}$ in the sense of *loc. cit.*

2.1.6 Proposition. — *The conjugacy class $\{\iota\}$ is relevant to \mathbf{G} if and only if \mathcal{M}_ι is relevant to \mathbf{G} in the sense of [Bor79, 3.4]. Moreover, in this case, the centralizer \mathbf{G}_ι of any F -rational Levi-center-embedding $\iota : \mathbf{S}_{\{\iota\}} \hookrightarrow \mathbf{G}$ dual to $\{\iota\}$ is an inner form of $\mathbf{G}_{\{\iota\}}$.*

Proof. Assume first that $\{\iota\}$ is relevant and let $\iota : \mathbf{S}_{\{\iota\}} \hookrightarrow \mathbf{G}$ be an F -rational dual embedding. Choose a maximal F -torus \mathbf{T} of \mathbf{G}_ι , and a dual embedding $\hat{j} : \widehat{\mathbf{T}} \hookrightarrow \widehat{\mathbf{G}}$ that extends $\hat{\iota}$. Its stabilizer $({}^L\mathbf{G})_{\hat{j}}$ in ${}^L\mathbf{G}$ is contained in $({}^L\mathbf{G})_{\hat{\iota}}$, hence $Z(({}^L\mathbf{G})_{\hat{j}})^\circ$ contains $Z(({}^L\mathbf{G})_{\hat{\iota}})^\circ$ and therefore the Levi subgroup $\mathcal{M}_{\hat{j}} := C_{L\mathbf{G}}(Z(({}^L\mathbf{G})_{\hat{j}})^\circ)$ of ${}^L\mathbf{G}$ is contained in $\mathcal{M}_{\hat{\iota}}$. Since any Levi subgroup of ${}^L\mathbf{G}$ that contains a relevant Levi subgroup is relevant, it suffices to show that $\mathcal{M}_{\hat{j}}$ is relevant. Now observe that, by definition, $({}^L\mathbf{G})_{\hat{j}}$ is an extension of W_F by $\widehat{\mathbf{T}}$ such that the action of W_F

on $\hat{\mathbf{T}}$ induced by conjugation is the one inherited from the F -structure on \mathbf{T} . In particular we have $Z(({}^L\mathbf{G})_j)^\circ = \hat{j}(\hat{\mathbf{T}}^{W_{F,\alpha}})$, and we see that

$$\Sigma(\hat{\mathbf{T}}, \mathcal{M}_j^\circ) = \left\{ \alpha^\vee \in \Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}}), \langle \alpha^\vee, X_*(\hat{\mathbf{T}})^{W_F} \rangle = 0 \right\}.$$

We claim that for $\alpha \in \Sigma(\mathbf{T}, \mathbf{G})$ we have $\langle \alpha^\vee, X_*(\hat{\mathbf{T}})^{W_F} \rangle = 0 \Leftrightarrow \langle \alpha, X_*(\mathbf{T})^{W_F} \rangle = 0$. Indeed, let $W_{F,\alpha}$ be the finite subgroup of $\text{Aut}_{\mathbb{Q}}(X_*(\hat{\mathbf{T}})_{\mathbb{Q}})$ generated by the image of W_F and the reflection s_α . Then $\langle \alpha^\vee, X_*(\hat{\mathbf{T}})^{W_F} \rangle = 0 \Leftrightarrow \dim_{\mathbb{Q}}(X_*(\hat{\mathbf{T}})_{\mathbb{Q}}^{W_F}) = \dim_{\mathbb{Q}}(X_*(\hat{\mathbf{T}})_{\mathbb{Q}}^{W_{F,\alpha}})$, which by duality is equivalent to $\dim_{\mathbb{Q}}(X_*(\mathbf{T})_{\mathbb{Q}}^{W_F}) = \dim_{\mathbb{Q}}(X_*(\mathbf{T})_{\mathbb{Q}}^{W_{F,\alpha}})$ hence to $\langle \alpha, X_*(\mathbf{T})^{W_F} \rangle = 0$. Now, denoting by $\mathbf{T}^{\text{split}}$ the maximal split subtorus of \mathbf{T} , we obtain $\Sigma(\hat{\mathbf{T}}, \mathcal{M}_j^\circ) = \{\alpha \in \Sigma(\mathbf{T}, \mathbf{G}), \alpha|_{\mathbf{T}^{\text{split}}} \equiv 1\}^\vee$. It follows that \mathcal{M}_j is dual to the F -Levi subgroup $C_{\mathbf{G}}(\mathbf{T}^{\text{split}})$ of \mathbf{G} and is therefore relevant.

Conversely, assume now that \mathcal{M}_i is relevant. After replacing \hat{i} by a conjugate, we may assume that \mathcal{M}_i is a standard Levi subgroup of ${}^L\mathbf{G}$, and in particular of the form $\hat{\mathbf{M}}_i \rtimes W_F$ for some W_F -stable Levi subgroup $\hat{\mathbf{M}}_i$ of $\hat{\mathbf{G}}$. Since \mathcal{M}_i is relevant to \mathbf{G} , $\hat{\mathbf{M}}_i \rtimes W_F$ is the L -group of some F -Levi subgroup \mathbf{M}_i of \mathbf{G} . On the other hand, \hat{i} factors through $\hat{\mathbf{M}}_i$ and provides a Levi-center-embedding for this group. Since $({}^L\mathbf{G})_i$ is contained in \mathcal{M}_i , the stabilizer $({}^L\mathbf{M}_i)_i = ({}^L\mathbf{G})_i$ surjects to W_F so that the $\hat{\mathbf{M}}_i$ -conjugacy class of \hat{i} is W_F -stable. So we are now left to show that \hat{i} is relevant for \mathbf{M}_i . Equivalently, we may and will restrict to the case where $\mathcal{M}_i = {}^L\mathbf{G}$, that is $Z(({}^L\mathbf{G})_i)^\circ = Z({}^L\mathbf{G})^\circ$.

We will now reduce further to the case where \mathbf{G} is an adjoint group. To this aim, denote by $\pi : \mathbf{G} \rightarrow \mathbf{G}_{\text{ad}}$ the adjoint quotient map (defined over F) and by $\hat{\pi} : \widehat{\mathbf{G}}_{\text{ad}} = \widehat{\mathbf{G}}_{\text{sc}} \rightarrow \widehat{\mathbf{G}}$ its dual (W_F -equivariant) map. Consider the connected fiber product $\hat{\mathbf{S}}_{\text{ad}} := (\widehat{\mathbf{S}} \times_{\widehat{\mathbf{G}}} \widehat{\mathbf{G}}_{\text{ad}})^\circ$. This is a torus with finite W_F -action and the second projection $\hat{\iota}_{\text{ad}} : \hat{\mathbf{S}}_{\text{ad}} \rightarrow \widehat{\mathbf{G}}_{\text{ad}}$ is a Levi-center-embedding whose stabilizer $(\widehat{\mathbf{G}}_{\text{ad}})_{\hat{\iota}_{\text{ad}}}$ is the inverse image $\hat{\pi}^{-1}(\widehat{\mathbf{G}}_i)$ of that of \hat{i} . Moreover, if we write an element $\hat{g} \in \widehat{\mathbf{G}}$ in the form $\hat{g} = \hat{z}\hat{\pi}(\hat{h})$ according to the decomposition $\widehat{\mathbf{G}} = Z(\widehat{\mathbf{G}})\hat{\pi}(\widehat{\mathbf{G}}_{\text{sc}})$, then we see that $(\hat{g}\hat{i})_{\text{ad}} = \hat{h}(\hat{\iota}_{\text{ad}})$. It follows that $\{\hat{i}\}$ determines a $\widehat{\mathbf{G}}_{\text{ad}}$ -conjugacy class $\{\hat{\iota}_{\text{ad}}\}$. Since $\hat{\pi}$ is W_F -equivariant, $\{\hat{\iota}_{\text{ad}}\}$ is W_F -stable, and its stabilizer $({}^L\mathbf{G}_{\text{ad}})_{\hat{\iota}_{\text{ad}}}$ is the preimage of $({}^L\mathbf{G})_i$ along $\hat{\pi} \rtimes \text{Id}_{W_F}$. Also $\hat{\pi}$ induces a W_F -equivariant morphism $\widehat{\mathbf{S}}_{\{\hat{\iota}_{\text{ad}}\}} \rightarrow \widehat{\mathbf{S}}_{\{\hat{i}\}}$ which, dually, induces an F -morphism $\mathbf{S}_{\{\hat{i}\}} \rightarrow \mathbf{S}_{\{\hat{\iota}_{\text{ad}}\}}$. Now we claim that

$$\{\hat{i}\} \text{ is relevant to } \mathbf{G} \text{ if and only if } \{\hat{\iota}_{\text{ad}}\} \text{ is relevant to } \mathbf{G}_{\text{ad}}.$$

Indeed, suppose there is an F -rational Levi-center-embedding $\iota_{\text{ad}} : \mathbf{S}_{\{\hat{\iota}_{\text{ad}}\}} \hookrightarrow \mathbf{G}_{\text{ad}}$ in \mathbf{G}_{ad} dual to $\{\hat{\iota}_{\text{ad}}\}$. Then consider the torus $\mathbf{S} := (\mathbf{S}_{\{\hat{\iota}_{\text{ad}}\}} \times_{\mathbf{G}_{\text{ad}}} \mathbf{G})^\circ$. The second projection provides an F -rational Levi-center embedding $\iota : \mathbf{S} \hookrightarrow \mathbf{G}$ and we need to prove it is dual to $\{\hat{i}\}$. This is a problem over \bar{F} and we need to go through the duality procedure of 2.1.3. So let us choose a maximal torus $\hat{\mathbf{T}}$ in $\widehat{\mathbf{G}}_i$ with dual \mathbf{T} over \bar{F} . It provides a maximal torus $\widehat{\mathbf{T}}_{\text{ad}} = \hat{\pi}^{-1}(\hat{\mathbf{T}})$ in $(\widehat{\mathbf{G}}_{\text{ad}})_{\hat{\iota}_{\text{ad}}}$ whose dual over \bar{F} we denote by \mathbf{T}_{ad} . Also $\hat{\pi}$ provides a dual morphism $\mathbf{T} \xrightarrow{\pi} \mathbf{T}_{\text{ad}}$. Now choose an embedding $j : \mathbf{T} \hookrightarrow \mathbf{G}_{\bar{F}}$ dual to $\hat{\mathbf{T}} \subset \widehat{\mathbf{G}}$ that factors through $(\mathbf{G}_i)_{\bar{F}}$. Then $\pi \circ j$ factors over an embedding $j_{\text{ad}} : \mathbf{T}_{\text{ad}} \hookrightarrow (\mathbf{G}_{\text{ad}})_{\bar{F}}$ dual to $\widehat{\mathbf{T}}_{\text{ad}} \subset \widehat{\mathbf{G}}_{\text{ad}}$ and that factors through $((\mathbf{G}_{\text{ad}})_{\iota_{\text{ad}}})_{\bar{F}}$. As in 2.1.3, the embedding ι_{ad} identifies $(\mathbf{S}_{\{\hat{\iota}_{\text{ad}}\}})_{\bar{F}}$ with the subtorus $\left(\bigcap_{\alpha \in \Sigma_{\hat{\iota}_{\text{ad}}}} \ker(\alpha^\vee) \right)^\circ$ of \mathbf{T}_{ad} involving the subroot system $\Sigma_{\hat{\iota}_{\text{ad}}}$ of $\Sigma(\widehat{\mathbf{T}}_{\text{ad}}, \widehat{\mathbf{G}}_{\text{ad}})$. This subroot system coincides with Σ_i through the canonical identification $\Sigma(\widehat{\mathbf{T}}_{\text{ad}}, \widehat{\mathbf{G}}_{\text{ad}}) = \Sigma(\hat{\mathbf{T}}, \widehat{\mathbf{G}})$. Now our definition of \mathbf{S} and ι show that ι identifies $\mathbf{S}_{\bar{F}}$ with the subtorus $\left(\bigcap_{\alpha \in \Sigma_i} \ker(\alpha^\vee) \right)^\circ$ of \mathbf{T} , hence

ι is dual to $\{\hat{\iota}\}$ as desired. The other implication is seen in a similar way but we omit the proof since we do not need it here.

So we are now left to prove that if \mathbf{G} is an adjoint group and $Z(({}^L\mathbf{G})_i)^\circ = \{1\}$, then $\{\hat{\iota}\}$ is relevant. Since \mathbf{G} is adjoint, there is $\eta \in H^1(\Gamma_F, \mathbf{G})$ such that the associated pure inner form \mathbf{G}_η over F is quasi-split. Then $\eta^{-1} \in H^1(\Gamma_F, \mathbf{G}_\eta)$ and we have $(\mathbf{G}_\eta)_{\eta^{-1}} = \mathbf{G}$. Through Kottwitz' duality we can view η^{-1} as a character of the finite group $Z({}^L\mathbf{G})$. Since $Z(({}^L\mathbf{G})_i)^\circ = \{1\}$ we may extend η^{-1} to a character of the finite group $Z({}^L\mathbf{G})_i$ that we denote by ζ^{-1} . Going through Kottwitz' duality again, we get a cohomology class $\zeta^{-1} \in H^1(F, \mathbf{G}_{\{\hat{\iota}\}})$. Now by Proposition 2.1.5 there is an F -rational Levi-center-embedding $\iota : \mathbf{S}_{\{\hat{\iota}\}} \hookrightarrow \mathbf{G}_\eta$ with a natural F -rational isomorphism $\mathbf{G}_{\eta, \iota} \simeq \mathbf{G}_{\{\hat{\iota}\}}$. Let us choose a 1-cocycle $\zeta^{-1} : \Gamma_F \rightarrow \mathbf{G}_{\eta, \iota}$ that represents the cohomology class ζ^{-1} . Then ι is still F -rational for the F -structure of \mathbf{G}_η twisted by ζ^{-1} , *i.e.*, ι is an F -rational Levi-center-embedding $\mathbf{S}_{\{\hat{\iota}\}} \hookrightarrow (\mathbf{G}_\eta)_{\zeta^{-1}}$. However, we know by Proposition 2.1.5 that the map $H^1(F, \mathbf{G}_{\eta, \iota}) \rightarrow H^1(F, \mathbf{G}_\eta)$ is induced by the inclusion $Z({}^L\mathbf{G}) \subset Z(({}^L\mathbf{G})_i)$ through Kottwitz' duality. Therefore we have $\zeta^{-1} = \eta^{-1}$ in $H^1(F, \mathbf{G}_\eta)$, so that $(\mathbf{G}_\eta)_{\zeta^{-1}} \simeq \mathbf{G}$ and ι finally provides the desired F -rational Levi-center-embedding into \mathbf{G} .

We now turn to the second assertion of the proposition. Our argument has provided one ι with centralizer \mathbf{G}_ι an inner form of $\mathbf{G}_{\{\hat{\iota}\}}$. The fact that this property remains true for all F -rational embeddings dual to $\{\hat{\iota}\}$ follows from the discussion above Lemma 2.1.7 below. \square

Now that we have studied the existence of F -rational dual Levi-center-embeddings, we may try to classify all of them. Obviously $\mathbf{G}(F)$ acts by conjugation on these F -rational embeddings. So, let us fix one of them, ι , and let ι' be another one. Then pick some $g \in \mathbf{G}(\overline{F})$ such that $\iota' = \text{Ad}_g \circ \iota$. Then for any $\gamma \in \Gamma_F$ we also have $\iota' = \gamma \iota' = \text{Ad}_{\gamma(g)} \circ \gamma \iota = \text{Ad}_{\gamma(g)} \circ \iota$, so that $g^{-1}\gamma(g) \in \mathbf{G}_\iota(\overline{F})$. We then see that

- $(\gamma \mapsto g^{-1}\gamma(g)) \in Z^1(F, \mathbf{G}_\iota)$ and its image $\eta_{\iota, \iota'}$ in $H^1(F, \mathbf{G}_\iota)$ is independent of the choice of g .
- Ad_g is an inner twisting $(\mathbf{G}_\iota)_{\overline{F}} \xrightarrow{\sim} (\mathbf{G}_{\iota'})_{\overline{F}}$ with associated inner cocycle $\gamma \mapsto g^{-1}\gamma(g)$.

2.1.7 Lemma. — *The map $\iota' \mapsto \eta_{\iota, \iota'}$ induces a bijection between the set of $\mathbf{G}(F)$ -conjugacy classes of F -rational embeddings in $\{\hat{\iota}\}$ and $\ker(H^1(F, \mathbf{G}_\iota) \rightarrow H^1(F, \mathbf{G}))$.*

Proof. Indeed, it is easily seen that $\eta_{\iota, \iota'}$ only depends on the $\mathbf{G}(F)$ -conjugacy class of ι' , and by construction it lies in the above kernel. Conversely, let η belong to this kernel. Then it can be represented by a 1-cocycle of the form $\gamma \mapsto g^{-1}\gamma(g)$ for some $g \in \mathbf{G}(\overline{F})$, and the embedding $\iota' = \text{Ad}_g \circ \iota$ is thus F -rational. This element g is not unique, but any other one is of the form $h g k$ with $h \in \mathbf{G}(F)$ and $k \in \mathbf{G}_\iota(\overline{F})$ and thus leads to a $\mathbf{G}(F)$ -conjugate rational embedding. We thus have constructed the inverse map. \square

2.2 Levi factorization of a parameter

We start with a continuous L -homomorphism $\phi : P_F \rightarrow {}^L\mathbf{G}$ that admits an extension to W_F .

2.2.1 The group \mathbf{L}_ϕ . — The centralizer $\hat{\mathbf{L}}_\phi := C_{\hat{\mathbf{G}}}(Z(C_{\hat{\mathbf{G}}}(\phi))^\circ)$ of the connected center $Z(C_{\hat{\mathbf{G}}}(\phi))^\circ$ of $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$ which contains $C_{\hat{\mathbf{G}}}(\phi)$. If $\varphi : W_F \rightarrow {}^L\mathbf{G}$ extends ϕ , then the conjugation action Ad_φ of W_F on $C_{\hat{\mathbf{G}}}(\phi)$ preserves its connected center and therefore also $\hat{\mathbf{L}}_\phi$. Since for any other extensions φ' the ratio $\varphi^{-1}\varphi'$ takes values in $C_{\hat{\mathbf{G}}}(\phi) \times \{1\}$, the outer action $W_F \xrightarrow{\text{Ad}_\varphi} \text{Out}(\hat{\mathbf{L}}_\phi)$ is independent of the choice of φ . We know from [Dat17, Lemma 2.1.1] that this action is finite. Hence we may denote by \mathbf{L}_ϕ a quasi-split group over F endowed with a W_F -equivariant isomorphism $\psi_0(\mathbf{L}_\phi) \xrightarrow{\sim} \psi_0(\hat{\mathbf{L}}_\phi)^\vee$. Note that $\psi_0(\hat{\mathbf{L}}_\phi)$ only depends on the $\hat{\mathbf{G}}$ -conjugacy class of ϕ in the sense that if ϕ' is conjugate to ϕ , there is a canonical isomorphism $\psi_0(\hat{\mathbf{L}}_\phi) \xrightarrow{\sim} \psi_0(\hat{\mathbf{L}}_{\phi'})$ given by any \hat{g} that conjugates ϕ to ϕ' . Note also that the inclusion $Z(\hat{\mathbf{L}}_\phi)^{W_F} \subset Z(\hat{\mathbf{G}})^{W_F}$ induces by Kottwitz' duality a map $H^1(F, \mathbf{L}_\phi) \rightarrow H^1(F, \mathbf{G})$. We put

$$H^1(F, \mathbf{L}_\phi, \mathbf{G}) := \ker(H^1(F, \mathbf{L}_\phi) \rightarrow H^1(F, \mathbf{G})).$$

2.2.2 The group \mathcal{L}_ϕ and the L -group of \mathbf{L}_ϕ . — Consider the subgroup $\mathcal{L}_\phi := \hat{\mathbf{L}}_\phi \cdot \varphi(W_F)$ of ${}^L\mathbf{G}$. As the notation suggests, it is independent of the choice of a parameter φ extending ϕ . It sits in a split exact sequence $\hat{\mathbf{L}}_\phi \hookrightarrow \mathcal{L}_\phi \twoheadrightarrow W_F$ and we may ask whether it is isomorphic to ${}^L\mathbf{L}_\phi$. To this aim, fix a pinning ε_ϕ of $\hat{\mathbf{L}}_\phi$ and consider the stabilizer $\mathcal{L}_{\phi, \varepsilon_\phi}$ of ε_ϕ in \mathcal{L}_ϕ . It sits in an exact sequence $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi} \twoheadrightarrow W_F$.

Lemma. — *The extension $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi} \twoheadrightarrow W_F$ splits continuously, and the set of its splittings $W_F \rightarrow \mathcal{L}_{\phi, \varepsilon_\phi}$ is principal homogeneous under $Z^1(W_F, Z(\hat{\mathbf{L}}_\phi))$.*

Proof. Only the existence of a splitting requires a proof, the second assertion being easy. Recall first that, by [Dat17, Lemma 2.1.1], the extension under consideration comes from a finite quotient of W_F . By Langland's Lemma 4 in [Lan79], the image of $H_{cts}^2(\Gamma_F, Z(\hat{\mathbf{L}}_\phi)^\circ) \rightarrow H^2(W_F, Z(\hat{\mathbf{L}}_\phi)^\circ)$ is $\{1\}$. This reduces the problem to showing that the extension

$$\pi_0(Z(\hat{\mathbf{L}}_\phi)) \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi} / Z(\hat{\mathbf{L}}_\phi)^\circ \twoheadrightarrow W_F$$

splits. This in turn follows from the argument in Kaletha's Lemma 5.2.6 in [Kal19]. In order to explain this, observe that the short exact sequence of the lemma is a pull-back of the short exact sequence

$$Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi} \twoheadrightarrow \mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi} / Z(\hat{\mathbf{L}}_\phi),$$

where the index ε_ϕ indicates the stabilizer of the pinning ε_ϕ . To make the latter more explicit, we may assume that the pinning $\varepsilon_\phi = (\hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi, \{X_\alpha\}_{\alpha \in \Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi)})$ is the restriction of a W_F -stable pinning $\varepsilon = (\hat{\mathbf{T}}, \hat{\mathbf{B}}, \{X_\alpha\}_{\alpha \in \Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}})})$ of $\hat{\mathbf{G}}$ (after conjugating (ϕ, ε_ϕ) by some appropriate $\hat{g} \in \hat{\mathbf{G}}$). Then the inclusion $\mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi} \subset \mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi, \hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi)$ induces an isomorphism

$$\mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi} / Z(\hat{\mathbf{L}}_\phi) \xrightarrow{\sim} \mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi, \hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi) / \hat{\mathbf{T}} = (\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F)_{\Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi)}$$

where the index $\Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi)$ denotes the stabilizer of this set of characters of $\hat{\mathbf{T}}$. Now, consider the set-theoretic section $\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F \rightarrow \mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{T}})$ given by Tits' liftings with respect to the pinning ε . By [Spr98, Prop 9.3.5], it restricts to a map $(\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F)_{\Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi)} \rightarrow \mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi}$.

The latter may not be a homomorphism of groups, but the content of Kaletha's study of the Tits liftings in the proof of [Kal19, Lemma 5.2.6] is that the composed map

$$(\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F)_{\Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi)} \longrightarrow \mathcal{N}_{L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi} / Z(\hat{\mathbf{L}}_\phi)^\circ$$

is a homomorphism. This provides a splitting for the first displayed exact sequence of this proof, as desired. \square

Let $\psi : W_F \longrightarrow \mathcal{L}_{\phi, \varepsilon_\phi}$ be a continuous splitting as in the lemma. We get an isomorphism of extensions $\text{Id} \times \psi : {}^L\mathbf{L}_\phi \xrightarrow{\sim} \mathcal{L}_\phi$, where the L -group is formed by using the section $\text{Out}(\hat{\mathbf{L}}_\phi) \hookrightarrow \text{Aut}(\hat{\mathbf{L}}_\phi)$ associated to ε_ϕ . Then $\varphi_L := (\text{Id} \times \psi)^{-1} \circ \varphi$ is a Langlands parameter for \mathbf{L}_ϕ , whose restriction to P_F we denote by $\phi_L \in \Phi(P_F, \mathbf{L}_\phi)$. We thus get a factorization of ϕ

$$\phi : P_F \xrightarrow{\phi_L} {}^L\mathbf{L}_\phi \xrightarrow{\xi_\psi} {}^L\mathbf{G}$$

with ξ_ψ the composition of $\text{Id} \times \psi$ and the inclusion $\mathcal{L}_\phi \subset {}^L\mathbf{G}$. Then we see that ξ_ψ induces an isomorphism $C_{\hat{\mathbf{L}}_\phi}(\phi_L) \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$, which makes it fall into the framework of [Dat17, Expectation 1.3.2], which predicts (at least when \mathbf{G} is quasi-split) the existence of an equivalence of categories $\prod_{\eta \in H^1(F, \mathbf{L}_{\phi, \mathbf{G}})} \text{Rep}^{\phi_L}(L_{\phi, \eta}) \xrightarrow{\sim} \text{Rep}^\phi(G)$ where $\mathbf{L}_{\phi, \eta}$ is the pure inner form of \mathbf{L}_ϕ associated to η . Interestingly, this set $H^1(F, \mathbf{L}_\phi, \mathbf{G})$ and the associated pure inner forms of $\hat{\mathbf{L}}_\phi$ also appear when we try to go from $\hat{\mathbf{L}}_\phi$ to twisted Levi subgroups of \mathbf{G} .

2.2.3 Twisted Levi subgroups of \mathbf{G} . — With the outer action map, also the action maps $W_F \xrightarrow{\text{Ad}_\varphi} \text{Aut}(Z(\hat{\mathbf{L}}_\phi)^\circ)$ and $W_F \xrightarrow{\text{Ad}_\varphi} \text{Aut}(\hat{\mathbf{L}}_{\phi, \text{ab}})$ are independent of the choice of φ . Moreover, the existence of φ tells us that the $\hat{\mathbf{G}}$ -conjugacy class of the embedding $Z(\hat{\mathbf{L}}_\phi)^\circ \subset \hat{\mathbf{G}}$ is W_F -stable.

Notation. — We denote by \mathbf{S}_ϕ the F -torus dual to the complex torus $\hat{\mathbf{L}}_{\phi, \text{ab}}$ with its W_F -action, and by I_ϕ the $\mathbf{G}(\overline{F})$ -conjugacy class of Levi-center-embeddings $(\mathbf{S}_\phi)_{\overline{F}} \hookrightarrow \mathbf{G}_{\overline{F}}$ which is dual to the Levi-center-embedding $Z(\hat{\mathbf{L}}_\phi)^\circ \subseteq \hat{\mathbf{G}}$ in the sense of 2.1.3.

Recall from 1.1 that we say that ϕ is a *wild inertia parameter* of \mathbf{G} if it admits an extension $\varphi' : W'_F \longrightarrow {}^L\mathbf{G}$ that is *relevant* to \mathbf{G} .

Proposition. — *With the above notation:*

- i) *If ϕ is a wild inertia parameter of \mathbf{G} , then I_ϕ contains an F -rational embedding ι . Moreover, the following holds:*
 - (a) *The centralizer \mathbf{G}_ι of ι is an inner form of \mathbf{L}_ϕ , and if \mathbf{G} is quasi-split, one can choose ι such that \mathbf{G}_ι is isomorphic to \mathbf{L}_ϕ .*
 - (b) *For any F -rational $\iota' \in I_\phi$ there is $\eta_{\iota, \iota'} \in H^1(F, \mathbf{G}_\iota)$ such that $\mathbf{G}_{\iota'}$ is isomorphic to the pure inner form $\mathbf{G}_{\iota, \eta_{\iota, \iota'}}$.*
 - (c) *The map $\iota' \mapsto \eta_{\iota, \iota'}$ induces a bijection between the set of G -conjugacy classes of F -rational embeddings in I_ϕ and the set $H^1(F, \mathbf{G}_\iota, \mathbf{G})$.*
- ii) *Assume that $C_{\hat{\mathbf{G}}}(\phi)$ is connected and that there exists an extension φ of ϕ that preserves a pinning of $C_{\hat{\mathbf{G}}}(\phi)$. If I_ϕ contains an F -rational embedding ι , then ϕ is a wild inertia parameter for \mathbf{G} .*

Proof. Denote by $\hat{\iota}$ the Levi-center-embedding $\hat{\iota} : Z(\hat{\mathbf{L}}_\phi)^\circ \subseteq \hat{\mathbf{G}}$, and recall the notation $\mathcal{M}_\iota = C_{L\mathbf{G}}(Z(({}^L\mathbf{G})_\iota)^\circ)$ from Proposition 2.1.6. By definition of the W_F -action on $Z(\hat{\mathbf{L}}_\phi)$, we have $({}^L\mathbf{G})_\iota = \mathcal{L}_\phi$, hence $\mathcal{M}_\iota = \mathcal{M}_\phi := C_{L\mathbf{G}}(Z(\mathcal{L}_\phi)^\circ)$. Then Proposition 2.1.6 shows that I_ϕ contains an F -rational embedding if, and only if, the Levi subgroup \mathcal{M}_ϕ of ${}^L\mathbf{G}$ is relevant to \mathbf{G} .

i) Fix a relevant Langlands parameter $\varphi' : W'_F \rightarrow {}^L\mathbf{G}$ that extends ϕ . We claim that $\varphi'(W'_F) \subset \mathcal{L}_\phi$. Indeed, the inclusion $\varphi'(W'_F) \subset \mathcal{L}_\phi$ holds by definition, and the inclusion $\varphi'(\mathrm{SL}_2) \subset \mathcal{L}_\phi$ holds too since $\varphi'(\mathrm{SL}_2)$ is contained in $C_{\hat{\mathbf{G}}}(\phi)$, hence commutes with $Z(C_{\hat{\mathbf{G}}}(\phi))^\circ$ and is thus contained in $\hat{\mathbf{L}}_\phi$. Because \mathcal{M}_ϕ contains \mathcal{L}_ϕ , it follows that φ' factors through \mathcal{M}_ϕ and, since φ' is relevant, \mathcal{M}_ϕ is relevant to \mathbf{G} . The first claims of i) and i)(a) now follow from Proposition 2.1.6, the remaining claims of i) follow from Proposition 2.1.5 and Lemma 2.1.7.

ii) Fix an extension $\varphi : W_F \rightarrow {}^L\mathbf{G}$ of ϕ that normalizes a pinning of $C_{\hat{\mathbf{G}}}(\phi)^\circ$, and define $\varphi' = \varphi \times \alpha : W_F \times \mathrm{SL}_2 \rightarrow \mathcal{L}_\phi$ with α an isomorphism on a principal SL_2 in $C_{\hat{\mathbf{G}}}(\varphi)^\circ$. We then have $C_{\hat{\mathbf{G}}}(\varphi')^\circ = C_{C_{\hat{\mathbf{G}}}(\varphi)^\circ}(\alpha)^\circ = Z(C_{\hat{\mathbf{G}}}(\varphi)^\circ)^\circ$. Writing $C_{\hat{\mathbf{G}}}(\varphi)^\circ = (C_{\hat{\mathbf{G}}}(\phi)^\circ)^{\varphi(W_F)^\circ}$, it follows from the next lemma that $Z(C_{\hat{\mathbf{G}}}(\varphi)^\circ)^\circ = Z(C_{\hat{\mathbf{G}}}(\phi)^\circ)^{\varphi(W_F)^\circ}$. Since we assume $C_{\hat{\mathbf{G}}}(\phi)^\circ = C_{\hat{\mathbf{G}}}(\phi)$, this means that $C_{\hat{\mathbf{G}}}(\varphi')^\circ$ is contained in (hence equal to) $Z(\mathcal{L}_\phi)^\circ$. It follows that for any Levi subgroup \mathcal{M} of ${}^L\mathbf{G}$ that contains $\varphi'(W_F)$, we have $Z(\mathcal{M})^\circ \subseteq Z(\mathcal{L}_\phi)^\circ \subseteq Z(\mathcal{M}_\phi)$ and, therefore, $\mathcal{M} \supseteq \mathcal{M}_\phi$. By assumption, \mathcal{M}_ϕ is relevant, so \mathcal{M} is relevant, and so is φ' . \square

Lemma. — Let \mathbf{H} be a complex reductive group and Γ a group acting on \mathbf{H} and preserving a pinning of \mathbf{H} . Then $Z(\mathbf{H}^{\Gamma,\circ})^\circ = Z(\mathbf{H})^{\Gamma,\circ}$.

Proof. The inclusion $Z(\mathbf{H}^{\Gamma,\circ})^\circ \supset Z(\mathbf{H})^{\Gamma,\circ}$ is clear. To get the other inclusion it is enough to show that $Z(\mathbf{H}^{\Gamma,\circ}) \subset Z(\mathbf{H})$. Observe that any isogeny $\mathbf{H}' \rightarrow \mathbf{H}$ is Γ -equivariant for the action of Γ on \mathbf{H}' obtained by lifting a Γ -stable pinning from \mathbf{H} to \mathbf{H}' and identifying $\mathrm{Out}(\mathbf{H}) = \mathrm{Out}(\mathbf{H}')$. In such a situation, the image of $(\mathbf{H}')^\Gamma$ has finite index in \mathbf{H}^Γ so that the statement of the lemma is true for \mathbf{H} if and only if it is true for \mathbf{H}' . Since this statement is clear for tori, the isogeny $\mathbf{H}' = \mathbf{H}_{\mathrm{sc}} \times Z(\mathbf{H}) \rightarrow \mathbf{H}$ allows us to reduce to the case where \mathbf{H} is semi-simple and simply connected. Then Γ permutes the set of simple factors of \mathbf{H} , so we may restrict to the case with one orbit, and then restrict to a simple factor with the action of its stabilizer. Hence we may assume that \mathbf{H} is simple and replace Γ by its image in $\mathrm{Out}(\mathbf{H})$ which is either $\mathbb{Z}/2\mathbb{Z}$ or S_3 . At this point we could conclude with a case by case inspection. But we can also invoke Steinberg's Thm 8.1 in [Ste68], which ensures that $\mathbf{H}^\Gamma = \mathbf{H}^{\Gamma,\circ}$ is a reductive group with maximal torus $\mathbf{T}^\Gamma = \mathbf{T}^{\Gamma,\circ}$, where \mathbf{T} is part of a Γ -stable pinning. In particular $Z(\mathbf{H}^\Gamma)^\circ \subset \mathbf{T}^\Gamma$. Now let $(\mathbf{T}, \mathbf{B}, (X_\alpha)_{\alpha \in \Delta(\mathbf{T}, \mathbf{B})})$ be a Γ -stable pinning of \mathbf{H} , where X_α is a non-zero element of the weight α subspace in the Lie algebra \mathfrak{h} of H . Then $Z(\mathbf{H}^\Gamma)^\circ$ must act trivially on the elements $\sum_{\gamma \in \Gamma} X_{\gamma\alpha} \in \mathfrak{h}^\Gamma$ for $\alpha \in \Delta(\mathbf{T}, \mathbf{B})$. These elements are non-zero (here, compared to Steinberg's result, we need the fact that Γ preserves the pinning and not only the pair (\mathbf{T}, \mathbf{B})), therefore we have $Z(\mathbf{H}^\Gamma)^\circ \subset \bigcap_{\alpha \in \Delta} \ker(\alpha) = Z(\mathbf{H})$. \square

Remark.— For any F -rational $\iota \in I_\phi$, we may identify the group $X_\phi := X^*(Z(\mathcal{L}_\phi)/Z({}^L\mathbf{G}))_{\mathrm{tors}}$ with $H^1(F, \mathbf{G}_\iota, \mathbf{G})$ as in the proof of Proposition 2.1.5. Then, denoting by \mathcal{I}_ϕ the set of $G(F)$ -conjugacy classes of F -rational elements in I_ϕ , items ii) and iii) of the last proposition provide \mathcal{I}_ϕ with a structure of a X_ϕ -torsor.

2.2.4 Lemma. — Assume that \mathbf{G} is tamely ramified. Then \mathbf{L}_ϕ is tamely ramified, the subgroup $1 \times P_F$ of ${}^L\mathbf{G}$ is contained in $\mathcal{L}_{\phi, \varepsilon_\phi}$ and there is a splitting $\psi : W_F \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi}$ which is tame in the sense that $\psi|_{P_F} = 1 \times \text{Id}$.

Proof. We can write $\phi = \hat{\phi} \times \text{Id}$ with $\hat{\phi} : P_F \longrightarrow \hat{\mathbf{G}}$ a homomorphism. Then $C_{\hat{\mathbf{G}}}(\phi) = C_{\hat{\mathbf{G}}}(\hat{\phi})$ so that $\hat{\phi}(P_F) \subset \hat{\mathbf{L}}_\phi$. Since $1 \times P_F$ acts trivially on $\hat{\mathbf{G}}$, it follows that the action of $\phi(P_F)$ on $\hat{\mathbf{L}}_\phi$ is inner, hence \mathbf{L}_ϕ is tamely ramified. Moreover, since $\phi(P_F) \subset \mathcal{L}_\phi$ by construction, we get that $1 \times P_F \subset \mathcal{L}_\phi$, and because this group acts trivially on $\hat{\mathbf{G}}$, we even have $1 \times P_F \subset \mathcal{L}_{\phi, \varepsilon_\phi}$. Now, the extension $\mathcal{L}_{\phi, \varepsilon_\phi}$ considered above is the pullback of the extension $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi} / (1 \times P_F) \twoheadrightarrow W_F / P_F$ by the projection $W_F \twoheadrightarrow W_F / P_F$ and we need to show that the latter extension splits. By [DHKM25, Lemma 3.8], we know that for any complex torus $\hat{\mathbf{S}}$ with a finite action of W_F / P_F we have $H_{cts}^2(W_F / P_F, \hat{\mathbf{S}}) = \{1\}$ (an alternative argument relying on Langlands' Lemma 4 in [Lan79] can be found in the proof of [Kal19, Lemma 5.2.8]). On the other hand, the same argument as in Lemma 2.2.2 shows that the extension $\pi_0(Z(\hat{\mathbf{L}}_\phi)) \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi} / ((1 \times P_F)Z(\hat{\mathbf{L}}_\phi)^\circ) \twoheadrightarrow W_F / P_F$ splits. Indeed, it suffices to replace ${}^L\mathbf{G}$ by its quotient $\mathbf{G} \rtimes (W_F / P_F)$. \square

2.2.5 Lemma. — Assume that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$. Then \mathbf{G} is tamely ramified, $C_{\hat{\mathbf{G}}}(\phi) = \hat{\mathbf{L}}_\phi$, the subgroup $\phi(P_F)$ is contained in $\mathcal{L}_{\phi, \varepsilon_\phi}$, and $\hat{\phi}(P_F) \subset Z(\hat{\mathbf{L}}_\phi)$, where we write $\phi = \hat{\phi} \times \text{Id}$. Moreover, the following properties hold true :

i) There is a splitting $\varphi : W_F \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi}$ that extends ϕ .

ii) There is a 1-cocycle $\hat{\varphi} : W_F \longrightarrow Z(\hat{\mathbf{L}}_\phi)$ that extends $\hat{\phi}$.

Proof. The equality $C_{\hat{\mathbf{G}}}(\phi) = \hat{\mathbf{L}}_\phi$ is clear by definition of $\hat{\mathbf{L}}_\phi$. The inclusion $\phi(P_F) \subset \mathcal{L}_\phi$ holds by construction, and since $\phi(P_F)$ centralizes $\hat{\mathbf{L}}_\phi$, it normalizes ε_ϕ , whence the inclusion $\phi(P_F) \subset \mathcal{L}_{\phi, \varepsilon_\phi}$. Actually, $\phi(P_F)$ centralizes any maximal torus of $\hat{\mathbf{L}}_\phi$, so a $\hat{\mathbf{G}}$ -conjugate of $\phi(P_F)$ centralizes a reference maximal torus $\hat{\mathbf{T}}$ of $\hat{\mathbf{G}}$ (i.e., a part of a W_F -stable pinning ε of $\hat{\mathbf{G}}$). But since $\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes_\varepsilon \text{Out}(\hat{\mathbf{G}}) \hookrightarrow \text{Aut}(\hat{\mathbf{T}})$, where $\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}})$ denotes the Weyl group of $\hat{\mathbf{G}}$ with respect to $\hat{\mathbf{T}}$, the centralizer of $\hat{\mathbf{T}}$ in ${}^L\mathbf{G}$ is $\hat{\mathbf{T}} \times \ker(W_F \longrightarrow \text{Out}(\hat{\mathbf{G}}))$. It follows that the restriction of the action of W_F on $\hat{\mathbf{T}}$ to P_F is trivial, hence that \mathbf{G} is tamely ramified. Now, with $1 \times P_F$ and $\phi(P_F)$, also $\hat{\phi}(P_F)$ centralizes $\hat{\mathbf{L}}_\phi$, hence $\hat{\phi}(P_F) \subset Z(\hat{\mathbf{L}}_\phi)$.

We claim that properties i) and ii) are equivalent. Indeed, since \mathbf{G} is tamely ramified, Lemma 2.2.4 provides us with a splitting $\psi : W_F \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi}$ such that $\psi|_{P_F} = 1 \times \text{Id}$. Therefore, if φ is as in item i), we can write it in the form $\varphi = \hat{\varphi} \cdot \psi$ and $\hat{\varphi}$ is as in item ii). Conversely, the same formula shows the equivalence i) \Leftrightarrow ii).

Let us finally prove i). Since $\phi(P_F) \subset \mathcal{L}_{\phi, \varepsilon_\phi}$, the extension $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi} / \phi(P_F) \twoheadrightarrow W_F / P_F$ is a pullback of the extension $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi, \varepsilon_\phi} / \phi(P_F) \twoheadrightarrow W_F / P_F$. Therefore, there is a splitting as in i) if and only if the latter extension splits. Now, recalling that $\phi = \hat{\phi} \times \text{Id}_{P_F}$ and $\psi|_{P_F} = 1 \times \text{Id}_{P_F}$, we have a commutative diagram

$$\begin{array}{ccccc} Z(\hat{\mathbf{L}}_\phi) & \hookrightarrow & \mathcal{L}_{\phi, \varepsilon_\phi} / \phi(P_F) & \twoheadrightarrow & W_F / P_F \\ \hat{\phi} \uparrow & & \psi \uparrow & & \parallel \\ P_F & \hookrightarrow & W_F & \twoheadrightarrow & W_F / P_F \end{array}$$

that shows that the class of the upper line in $H^2(W_F/P_F, Z(\hat{\mathbf{L}}_\phi))$ is the pushforward $\hat{\phi}(\mathcal{E})$ of the canonical extension $\mathcal{E} = [W_F/[\overline{P_F}, P_F]] \in H^2(W_F/P_F, P_F^{\text{ab}})$. But the latter is known to vanish, e.g., by the main result of [Iwa55] (whose proof works also in equal characteristic). \square

2.2.6 *The category $\text{Rep}_R^\phi(G'_\phi)$.* — Let us assume that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup. In accordance with our notation in the introduction and in [Dat17], we write $\mathbf{G}_\phi = \mathbf{L}_\phi$ and we denote by \mathbf{G}'_ϕ an inner form of \mathbf{G}_ϕ . Then we have $C_{\hat{\mathbf{G}}}(\phi) = \widehat{\mathbf{G}}_\phi = \widehat{\mathbf{G}'_\phi}$, and any choice of ψ as in Lemma 2.2.4 provides a tamely ramified embedding ${}^L\mathbf{G}_\phi \hookrightarrow {}^L\mathbf{G}$, through which the morphism ϕ factors.

Corollary. — ϕ is a wild inertia parameter for \mathbf{G}'_ϕ .

Proof. This follows from Proposition 2.2.3 ii) applied to \mathbf{G}'_ϕ . Concretely, with $\hat{\phi}$ as in ii) of Lemma 2.2.5, we define φ' by $\varphi'|_{W_F} = \hat{\phi} \rtimes \text{Id}$ and $\varphi'|_{\text{SL}_2}$ given by a principal SL_2 of $C_{\hat{\mathbf{G}}}(\hat{\phi} \rtimes \text{Id})$. Then φ' does not factor through any proper Levi L -subgroup of ${}^L\mathbf{G}_\phi$. \square

Let us now fix a 1-cocycle $\hat{\phi} : W_F \rightarrow Z(C_{\hat{\mathbf{G}}}(\phi))$ as in Lemma 2.2.5 ii). Multiplication by $\hat{\phi}$ induces a bijection $Z^1(W_F, \widehat{\mathbf{G}'_\phi})_1 \xrightarrow{\sim} Z^1(W_F, \widehat{\mathbf{G}'_\phi})_{\hat{\phi}}$ where the index prescribes the restriction of cocycles to P_F . We are interested here in the representation theoretic counterpart of this bijection.

Let R be a commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra. Recall from the introduction the full subcategory $\text{Rep}_R^1(G'_\phi)$ of $\text{Rep}_R(G'_\phi)$ consisting of all depth-zero R -representations, i.e., whose objects are given by $\text{Rep}_R^1(G'_\phi) = \left\{ V \in \text{Rep}_R(G'_\phi), V = \sum_{x \in \mathcal{B}(\mathbf{G}'_\phi, F)} e_{G'_{\phi, x, 0+}} V \right\}$, where $\mathcal{B}(\mathbf{G}'_\phi, F)$ denotes the extended Bruhat–Tits building of \mathbf{G}'_ϕ and $e_{G'_{\phi, x, 0+}} \in \mathcal{H}_R(G'_\phi)$ is the averaging idempotent along the pro- p -radical $G'_{\phi, x, 0+}$ of the parahoric group at the point x .

Borel’s construction in [Bor79, 10.2] associates to the 1-cocycle $\hat{\phi}$ chosen above a character

$$\check{\varphi} : G'_\phi = \mathbf{G}'_\phi(F) \rightarrow \mathbb{C}^\times.$$

Any other choice $\hat{\phi}'$ differs from $\hat{\phi}$ by a cocycle $\hat{\delta} \in Z^1(W_F, Z(\hat{\mathbf{G}}_\phi))$ such that $\hat{\delta}|_{P_F} = 1 \times \text{Id}$. We then have $\check{\varphi}' = \check{\varphi}\check{\delta}$ for a character $\check{\delta}$ of G'_ϕ that is trivial on $G'_{\phi, x, 0+}$ for all $x \in \mathcal{B}(\mathbf{G}'_\phi, F)$ (see Lemma 2.4.1). It follows that for every $x \in \mathcal{B}(\mathbf{G}'_\phi, F)$, the restriction $(\check{\varphi})|_{G'_{\phi, x, 0+}}$ is independent of the choice of $\hat{\phi}$, and therefore the subcategory $\check{\varphi} \otimes \text{Rep}_\mathbb{C}^1(G'_\phi)$ of $\text{Rep}_\mathbb{C}(G'_\phi)$ is also independent of this choice. Now, the expected compatibility between the Langlands correspondence and twisting naturally leads us to put $\text{Rep}_\mathbb{C}^\phi(G'_\phi) := \check{\varphi} \otimes \text{Rep}_\mathbb{C}^1(G'_\phi)$. It is defined over any commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R by the following formula for its objects

$$\text{Rep}_R^\phi(G'_\phi) = \left\{ V \in \text{Rep}_R(G'_\phi), V = \sum_{x \in \mathcal{B}(\mathbf{G}'_\phi, F)} e_x^\phi V \right\}$$

where $e_x^\phi \in RG'_{\phi, x, 0+}$ is the idempotent associated to the restriction of any $\check{\varphi}$ to $G'_{\phi, x, 0+}$. Actually, the next lemma together with the fact that Borel’s procedure produces a group homomorphism $H^1(W_F, Z(\hat{\mathbf{G}}_\phi)) \rightarrow \text{Hom}(\mathbf{G}'_\phi(F), \mathbb{C}^\times)$, shows that we can choose $\hat{\phi}$ such that $\check{\varphi}$ has p -power order, and in particular is valued in $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]^\times$. We then have $\text{Rep}_R^\phi(G'_\phi) = \check{\varphi} \otimes \text{Rep}_R^1(G'_\phi)$.

2.2.7 Lemma. — Let \mathbf{H} be a tamely ramified reductive group over F . Given any $\hat{\varphi} \in Z^1(W_F, Z(\hat{\mathbf{H}}))$, one can find $\hat{\varphi}' \in Z^1(W_F, Z(\hat{\mathbf{H}}))$ such that $\hat{\varphi}'|_{P_F} = \hat{\varphi}|_{P_F}$ and $[\hat{\varphi}']$ has p -power order in the abelian group $H^1(W_F, Z(\hat{\mathbf{H}}))$.

Proof. We first claim that there exists an unramified 1-cocycle $\hat{\delta} \in Z^1(W_F/I_F, Z(\hat{\mathbf{H}})^{I_F})$ such that $[\hat{\varphi} \cdot \hat{\delta}^{-1}]$ has finite order in $H^1(W_F, Z(\hat{\mathbf{H}}))$. Indeed, consider the exact sequence

$$1 \longrightarrow H^1(W_F/I_F, Z(\hat{\mathbf{H}})^{I_F}) \xrightarrow{i} H^1(W_F, Z(\hat{\mathbf{H}})) \xrightarrow{p} H^1(I_F, Z(\hat{\mathbf{H}}))^{W_F/I_F}.$$

Since I_F is profinite and cohomology is continuous for the discrete topology on the coefficients, the last H^1 is a torsion group. So there is some integer N and some unramified cocycle $\hat{\delta}_1$ such that $[\hat{\varphi}]^N = i[\hat{\delta}_1]$. Let c be the order of the component group $\pi_0(Z(\hat{\mathbf{H}})^{I_F})$. Then, there is $\hat{\delta}_2 \in Z^1(W_F/I_F, (Z(\hat{\mathbf{H}})^{I_F})^\circ)$ such that $[\hat{\varphi}]^{Nc} = i[\hat{\delta}_2]$. But $Z^1(W_F/I_F, (Z(\hat{\mathbf{H}})^{I_F})^\circ) \simeq (Z(\hat{\mathbf{H}})^{I_F})^\circ$ is a divisible group, so we can find $\hat{\delta} \in Z^1(W_F/I_F, (Z(\hat{\mathbf{H}})^{I_F})^\circ)$ such that $[\hat{\varphi}]^{Nc} = i[\hat{\delta}]^{Nc}$, showing that $[\hat{\varphi} \hat{\delta}^{-1}]$ has finite order in $H^1(W_F, Z(\hat{\mathbf{H}}))$.

Now, write the order of $[\hat{\varphi} \hat{\delta}^{-1}]$ as $p^r \cdot M$ with M prime to p , choose $M' \in \mathbb{N}$ such that $MM' \equiv 1[p^r]$, and set $\hat{\varphi}' := (\hat{\varphi} \hat{\delta}^{-1})^{MM'}$. Then $[\hat{\varphi}']$ has order p^r in $H^1(W_F, Z(\hat{\mathbf{H}}))$ and $\hat{\varphi}'|_{P_F} = (\hat{\varphi}|_{P_F})^{MM'}$. Since P_F acts trivially on $Z(\hat{\mathbf{H}})$, we have $Z^1(P_F, Z(\hat{\mathbf{H}})) = H^1(P_F, Z(\hat{\mathbf{H}}))$, hence the order of $\hat{\varphi}|_{P_F}$ in $Z^1(P_F, Z(\hat{\mathbf{H}}))$ divides $p^r M$, hence it divides p^r , and the congruence $MM' \equiv 1[p^r]$ ensures that $(\hat{\varphi}|_{P_F})^{MM'} = \hat{\varphi}|_{P_F}$ in $Z^1(P_F, Z(\hat{\mathbf{H}}))$. \square

2.2.8 Levi-center-embeddings and root systems. — We still assume that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup. Fix an F -rational Levi-center-embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in the set I_ϕ of Proposition 2.2.3. From paragraph 2.2.6, we get a class of characters $\check{\varphi} : \mathbf{G}_\iota(F) \longrightarrow \mathbb{C}^\times$ modulo depth-0 characters, associated to ϕ .

Let \mathbf{S} be any tamely ramified maximal F -torus of \mathbf{G} containing $\iota(\mathbf{S}_\phi)$, and let $E \supseteq F$ be a tamely ramified Galois extension that splits \mathbf{S} . We then have a norm map $N_{E|F} : \mathbf{S}(E) \rightarrow \mathbf{S}(F)$ and an inclusion $\mathbf{S}(F) \subseteq \mathbf{G}_\iota(F)$.

Lemma. — For any character $\check{\varphi}$ of $\mathbf{G}_\iota(F)$ associated to ϕ , the (absolute) root system of \mathbf{G}_ι with respect to \mathbf{S} is given by

$$\Sigma(\mathbf{S}, \mathbf{G}_\iota) = \{\alpha \in \Sigma(\mathbf{S}, \mathbf{G}), \check{\varphi}(N_{E|F}(\alpha^\vee(E_{0+}^\times))) = \{1\}\}.$$

Proof. The inclusion $\mathbf{S} \subseteq \mathbf{G}_\iota$ gives rise to a W_F -stable conjugacy class of maximal torus embeddings $\hat{\mathbf{S}} \hookrightarrow C_{\hat{\mathbf{G}}}(\phi)$. Fix any such embedding and identify $\hat{\mathbf{S}}$ with a maximal torus in $C_{\hat{\mathbf{G}}}(\phi)$ thanks to this choice. Then, through the bijection $\alpha \leftrightarrow \alpha^\vee, \Sigma(\mathbf{S}, \mathbf{G}) \leftrightarrow \Sigma(\hat{\mathbf{S}}, \hat{\mathbf{G}})$, the subset $\Sigma(\mathbf{S}, \mathbf{G}_\iota)$ corresponds to $\Sigma(\hat{\mathbf{S}}, C_{\hat{\mathbf{G}}}(\phi))$, by the construction in 2.1.3.

On the other hand, $\hat{\mathbf{S}}$ contains $Z(C_{\hat{\mathbf{G}}}(\phi))$ and hence $\hat{\varphi}$ (as in 2.2.5.ii) and 2.2.6) factors through $\hat{\mathbf{S}}$, giving a Langlands parameter that we still denote by $\hat{\varphi} \in Z^1(W_F, \hat{\mathbf{S}})$. This is the Langlands parameter of the character $\check{\varphi} : \mathbf{S}(F) \hookrightarrow \mathbf{G}_\iota(F) \longrightarrow \mathbb{C}^\times$. Then the Langlands parameter of the character $\check{\varphi} \circ N_{E|F} : \mathbf{S}(E) \longrightarrow \mathbb{C}^\times$ is $\hat{\varphi}|_{W_E}$. Accordingly, the character $\check{\varphi} \circ N_{E|F} \circ \alpha^\vee : E^\times \longrightarrow \mathbb{C}^\times$ corresponds via the local class field reciprocity to the character $\alpha^\vee \circ \hat{\varphi}$ of W_E (where α^\vee is first seen as a cocharacter of \mathbf{S} , then as a character of $\hat{\mathbf{S}}$). Its restriction to E_{0+}^\times is therefore trivial if and only if $\alpha^\vee \circ \hat{\varphi}$ is a trivial character of $P_E = P_F$, which is equivalent to α^\vee being a root of $\hat{\mathbf{S}}$ in the centralizer $C_{\hat{\mathbf{G}}}(\phi)$, as desired. \square

2.2.9 *On the condition that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup.* — Starting from Lemma 2.2.5 above, our results have been conditional on the hypothesis that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup, and this hypothesis will be in force in the rest of the paper. Here we discuss how strong this hypothesis is. It is easy to see that it implies that $\hat{\phi}$ factors through a maximal torus of $\hat{\mathbf{G}}$, which implies in turn that $\hat{\phi}(P_F)$ is an abelian group. In general the reverse implications may not be true. For this reason, Steinberg introduced in [Ste75] the notion of “torsion prime” for $\hat{\mathbf{G}}$, which is a prime that either is bad for the root system of $\hat{\mathbf{G}}$, or divides the order of $\pi_1(\hat{\mathbf{G}}_{\text{der}})$. The following lemma follows from his results.

Lemma. — *Let P be a finite p -subgroup of $\hat{\mathbf{G}}$. Consider the following properties of P :*

- (a) *The centralizer $C_{\hat{\mathbf{G}}}(P)$ is a Levi subgroup of $\hat{\mathbf{G}}$.*
- (b) *The connected centralizer $C_{\hat{\mathbf{G}}}(P)^\circ$ is a Levi subgroup of $\hat{\mathbf{G}}$.*
- (c) *P is contained in a maximal torus of $\hat{\mathbf{G}}$.*
- (d) *P is abelian.*

Then the following holds.

- i) We always have $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.*
- ii) If p is good for the root system of $\hat{\mathbf{G}}$, then $(b) \Leftrightarrow (c)$.*
- iii) If p is not a torsion prime of $\hat{\mathbf{G}}$, then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$.*
- iv) If p does not divide the order of the Weyl group of $\hat{\mathbf{G}}$, then (a) holds true for all P .*

Note that each implication $(d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$ fails to be true in full generality.

Proof. i) Only the implication $(b) \Rightarrow (c)$ is not tautological. So assume $C_{\hat{\mathbf{G}}}(P)^\circ$ is a Levi subgroup of $\hat{\mathbf{G}}$. Then its centralizer in $\hat{\mathbf{G}}$ is its center, which is contained in all its maximal tori. Since P centralizes $C_{\hat{\mathbf{G}}}(P)^\circ$, we are done.

ii) Assume p good. When P is cyclic, the fact that $C_{\hat{\mathbf{G}}}(P)^\circ$ is a Levi subgroup is proved e.g. in [AS08, Prop. A.7]. For a more general P contained in a maximal torus $\hat{\mathbf{T}}$, let us split it as a product $P = P_1 \times \cdots \times P_r$ of cyclic p -groups. Each $C_{\hat{\mathbf{G}}}(P_i)^\circ$ is a Levi subgroup that contains $\hat{\mathbf{T}}$, hence their intersection $\bigcap_i C_{\hat{\mathbf{G}}}(P_i)^\circ$ is also a Levi subgroup containing $\hat{\mathbf{T}}$. Since $C_{\hat{\mathbf{G}}}(P) = \bigcap_i C_{\hat{\mathbf{G}}}(P_i)$, we infer that $C_{\hat{\mathbf{G}}}(P)^\circ = \bigcap_i C_{\hat{\mathbf{G}}}(P_i)^\circ$ is a Levi subgroup.

iii) Assume p is not a torsion prime of $\hat{\mathbf{G}}$. We only have to prove $(d) \Rightarrow (a)$, so let us assume that P is abelian. When P is cyclic, hence contained in a torus, we already know that $C_{\hat{\mathbf{G}}}(P)^\circ$ is a Levi subgroup by ii). On top of that, since p does not divide $|\pi_1(\hat{\mathbf{G}}_{\text{der}})|$, we have $C_{\hat{\mathbf{G}}}(P) = C_{\hat{\mathbf{G}}}(P)^\circ$ by [Ste75, Cor. 2.16]. For a general abelian P , let us argue by induction on its rank r , i.e. the number of cyclic factors of P . We can write $P = P' \times P''$ with P' cyclic and P'' of rank $r - 1$. Then P'' is contained in $\hat{\mathbf{L}} := C_{\hat{\mathbf{G}}}(P')$ and we have $C_{\hat{\mathbf{G}}}(P) = C_{\hat{\mathbf{L}}}(P'')$. Since $\hat{\mathbf{L}}$ is a Levi subgroup of $\hat{\mathbf{G}}$, the prime p is not a torsion prime of $\hat{\mathbf{L}}$. So, by induction $C_{\hat{\mathbf{L}}}(P'')$ is a Levi subgroup of $\hat{\mathbf{L}}$, hence also of $\hat{\mathbf{G}}$.

iv) Let us assume that p does not divide the order of the Weyl group of $\hat{\mathbf{G}}$, and let P be a finite p -subgroup of $\hat{\mathbf{G}}$. By [Ste75, Cor. 2.8], p is not a torsion prime of $\hat{\mathbf{G}}$, hence, by iii), the

centralizer of any abelian subgroup A of P is a Levi subgroup $\hat{\mathbf{M}}$. If A is normal in P , then P normalizes $\hat{\mathbf{M}}$. Since $N_{\hat{\mathbf{G}}}(\hat{\mathbf{M}})/\hat{\mathbf{M}}$ is a subquotient of the Weyl group of $\hat{\mathbf{G}}$, and thus has order prime to p , it follows that P is actually contained in $\hat{\mathbf{M}}$, and thus centralizes A . However, a non-abelian p -group Q always contains an abelian, normal, but non-central, subgroup; for example, the subgroup generated by the center $Z(Q)$ and any non-central element of the second-center (the inverse image of the center of $Q/Z(Q)$). Therefore P must be abelian. \square

From the point of view of the representation theory of G , the hypothesis that p is good is quite mild. For a quasi-simple group, it is empty in type A_n and it means $p \neq 2$ in types B_n, C_n and D_n , $p > 3$ in type G_2, E_6 and E_7 , and $p > 5$ in type E_8 . For such groups, not being a torsion prime is equivalent to being good except in type A_n . There, it is at least satisfied if p does not divide $n + 1$, but, for example GL_n has no torsion prime. The hypothesis that p does not divide the order of the Weyl group is obviously a much stronger one, since it excludes GL_n for $n \geq p$.

2.3 Ramification groups and twisted Levi sequences

We denote by I_F^r , $r \in \mathbb{R}_+$, the ramification subgroups of the Galois group Γ_F in the upper numbering. We also put $I_F^{r+} := \overline{\bigcup_{s>r} I_F^s}$. So we have $I_F^0 = I_F$ and $I_F^{0+} = P_F$. As seems to be customary, we use the notation $\tilde{\mathbb{R}} := \mathbb{R} \sqcup \{r+, r \in \mathbb{R}\}$, which is ordered by letting $r < r+ < s$ for any $r < s \in \mathbb{R}$.

We fix a wild inertia parameter $\phi : P_F \rightarrow {}^L\mathbf{G}$ and we assume that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup.

By Lemma 2.2.5, this implies that \mathbf{G} is tamely ramified and that $\hat{\phi}(P_F)$ is a finite abelian p -group contained in the center of $C_{\hat{\mathbf{G}}}(\phi) = \hat{\mathbf{G}}_\phi = \hat{\mathbf{L}}_\phi$. Recall that \mathbf{S}_ϕ denotes the F -torus that is dual to $\hat{\mathbf{S}}_\phi = (\hat{\mathbf{G}}_\phi)_{\mathrm{ab}}$ with its canonical Galois action. We are going to define a filtration of \mathbf{S}_ϕ by F -subtori. For this, we assume from now on that the following hypothesis is satisfied:

(H1) p is not a torsion prime of $\hat{\mathbf{G}}$.

2.3.1 The groups $\mathbf{G}_{\phi,r}$ and $\mathbf{S}_{\phi,r}$. — Fix $r \in \tilde{\mathbb{R}}_{>0}$. We put $\hat{\mathbf{G}}_{\phi,r} := C_{\hat{\mathbf{G}}}(\phi(I_F^r))$ and $\hat{\mathbf{S}}_{\phi,r} := (\hat{\mathbf{G}}_{\phi,r})_{\mathrm{ab}}$. By Hypothesis (H1) and Lemma 2.2.9, $\hat{\mathbf{G}}_{\phi,r}$ is a Levi subgroup of $\hat{\mathbf{G}}$ that contains $C_{\hat{\mathbf{G}}}(\phi)$. Therefore, the group $\mathcal{G}_{\phi,r} := \hat{\mathbf{G}}_{\phi,r} \cdot \varphi(W_F)$ does not depend on the choice φ of an extension of ϕ to W_F and sits in an exact sequence $\hat{\mathbf{G}}_{\phi,r} \hookrightarrow \mathcal{G}_{\phi,r} \twoheadrightarrow W_F$ which provides a finite outer action $W_F \rightarrow \mathrm{Out}(\hat{\mathbf{G}}_{\phi,r})$ and thus defines a quasi-split reductive F -group $\mathbf{G}_{\phi,r}$. Since $\hat{\phi}(P_F)$ is contained in $C_{\hat{\mathbf{G}}}(\phi)$ hence also in $\hat{\mathbf{G}}_{\phi,r}$, the outer action factors through W_F/P_F and accordingly $\mathbf{G}_{\phi,r}$ is tamely ramified. Also this outer action descends to $\hat{\mathbf{S}}_{\phi,r}$, providing a dual tamely ramified F -torus $\mathbf{S}_{\phi,r}$ with a canonical isomorphism $\mathbf{S}_{\phi,r} \xrightarrow{\sim} Z(\mathbf{G}_{\phi,r})^\circ$.

Let us choose a pinning $\varepsilon_{\phi,r}$ of $\hat{\mathbf{G}}_{\phi,r}$ and consider the stabilizer $\mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$ of this pinning in $\mathcal{G}_{\phi,r}$, which sits in an exact sequence $Z(\hat{\mathbf{G}}_{\phi,r}) \hookrightarrow \mathcal{G}_{\phi,r,\varepsilon_{\phi,r}} \twoheadrightarrow W_F$. Observe that $1 \times P_F$ is contained in $\mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$. By the same arguments as in the proof of Lemma 2.2.4, we have :

Lemma. — *There exists a splitting $\psi_r : W_F \rightarrow \mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$ of the exact sequence $Z(\hat{\mathbf{G}}_{\phi,r}) \hookrightarrow \mathcal{G}_{\phi,r,\varepsilon_{\phi,r}} \twoheadrightarrow W_F$ such that $\psi_r|_{P_F} = 1 \times \mathrm{Id}$.*

Such a ψ_r induces an isomorphism ${}^L\mathbf{G}_{\phi,r} \xrightarrow{\sim} \mathcal{G}_{\phi,r}$ that is the identity on $1 \times P_F$, hence allows us to see ϕ as a wild inertia parameter for $\mathbf{G}_{\phi,r}$.

Now, the inclusion $\hat{\mathbf{G}}_\phi \subseteq \hat{\mathbf{G}}_{\phi,r}$ induces a W_F -equivariant epimorphism $\hat{\mathbf{S}}_\phi \twoheadrightarrow \hat{\mathbf{S}}_{\phi,r}$, which on the dual side induces an F -rational embedding $\mathbf{S}_{\phi,r} \hookrightarrow \mathbf{S}_\phi$. Note that the latter embedding only depends on ϕ , and on no other choice. For any F -rational Levi-center-embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in the set I_ϕ of Proposition 2.2.3, the restriction $\iota|_{\mathbf{S}_{\phi,r}}$ is an F -rational Levi-center-embedding, and the twisted Levi subgroup $C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$ of \mathbf{G} has L -group ${}^L\mathbf{G}_{\phi,r}$.

2.3.2 Lemma. — ϕ is a wild inertia parameter of $C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$.

Proof. This follows from ii) of Proposition 2.2.3 applied to the group $\mathbf{G}'_{\phi,r} := C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$. Indeed, since $C_{\hat{\mathbf{G}}_{\phi,r}}(\phi) = C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}_{\phi,r}$, it is connected, and any pinning is normalized by a suitable extension of ϕ , according to Lemma 2.2.5 i). On the other hand, ι induces a F -rational Levi-center-embedding $\mathbf{S}_\phi \hookrightarrow \mathbf{G}'_{\phi,r} = C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$ that is dual to the Levi-center-embedding $\hat{\iota} : Z(C_{\hat{\mathbf{G}}}(\phi))^\circ \hookrightarrow \hat{\mathbf{G}}_{\phi,r}$. So we may apply Proposition 2.2.3 ii). \square

2.3.3 Lemma. — Let $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ be an F -rational Levi-center-embedding in the set I_ϕ of Proposition 2.2.3, and let \mathbf{S} be a maximal F -torus of \mathbf{G}_ι split by some tamely ramified Galois extension E of F . Then for any character $\check{\varphi}$ of $\mathbf{G}_\iota(F)$ associated to ϕ as in 2.2.6, we have

$$\Sigma(\mathbf{S}, C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))) = \{\alpha \in \Sigma(\mathbf{S}, \mathbf{G}), \check{\varphi}(N_{E|F}(\alpha^\vee(E_r^\times))) = \{1\}\}.$$

Proof. As in the proof of Lemma 2.2.8, fix a dual embedding $\hat{\mathbf{S}} \subset C_{\hat{\mathbf{G}}}(\phi)$. Then, by the construction in 2.1.3, the bijection $\alpha \leftrightarrow \alpha^\vee, \Sigma(\mathbf{S}, \mathbf{G}) \leftrightarrow \Sigma(\hat{\mathbf{S}}, \hat{\mathbf{G}})$ takes $\Sigma(\mathbf{S}, C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r})))$ to $\Sigma(\hat{\mathbf{S}}, C_{\hat{\mathbf{G}}}(\phi(I_F^r))) = \{\alpha^\vee \in \Sigma(\hat{\mathbf{S}}, \hat{\mathbf{G}}), \alpha^\vee \circ \hat{\phi}(I_F^r) = \{1\}\}$. It remains to follow the proof of [Kal19, Lemma 3.6.1]. Indeed $\alpha^\vee \circ \hat{\phi}|_{W_E}$ corresponds to $\check{\varphi} \circ N_{E|F} \circ \alpha^\vee$ via the local class field reciprocity $E^\times \xrightarrow{\sim} W_E^{\text{ab}}$, while the latter also takes E_r^\times to the image of $I_E^r = I_F^r$ in W_E^{ab} . The lemma follows. \square

2.3.4 The twisted Levi sequence associated to ϕ and ι . — We denote by $0 < r_0 < \dots < r_{d-1}$ the jumps of the decreasing filtration $(\mathbf{S}_{\phi,r})_{r>0}$ of \mathbf{S}_ϕ . Namely we have

$$\{r_0, \dots, r_{d-1}\} = \{r > 0, \mathbf{S}_{\phi,r+} \subsetneq \mathbf{S}_{\phi,r}\} = \{r > 0, C_{\hat{\mathbf{G}}}(\phi(I_F^{r+})) \supsetneq C_{\hat{\mathbf{G}}}(\phi(I_F^r))\}.$$

Note that $\mathbf{S}_{\phi,r} = \mathbf{S}_\phi$ for $r \leq r_0$ while $\mathbf{S}_{\phi,r} = Z(\mathbf{G})^\circ$ for $r > r_{d-1}$. We also put $r_{-1} := 0$ and $r_d := \text{depth}(\phi) := \inf\{r > 0, \phi(I_F^r) = \{1\}\}$, which satisfies $r_d \geq r_{d-1}$.

Now fix an F -rational Levi-center-embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in I_ϕ . In order to simplify the notation a bit, we set

$$\mathbf{G}_\iota^i := C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r_i})) = C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r_{i-1}+})) \text{ for } i = 0, \dots, d-1 \text{ and } \mathbf{G}_\iota^d := \mathbf{G}$$

We thus obtain a tamely ramified twisted Levi sequence in \mathbf{G}

$$\vec{\mathbf{G}}_\iota := (\mathbf{G}_\iota = \mathbf{G}_\iota^0 \subset \dots \subset \mathbf{G}_\iota^d = \mathbf{G}).$$

2.4 Characters and idempotents

In this section, we will use the Langlands correspondence for characters described by Borel in [Bor79, 10.2] to construct certain characters of $\mathbf{G}_i^i(F)$ that are suitable to apply Yu's procedure in [Yu01] and obtain characters of certain open pro- p subgroups of G . Yu's work involves the group side analogue of the ramification filtration, namely the Moy–Prasad filtrations [MP94, MP96]. For each point x in the (enlarged) Bruhat–Tits building $\mathcal{B}(\mathbf{G}, F)$ of \mathbf{G} we thus have a filtration $(G_{x,r} = \mathbf{G}(F)_{x,r})_{r \geq 0}$ of the stabilizer $G_x = \mathbf{G}(F)_x$ of x by compact, open, normal subgroups. If we set $G_{x,r+} := \bigcup_{s > r} G_{x,s}$, then $G_{x,0+}$ is known to be the pro- p -radical of the parahoric group $G_{x,0}$. We will need the following relation between both filtrations, which follows easily from Yu's [Yu09, Theorem 7.10].

2.4.1 Lemma. — *Let \mathbf{G} be a tamely ramified reductive group over F and let $\check{\varphi} : \mathbf{G}(F) \rightarrow \mathbb{C}^\times$ be the character associated to some $\hat{\varphi} \in H^1(W_F, Z(\hat{\mathbf{G}}))$. Then $\check{\varphi}$ is trivial on $\mathbf{G}_{\text{sc}}(F)$ and for every $x \in \mathcal{B}(\mathbf{G}, F)$ and every $r \in \tilde{\mathbb{R}}_{\geq 0}$ we have $\check{\varphi}|_{G_{x,r}} \equiv 1 \Leftrightarrow \hat{\varphi}|_{I_F^r} \equiv 1$.*

Proof. We need to go through Borel's procedure in [Bor79, 10.2]. So let $\tilde{\mathbf{G}} \rightarrow \mathbf{G}$ be a z -extension, i.e., a central extension $\tilde{\mathbf{G}}$ of \mathbf{G} by an induced torus \mathbf{Z} such that the derived subgroup of $\tilde{\mathbf{G}}$ is simply connected. On the dual side we get a W_F -equivariant embedding of $Z(\hat{\mathbf{G}})$ into the torus $Z(\hat{\tilde{\mathbf{G}}})$. Pushing $\hat{\varphi}$ by this embedding we get a Langlands parameter for the tamely ramified torus $\tilde{\mathbf{G}}_{\text{ab}}$, whence a character $\tilde{\theta}$ of $\tilde{\mathbf{G}}_{\text{ab}}(F)$. By [Yu09, Thm 7.10] we have $\tilde{\theta}|_{\tilde{\mathbf{G}}_{\text{ab}}(F)_r} \equiv 1 \Leftrightarrow \hat{\varphi}|_{I_F^r} \equiv 1$. Now, $\check{\varphi}$ is defined as follows. The map $\tilde{\mathbf{G}}(F) \rightarrow \mathbf{G}(F)$ is surjective and the character $\tilde{\theta} : \tilde{\mathbf{G}}(F) \rightarrow \tilde{\mathbf{G}}_{\text{ab}}(F) \rightarrow \mathbb{C}^\times$ is trivial on the kernel $\mathbf{Z}(F)$ of this map and on $\tilde{\mathbf{G}}_{\text{der}}(F) = \mathbf{G}_{\text{sc}}(F)$. Therefore $\tilde{\theta}$ descends to the desired character $\check{\varphi}$ of $\mathbf{G}(F)$, which is trivial on (the image of) $\mathbf{G}_{\text{sc}}(F)$. Now, for every $x \in \mathcal{B}(\mathbf{G}, F)$ and every $\tilde{x} \in \mathcal{B}(\tilde{\mathbf{G}}, F)$ that projects onto x , Lemma 3.5.3 of [Kal19] tells us that the maps $\tilde{\mathbf{G}}(F)_{\tilde{x},r} \rightarrow \tilde{\mathbf{G}}_{\text{ab}}(F)_r$ and $\tilde{\mathbf{G}}(F)_{\tilde{x},r} \rightarrow \mathbf{G}(F)_{x,r}$ are both surjective. This implies the equivalence claimed in the lemma. \square

2.4.2 Remark. — *Conversely, any character $\check{\varphi} : \mathbf{G}(F) \rightarrow \mathbb{C}^\times$ that is trivial on $\mathbf{G}_{\text{sc}}(F)$ comes from some $\hat{\varphi} \in H^1(W_F, Z(\hat{\mathbf{G}}))$ via Borel's procedure. Indeed, with the notation of the above proof, the surjectivity of $\tilde{\mathbf{G}}(F) \rightarrow \mathbf{G}(F)$ allows one to inflate $\check{\varphi}$ to a character $\tilde{\theta}$ of $\tilde{\mathbf{G}}(F)$ that is trivial on $\mathbf{Z}(F)\tilde{\mathbf{G}}_{\text{der}}(F)$. In particular $\tilde{\theta}$ factors through the surjective map $\tilde{\mathbf{G}}(F) \rightarrow \tilde{\mathbf{G}}_{\text{ab}}(F)$, giving a character of $\tilde{\mathbf{G}}_{\text{ab}}(F)$ which, by Langlands' correspondence for tori, comes from some $\hat{\varphi} \in H^1(W_F, Z(\hat{\mathbf{G}}))$. But the pushforward of $\hat{\varphi}$ into $H^1(W_F, \hat{\mathbf{Z}})$ has to be trivial, hence $\hat{\varphi}$ comes from $H^1(W_F, Z(\hat{\mathbf{G}}))$.*

Recall the definitions of $\hat{\mathbf{G}}_{\phi,r}$ and $\mathcal{G}_{\phi,r}$ from 2.3.1, as well as that of $\mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$ for a pinning $\varepsilon_{\phi,r}$ of $\hat{\mathbf{G}}_{\phi,r}$. Observe that $\phi(I_F^r)$ is contained in $\mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$.

2.4.3 Lemma. — *The following hypotheses are equivalent.*

- i) *There is a splitting $\varphi_r : W_F \rightarrow \mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$ such that $\varphi_r|_{I_F^r} = \phi|_{I_F^r}$.*
- ii) *There is $\hat{\varphi}_r \in Z^1(W_F, Z(\hat{\mathbf{G}}_{\phi,r}))$ such that $\hat{\varphi}_r|_{I_F^r} = \hat{\phi}|_{I_F^r}$.*
- iii) *The image $\hat{\phi}(\mathcal{E}_r) \in H^2(W_F/I_F^r, Z(\hat{\mathbf{G}}_{\phi,r}))$ of the canonical extension $\mathcal{E}_r = [W_F/\overline{[I_F^r, I_F^r]}] \in H^2(W_F/I_F^r, I_F^{r,\text{ab}})$ vanishes.*

Further, these hypotheses are satisfied if p does not divide $|\pi_0(Z(\hat{\mathbf{G}}))| = |\pi_1(\mathbf{G}_{\text{der}})|$.

Proof. Thanks to Lemma 2.3.1, the equivalence between the three hypotheses is proved as in Lemma 2.2.5. For the last assertion, observe that if p does not divide the order $|\pi_0(Z(\hat{\mathbf{G}}))|$ of $\pi_0(Z(\hat{\mathbf{G}}))$, then it does not divide $|\pi_0(Z(\hat{\mathbf{G}}_{\phi,r}))|$ either, since $Z(\hat{\mathbf{G}}_{\phi,r}) = Z(\hat{\mathbf{G}}_{\phi,r})^\circ Z(\hat{\mathbf{G}})$. Therefore, in this case, we have $\hat{\phi}(P_F) \subset Z(\hat{\mathbf{G}}_{\phi,r})^\circ$, hence $\hat{\phi}(\mathcal{E}_r)$ lies in the image of the map $H^2(W_F/I_F^r, Z(\hat{\mathbf{G}}_{\phi,r})^\circ) \rightarrow H^2(W_F/I_F^r, Z(\hat{\mathbf{G}}_{\phi,r}))$. By Lemma 2.4.4 below, it follows that $\hat{\phi}(\mathcal{E}_r)$ vanishes. \square

2.4.4 Lemma. — *Let \mathbf{S} be a tamely ramified torus and $r \in \tilde{\mathbb{R}}_{>0}$. Then the image of $H_{\text{cts}}^2(\Gamma_F/I_F^r, \hat{\mathbf{S}})$ in $H^2(W_F/I_F^r, \hat{\mathbf{S}})$ is trivial.*

Proof. Start with $\eta \in H_{\text{cts}}^2(\Gamma_F/I_F^r, \hat{\mathbf{S}})$ and let $\bar{\eta}$ denote its image in $H^2(W_F/I_F^r, \hat{\mathbf{S}})$. By definition of continuous cohomology, η comes from an element $\eta \in H^2(\Gamma_{E/F}, \hat{\mathbf{S}})$ with E a finite extension that splits \mathbf{S} and that is r -ramified in the sense that I_F^r maps to $\{1\}$ in $\Gamma_{E/F}$. We may and will assume that \mathbf{S} is also split by the maximal tamely ramified subextension E^{tr} of E over F . As in the proof of [Lan79, Lemma 4], we can choose an exact sequence $\hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{S}}_1 \twoheadrightarrow \hat{\mathbf{S}}_2$ with \mathbf{S}_1 an induced torus for $\Gamma_{E^{tr}/F}$. Note that each \mathbf{S}_i is then tamely ramified. Let us look at the exact sequence

$$H^1(W_F/I_F^r, \hat{\mathbf{S}}_1) \longrightarrow H^1(W_F/I_F^r, \hat{\mathbf{S}}_2) \longrightarrow H^2(W_F/I_F^r, \hat{\mathbf{S}}) \longrightarrow H^2(W_F/I_F^r, \hat{\mathbf{S}}_1)$$

Since $H^2(\Gamma_{E/F}, \hat{\mathbf{S}}_1) = \{1\}$ (because \mathbf{S}_1 is an induced torus also for E/F), the image of $\bar{\eta}$ in $H^2(W_F/I_F^r, \hat{\mathbf{S}}_1)$ is trivial. So if we can prove that the first map is surjective, we infer that $\bar{\eta}$ itself is trivial. But by [Yu09, Theorem 7.10] the local Langlands correspondence identifies $H^1(W_F/I_F^r, \hat{\mathbf{S}}_i)$ with the group of characters of the group $\mathbf{S}_i(F)/\mathbf{S}_i(F)_r$. Moreover, by [Kal19, Lemma 3.1.3] the dual embedding $\mathbf{S}_2 \hookrightarrow \mathbf{S}_1$ satisfies $\mathbf{S}_2(F)_r = \mathbf{S}_2(F) \cap \mathbf{S}_1(F)_r$. So this dual embedding induces an injective map $\mathbf{S}_2(F)/\mathbf{S}_2(F)_r \hookrightarrow \mathbf{S}_1(F)/\mathbf{S}_1(F)_r$ which shows the surjectivity of the map $H^1(W_F/I_F^r, \hat{\mathbf{S}}_1) \rightarrow H^1(W_F/I_F^r, \hat{\mathbf{S}}_2)$. \square

2.4.5 Characters. — We now fix an F -rational embedding $\iota \in I_\phi$ and we take up the notation $\vec{\mathbf{G}}_\iota$ of 2.3.4. From now on, we will make the following additional hypothesis:

$$(H2) \quad p \text{ does not divide } |\pi_0(Z(\hat{\mathbf{G}}))| = |\pi_1(\mathbf{G}_{\text{der}})|.$$

Note that (H1) and (H2) together mean that p is neither a torsion prime of \mathbf{G} , nor of $\hat{\mathbf{G}}$. Thanks to this hypothesis, Lemma 2.4.3 ensures the existence of a 1-cocycle $\hat{\varphi}_i : W_F \rightarrow Z(\hat{\mathbf{G}}_{\phi, r_{i-1}+})$ that extends $\hat{\phi}|_{I_F^{r_{i-1}+}}$ for each $i = 0, \dots, d$. Using Lemma 2.2.7, we may assume that $\hat{\varphi}_i$ has p -power order in $H^1(W_F, Z(\hat{\mathbf{G}}_{\phi, r_{i-1}+}))$. Then, since \mathbf{G}_ι^i is an inner form of $\mathbf{G}_{\phi, r_{i-1}+}$, the Langlands correspondence for characters [Bor79, 10.2] associates to $\hat{\varphi}_i$ a character

$$\check{\varphi}_i : \mathbf{G}_\iota^i(F) \rightarrow \mu_{p^\infty} \subset \mathbb{C}^\times,$$

which we may view as an R -valued character for any commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R . Lemma 2.4.1 has the following consequences, for every $x \in \mathcal{B}(\mathbf{G}_\iota^i, F)$:

- i) the restriction $(\check{\varphi}_i)|_{\mathbf{G}_\iota^i(F)_{x, r_{i-1}+}}$ only depends on $\hat{\phi}|_{I_F^{r_{i-1}+}}$, and not on the choice of $\hat{\varphi}_i$,

ii) for all $j \geq i$ we have $(\check{\varphi}_i)|_{\mathbf{G}_\ell^i(F)_{x,r_{j-1}+}} = (\check{\varphi}_j)|_{\mathbf{G}_\ell^i(F)_{x,r_{j-1}+}}$,

iii) the character $\psi_i := \check{\varphi}_i \check{\varphi}_{i+1}^{-1}$ of $\mathbf{G}_\ell^i(F)$ is trivial on $\mathbf{G}_\ell^i(F)_{x,r_i+}$ (where we set $\check{\varphi}_{d+1} = 1$).

2.4.6 *The subset \mathcal{B}_ℓ of the building.* — We write \mathcal{B} for the (extended) Bruhat–Tits building $\mathcal{B}(\mathbf{G}, F)$. If \mathbf{S} is a maximal F -torus of \mathbf{G} that splits over some tamely ramified finite field extension E of F , we set $\mathcal{B}(\mathbf{S}, F) := \mathcal{A}(\mathbf{G}, \mathbf{S}, E) \cap \mathcal{B}(\mathbf{G}, F)$, where $\mathcal{A}(\mathbf{G}, \mathbf{S}, E)$ is the apartment of $\mathcal{B}(\mathbf{G}, E)$ associated with \mathbf{S} and the intersection is taken in $\mathcal{B}(\mathbf{G}, E)$. As the notation suggests, this does not depend on the choice of E . Note that it need not be an apartment of \mathcal{B} , unless \mathbf{S} has maximal F -split rank. Now we associate to ι the following subset of \mathcal{B} :

$$\mathcal{B}_\ell := \bigcup_{\mathbf{S} \subset \mathbf{G}_\ell} \mathcal{B}(\mathbf{S}, F)$$

where \mathbf{S} runs over tamely ramified maximal F -tori of \mathbf{G}_ℓ . The set \mathcal{B}_ℓ is also the common image of all the embeddings $\mathcal{B}(\mathbf{G}_\ell, F) \hookrightarrow \mathcal{B}$ obtained as restriction of a $\text{Gal}(E/F)$ -equivariant admissible embedding (in the sense of [KP23, §14.2]) $\mathcal{B}(\mathbf{G}_\ell, E) \hookrightarrow \mathcal{B}(\mathbf{G}, E)$ for a Galois, tamely ramified field extension E such that $\mathbf{G}_{\ell,E}$ is the Levi component of an E -rational parabolic subgroup of \mathbf{G}_E . From now on when writing “admissible embedding” we mean an embedding of the kind just described. The set of such embeddings is a torsor under $X_*(\mathbf{S}_\ell)_{\mathbb{R}}^{W_F}$. We could have restricted the above union to maximally F -split (tamely ramified maximal) F -tori of \mathbf{G}_ℓ , thanks to [Yu01, Lemma 2.1]. For such a maximally split torus, the subset $\mathcal{B}(\mathbf{S}, F)$ is an apartment of $\mathcal{B}(\mathbf{G}_\ell, F)$, but it is not an apartment of \mathcal{B} unless \mathbf{G}_ℓ is an F -Levi subgroup.

2.4.7 *A construction of Yu.* — Let us fix $x \in \mathcal{B}_\ell$. For each $i = 0, \dots, d$, the intersection

$$G_{\ell,x,r}^i := \mathbf{G}_\ell^i(F)_{x,r} := G_{x,r} \cap \mathbf{G}_\ell^i(F)$$

is the Moy–Prasad group associated to r and the preimage of x by any admissible embedding $\mathcal{B}(\mathbf{G}_\ell^i, F) \hookrightarrow \mathcal{B}$. Note that $G_{\ell,x,r}^i$ normalizes $G_{\ell,x,s}^j$ whenever $i \leq j$, so that we can define an open subgroup of $G_{x,0+}$ by

$$K_{\ell,x}^{++} := G_{\ell,x,0+}^0 G_{\ell,x,r_0+}^1 \cdots G_{\ell,x,r_{d-1}+}^d.$$

By property ii) of 2.4.5, there exists a character $\check{\phi}_{\ell,x}^{++}$ of $K_{\ell,x}^{++}$ whose restriction to $G_{\ell,x,r_{i-1}+}^i$ agrees with $\check{\varphi}_i|_{G_{\ell,x,r_{i-1}+}^i}$. By property i) of 2.4.5, $\check{\phi}_{\ell,x}^{++}$ *only depends on ϕ* and not on the choice of the extensions $\hat{\varphi}_i$. Now let us consider the following bigger open subgroup of $G_{x,0+}$:

$$K_{\ell,x}^+ := G_{\ell,x,0+}^0 G_{\ell,x,(r_0/2)+}^1 \cdots G_{\ell,x,(r_{d-1}/2)+}^d.$$

In [Yu01, §4], Yu describes a construction of a character $\check{\phi}_{\ell,x}^+$ of $K_{\ell,x}^+$ that extends $\check{\phi}_{\ell,x}^{++}$, starting from the characters ψ_i of 2.4.5. To explain this, we first observe that $\check{\phi}_{\ell,x}^{++}$ is also the product $\prod_{i=0}^d (\psi_{i,x}^{++})|_{K_{\ell,x}^{++}}$ where $\psi_{i,x}^{++}$ denotes the unique character of the group $G_{\ell,x,0+}^i G_{x,r_i+}$ that extends both $\psi_i|_{G_{\ell,x,0+}^i}$ and the trivial character of G_{x,r_i+} . Similarly, Yu defines $\check{\phi}_{\ell,x}^+$ as a product

$$\check{\phi}_{\ell,x}^+ := \prod_{i=0}^d (\psi_{i,x}^+)|_{K_{\ell,x}^+}$$

where $\psi_{i,x}^+$ is a certain character of $G_{\iota,x,0}^i G_{x,(r_i/2)+}$ that extends $\psi_{i,x}^{++}$ as in [Yu01, §4] and [FS25, Lemma 4.2.1] and as we recall below.

Yu's construction of $\psi_{i,x}^+$ uses the Moy–Prasad filtrations $(\mathfrak{g}_{x,r})_{r \in \mathbb{R}}$ on the Lie algebra $\mathfrak{g} := \text{Lie}(\mathbf{G})(F)$. We adopt Yu's notation $G_{x,(r/2)+:r+}$ for the quotient group $G_{x,(r/2)+}/G_{x,r+}$. This group is abelian and Moy and Prasad have defined an isomorphism $\mathfrak{g}_{x,(r/2)+:r+} \xrightarrow{\sim} G_{x,(r/2)+:r+}$. Now the Lie subalgebra $\mathfrak{g}_{\iota}^i = \text{Lie}(\mathbf{G}_{\iota}^i)(F)$ of \mathfrak{g} has as complement the sum $\mathfrak{n}_{\iota,x,(r/2)+:r+}^i$ of non-zero weight spaces of $\iota(\mathbf{S}_{\phi,r_i})$ acting on \mathfrak{g} through the adjoint representation. This induces a decomposition

$$\mathfrak{g}_{x,(r/2)+:r+} = \mathfrak{g}_{\iota,x,(r/2)+:r+}^i \oplus \mathfrak{n}_{\iota,x,(r/2)+:r+}^i.$$

Thanks to this decomposition, any character ψ of $G_{\iota,x,(r/2)+:r+}^i$ can be extended to a character $\tilde{\psi}$ of $G_{x,(r/2)+:r+}$ by letting it be trivial on $\mathfrak{n}_{\iota,x,(r/2)+:r+}^i$. In particular we obtain from $\psi|_{G_{\iota,x,(r_i/2)+}^i}$ a character $\tilde{\psi}_i$ of $G_{x,(r_i/2)+}$ which, in turn, can be glued with $\psi|_{G_{\iota,x,0}^i}$ to yield the desired character $\psi_{i,x}^+$ of $G_{\iota,x,0}^i G_{x,(r_i/2)+}$. Note that this character depends on the choices of $\hat{\varphi}_i$ and $\hat{\varphi}_{i+1}$ and, a priori, also the restriction $(\psi_{i,x}^+)|_{K_{\iota,x}^+}$ depends on these choices. However we have the following independence result.

2.4.8 Lemma. — *The character $\tilde{\phi}_{\iota,x}^+$ only depends on ϕ, ι, x , and not on the choice of $\hat{\varphi}_i$.*

Proof. We first note that the map $\psi \mapsto \tilde{\psi}$ described above is obviously multiplicative in ψ , and has the following property: If ξ is a character of G of depth $\leq r$, then $\xi|_{G_{\iota,x,(r/2)+}^i} = \xi|_{G_{x,(r/2)+}$. Indeed, ξ is trivial on root subgroups of G , hence $\xi|_{G_{x,(r/2)+}$ has to be trivial on $\mathfrak{n}_{\iota,x,(r/2)+:r+}^i$. More generally, if ξ_j is a character of G^j for some $j \geq i$, then $(\xi_j|_{G_{\iota,x,(r/2)+}^i})|_{G_{\iota,x,(r/2)+}^j} = \xi_j|_{G_{\iota,x,(r/2)+}^j}$ hence also $\xi_j|_{G_{\iota,x,(r/2)+}^i} = \xi_j|_{G_{\iota,x,(r/2)+}^j}$. So we may unambiguously denote this character by $\tilde{\xi}_j$.

Let us now check that the product $\prod_{i=1}^d (\psi_{i,x}^+)|_{K_{\iota,x}^+}$ is independent of the choices of cocycles $\hat{\varphi}_i$. So let $(\hat{\varphi}'_i)_{i=0,\dots,d}$ be another choice of cocycles leading to characters $\psi'_{i,x}$, and write $\check{\varphi}'_i = \check{\varphi}_i \xi_i$. Then ξ_i is a character of G_i of depth $\leq r_{i-1}$, hence $\xi_i|_{G_{\iota,x,(r_{i-1}/2)+}^i}$ extends to a character $\tilde{\xi}_i$ of $G_{x,(r_{i-1}/2)+}$ according to the procedure described before the lemma. Then we see that for all $i, j \leq d$ we have

$$(\psi'_{i,x})|_{G_{\iota,x,(r_{j-1}/2)+}^j} = \begin{cases} (\psi_{i,x}^+)|_{G_{\iota,x,(r_{j-1}/2)+}^j} \cdot (\xi_i \xi_{i+1}^{-1})|_{G_{\iota,x,(r_{j-1}/2)+}^j} & \text{if } j \leq i \\ (\psi_{i,x}^+)|_{G_{\iota,x,(r_{j-1}/2)+}^j} \cdot (\tilde{\xi}_i \tilde{\xi}_{i+1}^{-1})|_{G_{\iota,x,(r_{j-1}/2)+}^j} & \text{if } j > i \end{cases}$$

where we agree that $\xi_{d+1} = 1$. Taking products we obtain for all $j = 0, \dots, d$

$$\prod_{i=0}^d (\psi'_{i,x})|_{G_{x,(r_{j-1}/2)+}^j} = \prod_{i=0}^d (\psi_{i,x}^+)|_{G_{x,(r_{j-1}/2)+}^j} \cdot (\tilde{\xi}_j)^{-1}|_{G_{x,(r_{j-1}/2)+}^j} (\xi_j)|_{G_{x,(r_{j-1}/2)+}^j} = \prod_{i=0}^d (\psi_{i,x}^+)|_{G_{x,(r_{j-1}/2)+}^j}$$

as desired. \square

Since $K_{\iota,x}^+$ is a pro- p -group, the smooth character $\tilde{\phi}_{\iota,x}^+$ takes values in the ring $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$.

2.4.9 Iwahori decomposition. — The pair $(K_{\iota,x}^+, \check{\phi}_{\iota,x}^+)$ admits Iwahori decompositions with respect to certain pairs of opposite parabolic subgroups adapted to ι and x . More precisely, let $(\mathbf{P}, \bar{\mathbf{P}})$ be a pair of opposite F -rational parabolic subgroups with common Levi component \mathbf{M} that satisfies the following conditions:

- i) $\iota(\mathbf{S}_\phi) \subseteq \mathbf{M}$ (equivalently, $Z(\mathbf{M})^\circ \subseteq \mathbf{G}_\iota$),
- ii) x belongs to $\mathcal{B}(\mathbf{M}, F)$ (the union of all apartments of \mathcal{B} corresponding to tori of \mathbf{M}).

We denote by \mathbf{U} , resp., $\bar{\mathbf{U}}$, the unipotent radical of \mathbf{P} , resp., $\bar{\mathbf{P}}$. As usual, we also denote the groups of F -points of these algebraic groups by U, \bar{U} , etc.

Lemma. — Under the above assumptions, the groups $K_{\iota,x}^+$ and $K_{\iota,x}^\dagger$ have the Iwahori decomposition property with respect to P, \bar{P} , i.e., the multiplication map $(U \cap K_{\iota,x}^\bullet) \times (M \cap K_{\iota,x}^\bullet) \times (\bar{U} \cap K_{\iota,x}^\bullet) \longrightarrow K_{\iota,x}^\bullet$ for $K_{\iota,x}^\bullet \in \{K_{\iota,x}^+, K_{\iota,x}^\dagger\}$ is a bijection. Moreover, we have $(\check{\phi}_{\iota,x}^+)|_{U \cap K_{\iota,x}^+} \equiv 1$ and $(\check{\phi}_{\iota,x}^+)|_{\bar{U} \cap K_{\iota,x}^\dagger} \equiv 1$.

Proof. Let $K_{\iota,x}^\bullet \in \{K_{\iota,x}^+, K_{\iota,x}^\dagger\}$. By the properties of the big cell $\mathbf{UM}\bar{\mathbf{U}}$, the first claim is equivalent to the equality $K_{\iota,x}^\bullet = (U \cap K_{\iota,x}^\bullet)(M \cap K_{\iota,x}^\bullet)(\bar{U} \cap K_{\iota,x}^\bullet)$. But for each $i = 0, \dots, d$, the intersections $\mathbf{P} \cap \mathbf{G}_\iota^i$ and $\bar{\mathbf{P}} \cap \mathbf{G}_\iota^i$ are a pair of opposite parabolic subgroups of \mathbf{G}_ι^i with intersection $\mathbf{M} \cap \mathbf{G}_\iota^i$, since $Z(\mathbf{M})^\circ \subseteq \mathbf{G}_\iota^i$. Then, because x belongs to $\mathcal{B}(\mathbf{M} \cap \mathbf{G}_\iota^i, F) \subseteq \mathcal{B}(\mathbf{G}_\iota^i, F)$, we know that the Moy–Prasad group $G_{\iota,x,r_{i-1}/2+}^i$ has the Iwahori decomposition $G_{\iota,x,r_{i-1}/2+}^i = (U \cap G_{\iota,x,r_{i-1}/2+}^i)(M \cap G_{\iota,x,r_{i-1}/2+}^i)(\bar{U} \cap G_{\iota,x,r_{i-1}/2+}^i)$ and similarly for $G_{\iota,x,r_{i-1}/2}^i$. Now, using the fact that for $i < j$, the group $\bar{U} \cap G_{\iota,x,r_{i-1}/2+}^i$ normalizes $G_{\iota,x,r_{j-1}/2+}^j$ and the group $\bar{M} \cap G_{\iota,x,r_{i-1}/2+}^i$ normalizes $U \cap G_{\iota,x,r_{j-1}/2+}^j$, the desired decomposition follows inductively for $K_{\iota,x}^+$, and one proceeds similarly for $K_{\iota,x}^\dagger$.

Let us now prove that $\check{\phi}_{\iota,x}^+$ is trivial on $U \cap K_{\iota,x}^+$ and on $\bar{U} \cap K_{\iota,x}^\dagger$. Going back to the construction of $\check{\phi}_{\iota,x}^+$ it suffices to show that for each $i = 0, \dots, d$, the character $\psi_{i,x}^+$ of Paragraph 2.4.7 is trivial on the group $U \cap (G_{\iota,x,0+}^i G_{x,(r_i/2)+})$ (and similarly with \bar{U}). By the same argument as above, this group is $(U \cap G_{\iota,x,0+}^i)(U \cap G_{x,(r_i/2)+})$. On the one hand, the restriction of $\psi_{i,x}^+$ to $U \cap G_{\iota,x,0+}^i$ is trivial since it is also the restriction of the character ψ_i of G_ι^i and $U \cap G_\iota^i$ is contained in the derived subgroup of G_ι^i . On the other hand, the restriction of $\psi_{i,x}^+$ to $G_{x,(r_i/2)+}$ is the character denoted by $\tilde{\psi}_i$ in 2.4.7, which extends $(\psi_i)|_{G_{\iota,x,(r_i/2)+}^i}$ according to the decomposition $\mathfrak{g}_{x,(r_i/2)+:r_i+} = \mathfrak{g}_{\iota,x,(r_i/2)+:r_i+}^i \oplus \mathfrak{n}_{\iota,x,(r_i/2)+:r_i+}^i$ and via the Moy–Prasad isomorphism. Recall that the latter decomposition is induced by $\mathfrak{g} = \mathfrak{g}_\iota^i \oplus \mathfrak{n}_\iota^i$ where \mathfrak{g}_ι^i , resp., \mathfrak{n}_ι^i , is the trivial eigenspace, resp., the sum of all non-trivial eigenspaces, of $\iota(\mathbf{S}_{\phi,r_i})$ acting on \mathfrak{g} . On the other hand, the image $\mathfrak{u}_{x,(r_i/2)+:r_i+}$ of $U \cap G_{x,(r_i/2)+}$ in $\mathfrak{g}_{x,(r_i/2)+:r_i+}$ is induced by the Lie algebra \mathfrak{u} of \mathbf{U} which is a sum of (non-trivial) eigenspaces for $Z(\mathbf{M})^\circ$ acting on \mathfrak{g} . Now $\iota(\mathbf{S}_{\phi,r_i})$ and $Z(\mathbf{M})^\circ$ commute since \mathbf{M} contains $\iota(\mathbf{S}_{\phi,r_i})$. Therefore, \mathfrak{u} is stable under $\iota(\mathbf{S}_{\phi,r_i})$, and we have $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{g}_\iota^i) \oplus (\mathfrak{u} \cap \mathfrak{n}_\iota^i)$. Correspondingly, $\mathfrak{u}_{x,(r_i/2)+:r_i+}$ decomposes as the direct sum of $\mathfrak{u}_{x,(r_i/2)+:r_i+} \cap \mathfrak{n}_{\iota,x,(r_i/2)+:r_i+}^i$ and $\mathfrak{u}_{x,(r_i/2)+:r_i+} \cap \mathfrak{g}_{\iota,x,(r_i/2)+:r_i+}^i$. We have already seen that ψ_i is trivial on the latter intersection, and by definition $\tilde{\psi}_i$ is trivial on the former one, which finishes the proof. \square

2.4.10 *A source of pairs of opposite parabolic subgroups.* — For later reference, we describe here a useful source of pairs $(\mathbf{P}, \bar{\mathbf{P}})$ that satisfy conditions i) and ii) of 2.4.9.

Start with two points $x \neq x'$ in \mathcal{B}_ι , and pick a tamely ramified maximal torus $\mathbf{S} \subseteq \mathbf{G}_\iota$ such that $x, x' \in \mathcal{B}(\mathbf{S}, F)$. Let E be a tame Galois extension of F that splits \mathbf{S} . The affine space $\mathcal{A}(\mathbf{G}, \mathbf{S}, E)$ is principal homogeneous under the vector space $X_*(\mathbf{S})_{\mathbb{R}}$, so we can write $x' = x + \lambda$ for a unique $\lambda \in X_*(\mathbf{S})_{\mathbb{R}}$. Moreover, the action of $X_*(\mathbf{S})_{\mathbb{R}}$ on $\mathcal{A}(\mathbf{G}, \mathbf{S}, E)$ is compatible with the respective Galois actions of $\Gamma_{E/F}$. Since both x and x' are $\Gamma_{E/F}$ -fixed, we have $\lambda \in (X_*(\mathbf{S})_{\mathbb{R}})^{\Gamma_{E/F}}$. To λ is associated a pair of F -rational parabolic subgroups $(\mathbf{P}, \bar{\mathbf{P}}) := (\mathbf{P}_\lambda, \mathbf{P}_{-\lambda})$ with common Levi component $\mathbf{M} := \mathbf{M}_\lambda = \mathbf{M}_{-\lambda}$ and respective unipotent radicals denoted by $\mathbf{U} := \mathbf{U}_\lambda$ and $\bar{\mathbf{U}} := \mathbf{U}_{-\lambda}$ (see also [Vig97, §1.27]). By construction \mathbf{M} contains \mathbf{S} , hence also the connected center $\iota(\mathbf{S}_\phi)$ of \mathbf{G}_ι , and both x and x' belong to $\mathcal{B}(\mathbf{M}, F)$.

2.4.11 *Definition.*— We recall that $\mathcal{H}_R(G)$ is the R -algebra of compactly supported, locally constant R -valued distributions on G . We denote by $e_{\iota, x}$ the distribution that averages along the character $\check{\phi}_{\iota, x}^+$ of the pro- p , open subgroup $K_{\iota, x}^+$ of G . This is an idempotent of $\mathcal{H}_R(G)$ supported on $K_{\iota, x}^+$.

From the construction of $K_{\iota, x}^+$ and $e_{\iota, x}$, and in particular the fact that the latter does not depend on any further choice than ι , x and ϕ , we see that

$$(2.4.12) \quad \forall g \in G, \quad {}^g K_{\iota, x}^+ = K_{g\iota, gx}^+ \quad \text{and} \quad {}^g e_{\iota, x} = e_{g\iota, gx},$$

where $g\iota$ is the g -conjugate of ι , i.e., we have $(g\iota)(s) := g\iota(s)g^{-1}$ for all $s \in \mathbf{S}_\phi$.

Since the pair $(K_{\iota, x}^+, e_{\iota, x})$ depends on ϕ , we may write $K_{\phi, \iota, x}^+$ and $e_{\phi, \iota, x}$ whenever we want to emphasize this dependence.

2.5 Intertwining

2.5.1 We keep the data ϕ , $\iota \in I_\phi$, and $x \in \mathcal{B}_\iota$ of the previous subsection and we now introduce the following compact, open subgroups:

$$\begin{aligned} K_{\iota, x}^\dagger &:= G_{\iota, x, 0+}^0 G_{\iota, x, (r_0/2)}^1 \cdots G_{\iota, x, (r_{d-1}/2)}^d, \\ K_{\iota, x}^\circ &:= G_{\iota, x, 0}^0 G_{\iota, x, (r_0/2)}^1 \cdots G_{\iota, x, (r_{d-1}/2)}^d, \\ K_{\iota, x} &:= G_{\iota, x}^0 G_{\iota, x, (r_0/2)}^1 \cdots G_{\iota, x, (r_{d-1}/2)}^d. \end{aligned}$$

So we have inclusions $K_{\iota, x}^+ \subseteq K_{\iota, x}^\dagger \subseteq K_{\iota, x}^\circ \subseteq K_{\iota, x}$ and all subgroups are normal in $K_{\iota, x}$. Moreover, $K_{\iota, x}^\dagger$ is the pro- p -radical of $K_{\iota, x}^\circ$. We will write $K_{\phi, \iota, x}^?$ whenever we want to emphasize the dependence on ϕ . The aim of this subsection is to prove the following result.

2.5.2 *Proposition.* — *i) The group $K_{\phi, \iota, x}$ centralizes $e_{\phi, \iota, x}$.*
ii) If (ϕ', ι', x') is another triple of the same nature, then

$$(2.5.3) \quad e_{\phi, \iota, x} e_{\phi', \iota', x'} \neq 0 \Rightarrow \left(\phi \simeq \phi' \text{ and } K_{\phi, \iota, x}^\dagger \cdot \iota \cap K_{\phi', \iota', x'}^\dagger \cdot \iota' \neq \emptyset \right).$$

On the left hand side of (2.5.3), the product $e_{\phi,\iota,x}e_{\phi',\iota',x'}$ takes place in the R -algebra $\mathcal{H}_R(G)$ of compactly supported, locally constant R -valued distributions on G . Therefore,

$$(2.5.4) \quad e_{\phi,\iota,x}e_{\phi',\iota',x'} \neq 0 \Leftrightarrow (\check{\phi}_{\iota,x}^+)|_{K_{\phi,\iota,x}^+ \cap K_{\phi',\iota',x'}^+} = (\check{\phi}_{\iota',x'}^+)|_{K_{\phi,\iota,x}^+ \cap K_{\phi',\iota',x'}^+}.$$

On the right hand side of (2.5.3), $K_{\phi,\iota,x}^\dagger \cdot \iota$ denotes the $K_{\phi,\iota,x}^\dagger$ -orbit of the embedding ι inside I_ϕ , which is also the $K_{\phi,\iota,x}$ -orbit since G_ι centralizes ι . If we are in the situation that $\phi \simeq \phi'$, then we have $\mathbf{S}_\phi = \mathbf{S}_{\phi'}$ and $I_\phi = I_{\phi'}$, and the intersection $K_{\phi,\iota,x}^\dagger \cdot \iota \cap K_{\phi',\iota',x'}^\dagger \cdot \iota'$ is taken as subsets of I_ϕ .

The outline of the proof is the following : thanks to a lemma of Kaletha we prove that the characters ψ_i of 2.4.5 iii) are generic of depth r_i , in the sense of Yu in [Yu01, §9] and [Fin25, Definition 3.8]. This genericity condition is precisely what allows to control the intertwining as in ii).

2.5.5 Strata and intertwining. — We denote by \mathfrak{g}^* the dual of the Lie algebra \mathfrak{g} . In order to simplify the notation we merely write \mathfrak{g} for $\mathfrak{g}(F)$ and \mathfrak{g}^* for $\mathfrak{g}^*(F)$ if there is no ambiguity. If \mathcal{L} is any lattice in \mathfrak{g} , we put $\mathcal{L}^\bullet = \{f \in \mathfrak{g}^*, \langle f, \mathcal{L} \rangle \subset (F)_{0+}\}$. Then, following Moy and Prasad, we write $\mathfrak{g}_{x,-r}^* := (\mathfrak{g}_{x,r+})^\bullet$.

Let us fix a character $\Psi : F \longrightarrow R^\times$ of depth 0, and recall Adler's version of the Moy–Prasad isomorphism $\varphi_{x,r+} : \mathfrak{g}_{x,(r/2)+} / \mathfrak{g}_{x,r+} \xrightarrow{\sim} G_{x,(r/2)+} / G_{x,r+}$ from [Adl98, 1.6.6] for $r \in \mathbb{R}_{\geq 0}$. Any group J between $G_{x,r+}$ and $G_{x,(r/2)+}$ corresponds to a lattice \mathfrak{j} between $\mathfrak{g}_{x,r+}$ and $\mathfrak{g}_{x,(r/2)+}$. A character ψ of J is said to be *realized* by an element $X \in \mathfrak{g}_{x,-r}^*$ if we have $\psi(h) = \Psi(\langle X, \varphi_{x,r+}^{-1}(h) \rangle)$ for all $h \in J$. Such an X is not uniquely determined by ψ , but the stratum $X + \mathfrak{j}^\bullet$ is. The following result is certainly well known to the specialists, but we could not find a reference in this generality.

Lemma. — *Let x, r, J, ψ be as above, and let x', r', J', ψ' be another tuple of the same nature. Suppose that ψ is realized by some $X \in \mathfrak{g}_{x,-r}^*$ and that ψ' is realized by some $X' \in \mathfrak{g}_{x',-r'}^*$. Then we have $\psi|_{J \cap J'} = \psi'|_{J \cap J'}$ if and only if $(X + \mathfrak{j}^\bullet) \cap (X' + \mathfrak{j}'^\bullet) \neq \emptyset$.*

Proof. We first claim that there exist two “mock exponential maps” $\varphi_x : \mathfrak{g}_{x,0+} \longrightarrow G_{x,0+}$ and $\varphi_{x'} : \mathfrak{g}_{x',0+} \longrightarrow G_{x',0+}$ in the sense of [Adl98, §1.5] such that their restriction to $\mathfrak{g}_{x,0+} \cap \mathfrak{g}_{x',0+}$ agree.

This can be achieved as follows. We choose an apartment that contains both x and x' and denote by \mathbf{S} the corresponding maximal F -split torus. As in 2.4.10, the points x and x' provide us with a Levi subgroup $\mathbf{M} \subset \mathbf{G}$ and opposite parabolic subgroups $\mathbf{P} = \mathbf{M}\mathbf{U}$ and $\bar{\mathbf{P}} = \mathbf{M}\bar{\mathbf{U}}$ such that $G_{x,0+} = (G_{x,0+} \cap \bar{\mathbf{U}})(G_{x,0+} \cap \mathbf{M})(G_{x,0+} \cap \mathbf{U})$, $G_{x',0+} = (G_{x',0+} \cap \bar{\mathbf{U}})(G_{x',0+} \cap \mathbf{M})(G_{x',0+} \cap \mathbf{U})$, $G_{x,0+} \cap \mathbf{M} = M_{x,0+} = M_{x',0+} = G_{x',0+} \cap \mathbf{M}$, and the analogous equations for the Lie algebra. Further we have $U = \prod_{a \in \Sigma_{x,x'}(\mathbf{S})} U_a$ and $\bar{U} = \prod_{a \in \Sigma_{x,x'}(\mathbf{S})} U_{-a}$ as topological spaces and $\mathfrak{u} = \bigoplus_{a \in \Sigma_{x,x'}(\mathbf{S})} \mathfrak{u}_a$ and $\bar{\mathfrak{u}} = \bigoplus_{a \in \Sigma_{x,x'}(\mathbf{S})} \mathfrak{u}_{-a}$, where $\Sigma_{x,x'}(\mathbf{S})$ denotes the subset of the non-multipliable relative roots with respect to \mathbf{S} that occur as characters of \mathbf{S} acting on \mathfrak{u} , and where \mathfrak{u}_a denotes the sum of the a and $2a$ -eigenspaces, and U_a denotes the corresponding root subgroup with Lie algebra \mathfrak{u}_a .

Let $a \in \pm \Sigma_{x,x'}(\mathbf{S})$. Then the Moy–Prasad filtration submodules of \mathfrak{u}_a at x and x' agree up to a shift in the depth-parameterization, and likewise the sets of subgroups $\{U_{a,x,r} \cap U_{a,x',0+}\}_{r \in \mathbb{R}_{r>0}}$ and $\{U_{a,x',r} \cap U_{a,x,0+}\}_{r \in \mathbb{R}_{r>0}}$ agree. Following [Adl98, §1.3] and using these filtrations, we can

now define homeomorphisms $\varphi_{x,a} : \mathbf{u}_{a,x,0+} \longrightarrow U_{a,x,0+}$ and $\varphi_{x',a} : \mathbf{u}_{a,x',0+} \longrightarrow U_{a,x',0+}$ that agree with the restrictions of $\varphi_{x,r+}$ and $\varphi_{x',r+}$, respectively, for all $r \in \mathbb{R}_{\geq 0}$. By using the same coset representatives in the construction of *loc. cit.* for $\varphi_{x,a}$ and $\varphi_{x',a}$ for all the cosets contained in $\mathbf{u}_{a,x,0+} \cap \mathbf{u}_{a,x',0+}$ and their images, we can ensure that $\varphi_{x,a}$ and $\varphi_{x',a}$ agree on $\mathbf{u}_{a,x,0+} \cap \mathbf{u}_{a,x',0+}$. Let $\varphi_{M,x} : \mathfrak{m}_{x,0+} = \mathfrak{m}_{x',0+} \rightarrow M_{x,0+} = M_{x',0+}$ be a mock exponential map as provided by [Adl98, §1.5]. Then using the decompositions $\mathfrak{g}_{x,0+} = (\bigoplus_{a \in \Sigma_{x,x'}}(\mathfrak{s})\mathbf{u}_{-a,x,0+}) \oplus \mathfrak{m}_{x,0+} \oplus (\bigoplus_{a \in \Sigma_{x,x'}}(\mathfrak{s})\mathbf{u}_{a,x,0+})$ and $G_{x,0+} = (\prod_{a \in \Sigma_{x,x'}} U_{-a,x,0+})M_{x,0+}(\prod_{a \in \Sigma_{x,x'}} U_{a,x,0+})$ and the product of the above maps on each respective summand, we obtain a mock exponential map $\varphi_x : \mathfrak{g}_{x,0+} \rightarrow G_{x,0+}$ as in [Adl98, Remark 1.3.3 and §1.5]. Similarly we obtain the mock exponential map $\varphi_{x'} : \mathfrak{g}_{x',0+} \rightarrow G_{x',0+}$. By construction these two mock exponential maps agree on $\mathfrak{g}_{x,0+} \cap \mathfrak{g}_{x',0+}$.

Now, by definition, $\varphi_{x,r+}$ and $\varphi_{x',r'+}$ are induced by restriction from φ_x and $\varphi_{x'}$, respectively. It follows that $\varphi_x^{-1}(j) = \mathfrak{j}$ and, since $\Psi(\langle X, \mathfrak{g}_{x,r+} \rangle) = 1$, that $\psi(j) = \Psi(\langle X, \varphi_x^{-1}(j) \rangle)$ for all $j \in J$. Similarly $\varphi_{x'}^{-1}(j') = \mathfrak{j}'$ and $\psi(j') = \Psi(\langle X', \varphi_{x'}^{-1}(j') \rangle)$ for all $j' \in J'$, and $\psi(j') = \Psi(\langle X', \varphi_x^{-1}(j') \rangle)$ for all $j' \in J' \cap J$. Therefore, we have $\psi|_{J \cap J'} = \psi'|_{J \cap J'}$ if and only if $\Psi(\langle X, Y \rangle) = \Psi(\langle X', Y \rangle)$ for all $Y \in \mathfrak{j} \cap \mathfrak{j}'$. This is equivalent to $X - X' \in (\mathfrak{j} \cap \mathfrak{j}')^\bullet = \mathfrak{j}^\bullet + \mathfrak{j}'^\bullet$, which in turn is equivalent to $(X + \mathfrak{j}^\bullet) \cap (X' + \mathfrak{j}'^\bullet) \neq \emptyset$. \square

2.5.6 Generic elements. — Let \mathbf{L} be a tamely ramified twisted Levi subgroup of \mathbf{G} . We may identify $\mathfrak{l}^* = \text{Lie}^*(\mathbf{L})$ with the weight-0 subspace of \mathfrak{g}^* for the coadjoint action of the connected center of \mathbf{L} , and we write $(\mathfrak{l}^*)^{\mathbf{L}}$ for the invariants under the coadjoint action of \mathbf{L} . We recall from [Fin22, Lemma 2.3] that $(\mathfrak{l}^*)^{\mathbf{L}} \cap (\mathfrak{l}^*)_{x,r} = (\mathfrak{l}^*)^{\mathbf{L}} \cap (\mathfrak{l}^*)_{x',r}$ for $x, x' \in \mathcal{B}(\mathbf{L}, F)$ and $r \in \widetilde{\mathbb{R}}$. The proof of [Fin22, Lemma 2.3] also shows that

$$(2.5.6a) \quad (\mathfrak{l}^*)^{\mathbf{L}} \cap ((\mathfrak{l}^*)_{x,r} + (\mathfrak{l}^*)_{x',r}) = (\mathfrak{l}^*)^{\mathbf{L}} \cap (\mathfrak{l}^*)_{x,r}.$$

An element $X \in (\mathfrak{l}^*)^{\mathbf{L}} \subseteq \mathfrak{g}^*$ is called **G-generic**⁴ of depth $-r$ if $X \in (\mathfrak{l}^*)_{x,-r} \setminus (\mathfrak{l}^*)_{x,-r+}$ for some (equivalently, every) point $x \in \mathcal{B}(\mathbf{H}, F)$ and if for some (equivalently, every) maximal torus $\mathbf{S} \subseteq \mathbf{L}$ and every $\alpha \in \Sigma(\mathbf{S}, \mathbf{G}) \setminus \Sigma(\mathbf{S}, \mathbf{L})$ we have $v(\langle X, H_\alpha \rangle) = -r$, where E is a splitting field of \mathbf{S} , the valuation v extends that of F , and $H_\alpha := \text{Lie}(\alpha^\vee)(1) \in \mathfrak{s}(E)$. Note that we do not exclude the case $\mathbf{L} = \mathbf{G}$, where only the first condition, $X \in (\mathfrak{l}^*)_{x,-r} \setminus (\mathfrak{l}^*)_{x,-r+}$, is non-empty.

For $x \in \mathcal{B}(\mathbf{L}, F)$, we define $\mathfrak{j}_{x,r} := \mathfrak{l}_{x,r} \oplus \mathfrak{n}_{x,r/2}$ and $\mathfrak{j}_{x,r}^+ := \mathfrak{l}_{x,r} \oplus \mathfrak{n}_{x,(r/2)+}$. Here \mathfrak{n} denotes the sum of the non-invariant eigenspaces in \mathfrak{g} under the adjoint action of the center of \mathbf{L} . These lattices correspond to subgroups $J_{x,r}$ and $J_{x,r}^+$ between $G_{x,r}$ and $G_{x,r/2}$. With Yu's notation in [Yu01] we would write $J_{x,r} = (L, G)_{x,(r,r/2)}$ and $J_{x,r}^+ = (L, G)_{x,(r,r/2+)}$. We let $X \in (\mathfrak{l}^*)^{\mathbf{L}} \subseteq \mathfrak{g}^*$ be **G-generic** of depth $-r$. As in 2.5.5, the element X defines a character ψ_x of $J_{x,r}^+$ that is trivial on $G_{x,r+}$. Moreover this character is centralized by $J_{x,r}$ since $[J_{x,r}, J_{x,r}^+] \subset G_{x,r+}$. The following lemma follows from an adaptation to our setting of Yu's arguments in [Yu01, §8].

Lemma. — Let (\mathbf{L}, X, r, x) be as above and let $(\mathbf{L}', X', r', x')$ be another tuple of the same nature (so in particular X' is **G-generic** of depth $-r'$). If $(X + (\mathfrak{j}_{x,r}^+)^\bullet) \cap (X' + (\mathfrak{j}_{x',r'}^+)^\bullet) \neq \emptyset$, then $r = r'$ and there are $g \in J_{x,r}$ and $g' \in J_{x',r'}$ such that ${}^g\mathbf{L} = {}^{g'}\mathbf{L}'$ and ${}^gX - {}^{g'}X' \in ({}^g\mathfrak{l}^*)_{x'',-r+} \cap ({}^{g'}\mathfrak{l}'^*)_{x'',-r+}$ for every $x'' \in \mathcal{B}({}^g\mathbf{L}, F)$.

Proof. Let us first show that $r = r'$. Indeed, since $(\mathfrak{j}_{x,r}^+)^\bullet \subset \mathfrak{g}_{x,-r+}^*$, the strata $X + \mathfrak{g}_{x,-r+}^*$ and

⁴Here we only recall condition GE1 of [Yu01, §8] in the form stated in [Fin25, Definition 3.8] since under our hypotheses (H1) and (H2), [Yu01, Lemma 8.1] shows that condition GE2 is implied by GE1.

$X' + \mathfrak{g}_{x',-r'+}^*$ have a non-empty intersection. But $-r$ is the *depth* $d(X)$ of the non-nilpotent element X in the sense of [AD02, §3.3], and by [AD02, Lemma 3.3.7], this depth function is constant on the coset $X + \mathfrak{g}_{x,-r+}^*$. Similarly, the depth function is constant equal to $-r'$ on the coset $X' + \mathfrak{g}_{x',-r'+}^*$. Hence we conclude that $r = r'$.

Suppose first that $\mathbf{L} = \mathbf{L}' = \mathbf{G}$. Then using (2.5.6a) (and [Fin22, Lemma 2.3]) we have $X - X' \in (\mathfrak{g}_{x,-r+}^* + \mathfrak{g}_{x',-r'+}^*) \cap (\mathfrak{g}^*)^{\mathbf{G}} = \mathfrak{g}_{x,-r+}^* \cap (\mathfrak{g}^*)^{\mathbf{G}} = \mathfrak{g}_{x'',-r+}^* \cap (\mathfrak{g}^*)^{\mathbf{G}}$.

Let us return to the general case. The intersection $(X + (\mathfrak{j}_{x,r}^+)^{\bullet}) \cap (X' + (\mathfrak{j}_{x',r}^+)^{\bullet})$ is open in \mathfrak{g}^* . Since it is assumed to be non-empty, it contains a regular semi-simple element Y so that the centralizer \mathbf{S} of Y is a maximal torus of \mathbf{G} . Now Lemma 8.6 of [Yu01] provides us with an element $g \in G_{x,r/2}$ such that $g^{-1}Y \in X + \mathfrak{l}_{x,-r+}^*$. It follows that $\mathbf{S} \subset {}^g\mathbf{L}$ and that we have $v(\langle Y, H_{\alpha} \rangle) \geq -r$ for $\alpha \in \Sigma(\mathbf{S}, \mathbf{G})$, with equality if and only if $\alpha \notin \Sigma(\mathbf{S}, {}^g\mathbf{L})$. Similarly, there is an element $g' \in G_{x',r/2}$ such that $g'^{-1}Y \in X' + \mathfrak{l}_{x',-r+}^*$ and it follows that $\mathbf{S} \subset {}^{g'}\mathbf{L}'$ and that we have $v(\langle Y, H_{\alpha} \rangle) = -r$ if and only if $\alpha \notin \Sigma(\mathbf{S}, {}^{g'}\mathbf{L}')$.

We thus obtain the equality $\Sigma(\mathbf{S}, {}^g\mathbf{L}) = \Sigma(\mathbf{S}, {}^{g'}\mathbf{L}')$. This implies ${}^g\mathbf{L} = {}^{g'}\mathbf{L}'$, and also $({}^gX + {}^g\mathfrak{l}_{x,-r+}^*) \cap ({}^{g'}X' + {}^{g'}\mathfrak{l}_{x',-r+}^*) \neq \emptyset$. This situation is similar to the case $\mathbf{L} = \mathbf{L}' = \mathbf{G}$ treated above, hence we conclude that ${}^gX - {}^{g'}X' \in ({}^g\mathfrak{l}_{x'',-r+}^*) \cap ({}^{g'}\mathfrak{l}^*)^{{}^g\mathbf{L}}$. \square

We resume the setting associated to (ϕ, ι) . Taking invariants under the coadjoint action of $\iota(\mathbf{S}_{\phi, r_i})$ we obtain an increasing sequence of subspaces $\mathfrak{g}_l^{0*} \subset \dots \subset \mathfrak{g}_l^{d*} = \mathfrak{g}^*$ whose Moy–Prasad filtrations are induced from those of \mathfrak{g}^* . Finally, recall the character ψ_i of $\mathbf{G}_l^i(F)$ defined in 2.4.5 for each $i = 0, \dots, d-1$.

2.5.7 Lemma. — *There is a \mathbf{G}_l^{i+1} -generic element $X_i \in (\mathfrak{g}_l^{i*})^{\mathbf{G}_l^i}$ of depth $-r_i$ that represents $\psi_i|_{G_{l,x,r_i}^i}$ for all $x \in \mathcal{B}(\mathbf{G}_l^i, F)$. (In particular ψ_i has depth r_i).*

Proof. Let \mathbf{S} be a maximal F -torus of \mathbf{G}_l^i split by some tamely ramified Galois extension E . By Lemma 2.3.3, for all roots α in $\Sigma(\mathbf{S}, \mathbf{G}_l^i) \setminus \Sigma(\mathbf{S}, \mathbf{G}_l^i)$ we have $\check{\varphi}_i(N_{E|F}(\alpha^{\vee}(E_{r_i}^{\times}))) \neq \{1\}$. On the other hand, we also have $\check{\varphi}_{i+1}(N_{E|F}(\alpha^{\vee}(E_{r_i}^{\times}))) = \{1\}$ for such roots α , since $N_{E|F}(\alpha^{\vee}(E^{\times})) \subset (\mathbf{G}_l^{i+1})_{\text{sc}}(F)$. Since every maximally split torus in \mathbf{G}_l^i is contained in a tame maximal torus, we deduce that $\psi_i = \check{\varphi}_i \check{\varphi}_{i+1}^{-1}$ has depth $\geq r_i$. Combined with 2.4.5iii) it follows that the character ψ_i of $\mathbf{G}_l^i(F)$ has depth exactly r_i (at every point $x \in \mathcal{B}(\mathbf{G}_l^i, F)$) and the hypothesis of [Kal19, Lemma 3.6.8] are satisfied. This lemma asserts that for every $x \in \mathcal{B}(\mathbf{G}_l^i, F)$, there exists some \mathbf{G}_l^{i+1} -generic $X_i(x)$ of depth $-r_i$ that represents $\psi_i|_{G_{l,x,r_i}^i}$. However, it follows, for example, by the proof of [AFMO24b, Lemma 3.3.1] (cf. also [HM08, Lemma 2.51]), that we may choose X_i uniformly for all x . \square

2.5.8 Some auxiliary groups. — For $i = 1, \dots, d$ and $x \in \mathcal{B}_l$, let us introduce the open, compact subgroup $J_{l,x}^i$ of G_l^i that lies in between $G_{l,x,r_{i-1}}^i$ and $G_{l,x,r_{i-1}/2}^i$ and corresponds to the lattice

$$\mathfrak{j}_{l,x}^i = \mathfrak{g}_{l,x,r_{i-1}}^{i-1} \oplus (\mathfrak{n}_l^{i-1} \cap \mathfrak{g}_l^i)_{x,r_{i-1}/2}$$

of \mathfrak{g}_l^i through the Moy–Prasad isomorphism. This is the group $J_{l,x}^i = (G_l^{i-1}, G_l^i)_{x,r_{i-1},r_{i-1}/2}$ in Yu’s notation. Similarly we define $J_{l,x}^{i+}$ by replacing $r_{i-1}/2$ by $r_{i-1}/2+$.

Recall the character $\psi_{i,x}^+$ of the group $G_{l,x,0+}^i G_{x,r_i/2+}$ defined in 2.4.7. By construction, the restriction $(\psi_{i-1,x}^+)_{J_{l,x}^{i+}}$ is represented by the element X_{i-1} provided by Lemma 2.5.7.

2.5.9 Proof of Proposition 2.5.2. — i) We have $K_{\iota,x} = G_{\iota,x}^0 \prod_i J_{\iota,x}^i$, and since $G_{\iota,x}^0$ centralizes $e_{\phi,\iota,x}$ by (2.4.12), it is enough to prove that for each i and j , the group $J_{\iota,x}^i$ centralizes the character $\psi_{j,x}^+$. When $j \neq i-1$, this is immediate since $J_{\iota,x}^i$ is contained in the group $G_{\iota,x,0+}^j G_{\iota,x,r_j/2+}$ on which $\psi_{j,x}^+$ is defined. When $j = i-1$, this follows from $[J_{\iota,x}^i, G_{\iota,x,0+}^{i-1} G_{\iota,x,r_{i-1}/2+}] \subset (G_{\iota,x}^{i-1}, G_{\iota,x})_{x,r_{i-1}+,r_{i-1}/2+} \subset \ker(\psi_{i-1,x}^+)$, compare also [Yu01, Lemma 4.2].

ii) We will prove (2.5.3) by an inductive argument.

We start with two triples (ϕ, ι, x) and (ϕ', ι', x') such that

$$(2.5.10) \quad (\check{\phi}_{\iota,x})|_{K_{\phi,\iota,x}^+ \cap K_{\phi',\iota',x'}^+} = (\check{\phi}'_{\iota',x'})|_{K_{\phi,\iota,x}^+ \cap K_{\phi',\iota',x'}^+}.$$

Here we lighten the notation by omitting the exponent $+$ of $\check{\phi}_{\iota,x}^+$. We will decorate with the symbol $'$ all objects pertaining to the triple (ϕ', ι', x') . In particular the jumps of the filtration $\mathbf{S}_{\phi',\iota',x'}$ are denoted by $r'_0, \dots, r'_{d'-1}$ and $r'_{d'}$ is the depth of the character ϕ' .

We first reduce to the case where $r_{d-1} = r_d$. Indeed, if $r_{d-1} < r_d$, then it suffices to prove the result after replacing ϕ by $\phi \cdot (\varphi_d^{-1})|_{P_F}$ and ϕ' by $\phi' \cdot (\varphi_d^{-1})|_{P_F}$ because this operation does not affect (2.5.10) and does not change $K_{\phi,\iota,x}^+$ nor $K_{\phi',\iota',x'}^+$. We are thus left to prove the conclusion of (2.5.3) for these new ϕ and ϕ' . We now have $r_d = r_{d-1}$ as desired, but $r'_{d'-1}$ and $r'_{d'}$ might be distinct a priori.

Since $r_d = r_{d-1}$, the character $\check{\phi}_{\iota,x}$ is trivial on $G_{x,r_{d-1}+}$ and $(\check{\phi}_{\iota,x})|_{J_{\phi,\iota,x}^{d+}} = (\psi_{d-1,x}^+)|_{J_{\phi,\iota,x}^{d+}}$ is represented by the generic element X_{d-1} of Proposition 2.5.7. On the other hand, we have a priori two possibilities for $\check{\phi}'_{\iota',x'}$:

- either $r'_{d'} > r'_{d'-1}$ and $(\check{\phi}'_{\iota',x'})|_{G_{x',r'_{d'}}}$ is represented by some generic element $X'_{d'} \in (\mathfrak{g}^*)^{\mathbf{G}}$ of depth $-r'_{d'}$,
- or $r'_{d'} = r'_{d'-1}$ and $(\check{\phi}'_{\iota',x'})|_{J_{\phi',\iota',x'}^{d'+}}$ is represented by $X'_{d'-1}$ as provided by Proposition 2.5.7.

The first case is actually impossible. Indeed by (2.5.10) the characters $\check{\phi}_{\iota,x}$ and $\check{\phi}'_{\iota',x'}$ coincide on $J_{\phi,\iota,x}^{d+} \cap G_{x',r'_{d'}}$. So in the setting of the first case, Lemma 2.5.5 and Lemma 2.5.6 imply that $\mathbf{G}_{\iota'}^{d'-1} \not\subseteq \mathbf{G}$ is conjugate to $\mathbf{G}_{\iota'}^{d'} = \mathbf{G}$, which is absurd.

So we are in the second case, and by (2.5.10) the characters $\check{\phi}_{\iota,x}$ and $\check{\phi}'_{\iota',x'}$ coincide on $J_{\phi,\iota,x}^{d+} \cap J_{\phi',\iota',x'}^{d'+}$. Then, Lemma 2.5.5 and Lemma 2.5.6 tell us that $r_{d-1} = r'_{d'-1}$ and provide elements $j \in J_{\phi,\iota,x}^d$ and $j' \in J_{\phi',\iota',x'}^{d'}$ such that ${}^j \mathbf{G}_{\iota}^{d-1} = {}^{j'} \mathbf{G}_{\iota'}^{d'-1}$. Note that ${}^j \mathbf{G}_{\iota}^{d-1} = \mathbf{G}_{j\iota}^{d-1}$. Since $j \in K_{\phi,\iota,x}^+$, statement i) of the proposition and (2.4.12) show that $e_{\phi,\iota,x} = e_{\phi,j\iota,x}$. Therefore it is sufficient to prove the conclusion of (2.5.3) for the triples $(\phi, j\iota, x)$ and $(\phi', j'\iota', x')$. In other words, we may and will assume that $\mathbf{G}_{\iota}^{d-1} = \mathbf{G}_{\iota'}^{d'-1}$.

Let us put $\mathbf{H} := \mathbf{G}_{\iota}^{d-1} = \mathbf{G}_{\iota'}^{d'-1}$. The wild inertia parameter $\hat{\phi} : P_F \longrightarrow \hat{\mathbf{G}}$ factors through $\hat{\mathbf{H}}$, giving, according to Lemma 2.3.2, a wild inertia parameter of \mathbf{H} denoted by $\phi^{|H}$. Its associated F -torus $\mathbf{S}_{\phi^{|H}}$ is equal to \mathbf{S}_{ϕ} and the Levi-center embedding $\iota : \mathbf{S}_{\phi} \hookrightarrow \mathbf{G}$ factors through \mathbf{H} . Moreover, the associated set $\mathcal{B}_{\iota} \subset \mathcal{B}(\mathbf{H}, F)$ is the same as the one considered so far. We thus get a triple $(\phi^{|H}, \iota, x)$ pertaining to \mathbf{H} , whence groups ${}^H K_{\phi,\iota,x}$ and ${}^H K_{\phi,\iota,x}^+$ and a character $\check{\phi}_{\iota,x}^{|H}$. Actually we simply have

$${}^H K_{\phi,\iota,x}^+ = G_{\iota,x,0+}^0 \cdots G_{\iota,x,(r_{d-2}/2)+}^{d-1} \subset K_{\phi,\iota,x}^+ \text{ and } \check{\phi}_{\iota,x}^{|H} = (\check{\phi}_{\iota,x})|_{{}^H K_{\phi,\iota,x}^+}.$$

Similarly we have a triple $(\phi'^{|H}, \iota', x')$ pertaining to \mathbf{H} , groups ${}^H K_{\phi', \iota', x'} \subset K_{\phi', \iota', x'}$ and ${}^H K_{\phi', \iota', x'}^+ \subset K_{\phi', \iota', x'}^+$, as well as a character $\check{\phi}'^{|H}$ of ${}^H K_{\phi', \iota', x'}^+$ that coincides with the restriction of $\check{\phi}'_{\iota', x'}$. In particular (2.5.10) implies that the characters $\check{\phi}_{\phi, \iota, x}^{|H}$ and $\check{\phi}'^{|H}_{\phi', \iota', x'}$ coincide on the intersection ${}^H K_{\phi, \iota, x}^+ \cap {}^H K_{\phi', \iota', x'}^+$. Suppose now that the conclusion of (2.5.3) is known for the triples $(\phi^{|H}, \iota, x)$ and $(\phi'^{|H}, \iota', x')$. It then implies that the same conclusion holds for the triples (ϕ, ι, x) and (ϕ', ι', x') .

It follows that in order to finish the proof of (2.5.3), we may argue by induction, for example on the number $n(\phi, \mathbf{G}) = \dim(\mathbf{S}_\phi) - \dim(Z(\mathbf{G}))$. It remains however to initiate the induction process by considering the case $n(\phi, \mathbf{G}) = 0$. In this case we have $d = 0$, $K_{\phi, \iota, x}^\dagger = G_{x, 0+}$ and $\check{\phi}_{\iota, x}$ is the restriction of a character $\check{\varphi}_0$ of G . As we have done above, we may multiply both ϕ and ϕ' by $(\check{\varphi}_0^{-1})|_{P_F}$ so that we may assume now that ϕ is trivial. Then we need to show that ϕ' is trivial too, or equivalently that it has depth $r'_{d'} = 0$. However if ϕ' had depth $r'_{d'} > 0$, then $(\check{\phi}'_{\iota', x'})|_{G_{x', r'_{d'}}$ would be represented by a generic element $X'_{d'}$ of depth $-r'_{d'}$. Since the depth function of [AD02] is constant on the stratum $X'_{d'} + \mathfrak{g}_{x', -r'_{d'}+}^*$, the latter cannot intersect the stratum $\mathfrak{g}_{x, 0+}^*$, hence by Lemma 2.5.5 we would get a contradiction with (2.5.10). \square

2.5.11 *A variant.* — Fix ϕ and consider the open subgroup $K'_{\iota, x} := \prod_{i=1}^d J_{\iota, x}^{i+}$ of $K_{\iota, x}^+$. The following slight strengthening of 2.5.2 ii) will be useful in Section 3.

Proposition. — *If $(\check{\phi}_{\iota, x})|_{K'_{\iota, x} \cap K'_{\iota', x'}} = (\check{\phi}'_{\iota', x'})|_{K'_{\iota, x} \cap K'_{\iota', x'}}$, then $K_{\iota, x}^\dagger \cap K_{\iota', x'}^\dagger \neq \emptyset$*

Proof. This is the actual output of the foregoing proof when $\phi = \phi'$. \square

2.5.12 *A converse to Proposition 2.5.2 ii).* — Using (2.4.12), the following lemma shows that the implication (2.5.3) is actually an equivalence.

Lemma. — *Fix $\phi, \iota \in I_\phi$ and two points $x, x' \in \mathcal{B}_\iota$. Then the two characters $\check{\phi}_{\iota, x}^+$ and $\check{\phi}_{\iota, x'}^+$ agree on the intersection $K_{\iota, x}^+ \cap K_{\iota, x'}^+$.*

Proof. Only the case $x \neq x'$ is nontrivial. Let $(\mathbf{P}, \bar{\mathbf{P}})$ be the pair of opposite parabolic subgroups of \mathbf{G} with common Levi subgroup \mathbf{M} as defined in 2.4.10, and write $\mathbf{P} = \mathbf{M}\mathbf{U}$ and $\bar{\mathbf{P}} = \mathbf{M}\bar{\mathbf{U}}$ for their respective Levi decompositions. It follows from Lemma 2.4.9 that we have an Iwahori decomposition

$$K_{\iota, x}^+ \cap K_{\iota, x'}^+ = (U \cap K_{\iota, x}^+ \cap K_{\iota, x'}^+) (M \cap K_{\iota, x}^+ \cap K_{\iota, x'}^+) (\bar{U} \cap K_{\iota, x}^+ \cap K_{\iota, x'}^+)$$

and that both $\check{\phi}_{\iota, x}^+$ and $\check{\phi}_{\iota, x'}^+$ are trivial on $(U \cap K_{\iota, x}^+ \cap K_{\iota, x'}^+)$ and $(\bar{U} \cap K_{\iota, x}^+ \cap K_{\iota, x'}^+)$. From the proof of that lemma, setting $\mathbf{M}^i := \mathbf{M} \cap \mathbf{G}_\iota^i$, we also have

$$M \cap K_{\iota, x}^+ = M_{x, 0+}^0 M_{x, (r_0/2)+}^1 \cdots M_{\iota, x, (r_{d-1}/2)+}^d$$

and the same applies to x' . But since x' is a translate of x under $X_*(Z(\mathbf{M}))_{\mathbb{R}}^{\Gamma_{E/F}}$, we have $M_{x, (r_{i-1}/2)+}^i = M_{x', (r_{i-1}/2)+}^i$ for all i and, therefore, $M \cap K_{\iota, x}^+ = M \cap K_{\iota, x'}^+ = M \cap K_{\iota, x}^+ \cap K_{\iota, x'}^+$. It remains to see that the restrictions of $\check{\phi}_{\iota, x}^+$ and $\check{\phi}_{\iota, x'}^+$ to this group are equal. For this, it suffices to check that for all $i = 0, \dots, d$, the restrictions of the characters $\psi_{i, x}^+$ and $\psi_{i, x'}^+$ of

2.4.7 to this group are equal. The character $\psi_{i,x}^+$ is defined on $G_{\iota,x,0+}^i G_{x,(r_i/2)+}$. By the same argument as above, we have $M \cap (G_{\iota,x,0+}^i G_{x,(r_i/2)+}) = M_{x,0+}^i M_{x,(r_i/2)+} = M_{x',0+}^i M_{x',(r_i/2)+}$, and it suffices to see that $\psi_{i,x}^+$ and $\psi_{i,x'}^+$ agree on this group. On one hand, we have by definition $(\psi_{i,x}^+)|_{M_{x,0+}^i} = (\psi_i)|_{M_{x,0+}^i} = (\psi_{i,x'}^+)|_{M_{x,0+}^i}$. On the other hand, the restriction of $\psi_{i,x}^+$ to $G_{x,(r_i/2)+}$ is the character denoted by $\tilde{\psi}_i$ in 2.4.7, which extends $(\psi_i)|_{G_{\iota,x,(r_i/2)+}^i}$ trivially according to the decomposition $\mathfrak{g}_{x,(r_i/2)+:r_i+} = \mathfrak{g}_{\iota,x,(r_i/2)+:r_i+}^i \oplus \mathfrak{n}_{\iota,x,(r_i/2)+:r_i+}^i$ and via the Moy–Prasad isomorphism. Recall that the latter decomposition is induced by $\mathfrak{g} = \mathfrak{g}_\iota^i \oplus \mathfrak{n}_\iota^i$ where \mathfrak{g}_ι^i , resp., \mathfrak{n}_ι^i , is the trivial eigenspace, resp., the sum of all non-trivial eigenspaces, of $\iota(\mathbf{S}_{\phi,r_i})$ acting on \mathfrak{g} . Since $\iota(\mathbf{S}_{\phi,r_i})$ and $Z(\mathbf{M})^\circ$ commute with each other, taking the weight-0 part of the $Z(\mathbf{M})^\circ$ -action on the above decomposition provides us with the decomposition $\mathfrak{m} = \mathfrak{m}^i \oplus (\mathfrak{m} \cap \mathfrak{n}_\iota^i)$, and we see that $(\psi_{i,x}^+)|_{M_{x,(r_i/2)+}}$ is the character that extends $(\psi_i)|_{M_{x,(r_i/2)+}^i}$ trivially according to the decomposition $\mathfrak{m}_{x,(r_i/2)+:r_i+} = \mathfrak{m}_{x,(r_i/2)+:r_i+}^i \oplus (\mathfrak{m} \cap \mathfrak{n}_\iota^i)_{\iota,x,(r_i/2)+:r_i+}$ through the Moy–Prasad isomorphism. The same description applies to $(\psi_{i,x'}^+)|_{M_{x,(r_i/2)+}}$, finishing the proof. \square

2.5.13 *The Heisenberg property.* — By [Yu01, Lemma 1.3], we know that the quotient group $K_{\iota,x}^\dagger/K_{\iota,x}^+$ is abelian, hence the derived subgroup $[K_{\iota,x}^\dagger, K_{\iota,x}^\dagger]$ is contained in $K_{\iota,x}^+$ and we have a map

$$K_{\iota,x}^\dagger \times K_{\iota,x}^\dagger \longrightarrow \mu_{p^\infty}, (g, h) \mapsto \check{\phi}_{\iota,x}^+(ghg^{-1}h^{-1}).$$

Since $K_{\iota,x}^\dagger$ centralizes the character $\check{\phi}_{\iota,x}^+$, this map descends to a map

$$\theta : K_{\iota,x}^\dagger/K_{\iota,x}^+ \times K_{\iota,x}^\dagger/K_{\iota,x}^+ \longrightarrow \mu_{p^\infty}.$$

Proposition. — Assume that p is odd, on top of our running assumptions (H1) and (H2).

- i) The group $K_{\iota,x}^\dagger/K_{\iota,x}^+$ has exponent p and the map θ defines a perfect alternating pairing on this group, taking values in μ_p .
- ii) If $(\mathbf{P}, \bar{\mathbf{P}})$ are opposite parabolic subgroups of \mathbf{G} satisfying conditions i) and ii) of 2.4.9, then θ induces a perfect pairing between $(U \cap K_{\iota,x}^\dagger)/(U \cap K_{\iota,x}^+)$ and $(\bar{U} \cap K_{\iota,x}^\dagger)/(\bar{U} \cap K_{\iota,x}^+)$, and a perfect alternating pairing on $(M \cap K_{\iota,x}^\dagger)/(M \cap K_{\iota,x}^+)$.

Proof. i) By [Yu01, Lemma 1.3], we know that $K_{\iota,x}^\dagger/K_{\iota,x}^+$ is isomorphic to its Lie algebra counterpart $\mathfrak{k}_{\iota,x}/\mathfrak{k}_{\iota,x}^+$. By construction, the latter decomposes as

$$\mathfrak{k}_{\iota,x}/\mathfrak{k}_{\iota,x}^+ = \bigoplus_{i=1}^d (\mathfrak{g}_\iota^i \cap \mathfrak{n}_\iota^{i-1})_{x,(r_{i-1}/2):(r_{i-1}/2)+},$$

which is a direct sum of vector spaces over the residue field of F , hence has exponent p . The equality $[g, hh'] = [g, h] \cdot {}^h[g, h']$ shows that the map θ is \mathbb{Z} -bilinear, hence the image of this map is contained in the only subgroup μ_p of μ_{p^∞} of exponent p . Note that in the above decomposition of $\mathfrak{k}_{\iota,x}/\mathfrak{k}_{\iota,x}^+$ the summand $(\mathfrak{g}_\iota^i \cap \mathfrak{n}_\iota^{i-1})_{x,(r_{i-1}/2):(r_{i-1}/2)+} = \mathfrak{j}_{\iota,x}^i/\mathfrak{j}_{\iota,x}^{i+}$ identifies with the image of the subgroup $J_{\iota,x}^i$ in $K_{\iota,x}^\dagger/K_{\iota,x}^+$, so that another way to write this decomposition is as

$$(2.5.14) \quad K_{\iota,x}^\dagger/K_{\iota,x}^+ = \prod_{i=1}^d (J_{\iota,x}^i/J_{\iota,x}^{i+}).$$

Now, recall the factorization $\check{\phi}_{\iota,x}^+ = \prod_{k=0}^d (\psi_{k,x}^+) |_{K_{\iota,x}^+}$ of 2.4.7 and let $i, j \in \{1, \dots, d\}$ and $k \in \{0, \dots, d\}$. We claim that $[J_{\iota,x}^i, J_{\iota,x}^j] \subset \ker \psi_{k,x}^+$ unless $i = j = k + 1$. When $k + 1 \neq i, j$, this follows from the fact that both $J_{\iota,x}^i$ and $J_{\iota,x}^j$ are contained in the group $G_{\iota,x,0+}^k G_{x,r_k/2+}$ on which $\psi_{k,x}^+$ is defined. When $k + 1 = i$ and $i \neq j$, this follows from the inclusion $[J_{\iota,x}^i, J_{\iota,x}^j] \subset [J_{\iota,x}^{k+1}, G_{\iota,x,0+}^k G_{x,r_k/2+}] \subset (G_{\iota,x}^k, G)_{x,r_k+,r_k/2+} \subset \ker(\psi_{k,x}^+)$, and similarly for $k + 1 = j \neq i$.

As a consequence, the last displayed decomposition is orthogonal for the bilinear form θ , and the restriction of θ to the summand $J_{\iota,x}^i/J_{\iota,x}^{i+}$ is given by $\theta(\bar{j}, \bar{h}) = \psi_{i-1,x}^+(jhj^{-1}h^{-1})$. By [Yu01, Lemma 11.1] the latter bilinear form on $J_{\iota,x}^i/J_{\iota,x}^{i+}$ is non-degenerate, hence so is θ .

ii) We have $[(U \cap K_{\iota,x}^\dagger), (P \cap K_{\iota,x}^\dagger)] \subset (U \cap K_{\iota,x}^+)$. Since $\check{\phi}_{\iota,x}^+$ is trivial on $U \cap K_{\iota,x}^+$, we see that $P \cap K_{\iota,x}^\dagger$ is orthogonal to $U \cap K_{\iota,x}^\dagger$ for the bilinear form θ . From the Iwahori decomposition of $K_{\iota,x}^\dagger$ and $K_{\iota,x}^+$ with respect to $(\mathbf{P}, \bar{\mathbf{P}})$, see Lemma 2.4.9, we obtain a decomposition of \mathbb{F}_p -vector spaces

$$K_{\iota,x}^\dagger/K_{\iota,x}^+ = (U \cap K_{\iota,x}^\dagger)/(U \cap K_{\iota,x}^+) \oplus (M \cap K_{\iota,x}^\dagger)/(M \cap K_{\iota,x}^+) \oplus (U \cap K_{\iota,x}^\dagger)/(U \cap K_{\iota,x}^+)$$

and ii) now follows from the non-degeneracy of θ . \square

2.6 Systems of idempotents

In this section we fix a wild inertia parameter ϕ and a $\mathbf{G}(F)$ -conjugacy class $I \subseteq I_\phi$ of F -rational embeddings $\mathbf{S}_\phi \hookrightarrow \mathbf{G}$. Recall that we assume that $C_{\check{\mathbf{G}}}(\phi)$ is a Levi subgroup and that p satisfies (H1) and (H2).

2.6.1 An orthogonality property. — For $x \in \mathcal{B} = \mathcal{B}(\mathbf{G}, F)$, we put $I_x := \{\iota \in I, x \in \mathcal{B}_\iota\}$. If $\iota, \iota' \in I_x$, we declare that $\iota' \sim_x \iota$ if $\iota' \in K_{\iota,x}^\dagger \cdot \iota$. Using (2.4.12) we see that if $\iota' \in K_{\iota,x}^\dagger \cdot \iota$, then we have $K_{\iota',x}^\dagger = K_{\iota,x}^\dagger$, whence the transitivity and symmetry of the relation \sim_x , which is thus an equivalence relation on I_x .

Lemma. — For $\iota', \iota \in I_x$ we have $e_{\iota,x} e_{\iota',x} = \begin{cases} e_{\iota,x} & \text{if } \iota \sim_x \iota' \\ 0 & \text{else.} \end{cases}$

Proof. If $e_{\iota,x} e_{\iota',x} \neq 0$, then by ii) of Proposition 2.5.2 we have that $K_{\iota',x}^\dagger \cdot \iota' \cap K_{\iota,x}^\dagger \cdot \iota \neq \emptyset$, and hence $\iota \sim_x \iota'$. Thus $e_{\iota,x} e_{\iota',x} = 0$ if we are not in the case $\iota \sim_x \iota'$. If $\iota \sim_x \iota'$, then, by (2.4.12) and Proposition 2.5.2 i), we have $e_{\iota',x} = e_{\iota,x}$, hence $e_{\iota,x} e_{\iota',x} = e_{\iota,x}$. \square

2.6.2 A variant. — The following strengthening of the above lemma is not needed in Section 2 but will occasionally be useful in Section 3. Recall the open subgroup $K'_{\iota',x}$ of $K_{\iota',x}^+$ introduced in 2.5.11, and denote by $e'_{\iota',x}$ the idempotent associated to $\check{\phi}_{\iota',x}^+ |_{K'_{\iota',x}}$. It is coarser than $e_{\iota',x}$, in the sense that $e'_{\iota',x} e_{\iota',x} = e_{\iota',x}$, and the analog of (2.4.12) still holds, namely ${}^g e'_{\iota',x} = e'_{g\iota',gx}$ for all $g \in G$.

Lemma. — For $\iota', \iota \in I_x$ we have $e_{\iota,x} e'_{\iota',x} = \begin{cases} e_{\iota,x} & \text{if } \iota \sim_x \iota' \\ 0 & \text{else.} \end{cases}$

Proof. Same proof as the last lemma, with Proposition 2.5.11 instead of Proposition 2.5.2. \square

2.6.3 *The idempotents associated to ϕ and I .* — Since there is a finite number of idempotents $e \in \mathcal{H}_R(G)$ that are supported on G_x and satisfy $e \cdot e_{G_x, r_{\phi^+}} = e$, Lemma 2.6.1 shows that the sum

$$e_x := e_{I,x} := e_{\phi,I,x} := \sum_{\iota \in I_x / \sim_x} e_{\iota,x}$$

is finite and defines an idempotent of $\mathcal{H}_R(G)$ supported on $G_{x,0^+}$. It is non-zero if and only if I_x is non empty. By construction we have the following equivariance property:

$$(2.6.4) \quad \forall x \in \mathcal{B}, \forall g \in G, e_{gx} = {}^g e_x.$$

In particular, e_x is a central idempotent in $\mathcal{H}_R(G_x)$.

Lemma. — Fix $x, x' \in \mathcal{B}$, put $I_{x,x'} := I_x \cap I_{x'}$ and endow this set with the equivalence relation $\iota \sim_{x,x'} \iota' \Leftrightarrow (\iota \sim_x \iota' \text{ and } \iota \sim_{x'} \iota')$. Then we have

$$(2.6.5) \quad e_x e_{x'} = \sum_{\iota \in I_{x,x'} / \sim_{x,x'}} e_{\iota,x} e_{\iota,x'}.$$

Proof. Denote by $\bar{\iota} := K_{\iota,x}^+ \cdot \iota$ the \sim_x -equivalence class of $\iota \in I_x$, and similarly for $\iota' \in I_{x'}$. By Proposition 2.5.2.ii) we have $\bar{\iota} \cap \bar{\iota}' \neq \emptyset$ whenever $e_{\iota,x} e_{\iota',x'} \neq 0$. Hence $e_x e_{x'} = \sum_{\bar{\iota} \cap \bar{\iota}' \neq \emptyset} e_{\bar{\iota},x} e_{\bar{\iota}',x'}$. Now, the intersection $\bar{\iota} \cap \bar{\iota}'$ in I is contained in $I_{x,x'}$ and, if $\bar{\iota} \cap \bar{\iota}'$ is non-empty, it is actually a $\sim_{x,x'}$ -equivalence class. We thus have a map $(\bar{\iota}, \bar{\iota}') \mapsto \bar{\iota} \cap \bar{\iota}'$,

$$\{(\bar{\iota}, \bar{\iota}') \in I_x / \sim_x \times I_{x'} / \sim_{x'}, \bar{\iota} \cap \bar{\iota}' \neq \emptyset\} \longrightarrow I_{x,x'} / \sim_{x,x'},$$

which is easily seen to be a bijection. □

2.6.6 *A telescopic identity.* — We will provide a telescopic identity when moving along a geodesic. In the following lemma, we fix $\iota \in I$ and two points $x, x' \in \mathcal{B}_\iota$. Since \mathcal{B}_ι is convex, it contains the segment $[x, x']$.

Lemma. — Suppose $x'' \in [x, x']$. Then $e_{\iota,x} e_{\iota,x'} = e_{\iota,x} e_{\iota,x''} e_{\iota,x'}$ (product in $\mathcal{H}_R(G)$).

Proof. As explained in 2.4.10, the segment (x, x') determines a pair $(\mathbf{P}, \bar{\mathbf{P}})$ of F -rational opposite parabolic subgroups of \mathbf{G} with common Levi subgroup \mathbf{M} satisfying the conditions i) and ii) of 2.4.9 for each of the points x, x', x'' . The following equalities and inclusions

- $M \cap K_{\iota,x}^+ = M \cap K_{\iota,x''}^+ = M \cap K_{\iota,x'}^+$
- $U \cap K_{\iota,x}^+ \supset U \cap K_{\iota,x''}^+ \supset U \cap K_{\iota,x'}^+$
- $\bar{U} \cap K_{\iota,x}^+ \subset \bar{U} \cap K_{\iota,x''}^+ \subset \bar{U} \cap K_{\iota,x'}^+$

follow from similar equalities and inclusions for each $G_{\iota,x,(r_{i-1}/2)^+}$ in place of $K_{\iota,x}^+$. Using the Iwahori decomposition of Lemma 2.4.9, we then deduce that

$$K_{\iota,x''}^+ = (K_{\iota,x}^+ \cap K_{\iota,x''}^+) (K_{\iota,x''}^+ \cap K_{\iota,x'}^+).$$

Now denote by $e_{\iota,x,x''}$ and $e_{\iota,x'',x'}$ the idempotents associated to $(\check{\phi}_{\iota,x''}^+) |_{K_{\iota,x}^+ \cap K_{\iota,x''}^+}$ and $(\check{\phi}_{\iota,x''}^+) |_{K_{\iota,x''}^+ \cap K_{\iota,x'}^+}$, respectively. We obtain a factorization $e_{\iota,x''} = e_{\iota,x,x''} e_{\iota,x'',x'}$ in $\mathcal{H}_R(G_{x''})$. By Lemma 2.5.12, it follows that $e_{\iota,x} e_{\iota,x''} e_{\iota,x'} = e_{\iota,x} e_{\iota,x,x''} e_{\iota,x'',x'} e_{\iota,x'} = e_{\iota,x} e_{\iota,x'}$. □

Proposition. — Let $x, x' \in \mathcal{B}$ and $x'' \in [x, x']$. Suppose that there exists a facet of \mathcal{B} whose closure contains both x and x'' . Then $e_x e_{x''} e_{x'} = e_x e_{x'}$.

Proof. Since there exists some facet whose closure contains x and x'' , the group $G_{x''}$ contains $G_{x,0+}$, and it follows that $e_{x''}$ commutes with all $e_{\iota,x}$ since the former is central in $\mathcal{H}_R(G_{x''})$ and the latter are supported on $G_{x,0+}$. This commutation property provides the first and fourth equality in the following computation.

$$\begin{aligned}
e_x e_{x''} e_{x'} = e_{x''} e_x e_{x'} &= e_{x''} \left(\sum_{\iota \in I_{x,x'} / \sim_{x,x'}} e_{\iota,x} e_{\iota,x'} \right) = e_{x''} \left(\sum_{\iota \in I_{x,x'} / \sim_{x,x'}} e_{\iota,x} e_{\iota,x''} e_{\iota,x'} \right) \\
&= \sum_{\iota \in I_{x,x'} / \sim_{x,x'}} e_{\iota,x} e_{x''} e_{\iota,x''} e_{\iota,x'} = \sum_{\iota \in I_{x,x'} / \sim_{x,x'}} e_{\iota,x} e_{\iota,x''} e_{\iota,x'} \\
&= \sum_{\iota \in I_{x,x'} / \sim_{x,x'}} e_{\iota,x} e_{\iota,x'} = e_x e_{x'}.
\end{aligned}$$

In the second and the last equality, we used (2.6.5). In the third and sixth equality we used the last lemma, and in the fifth one we used Lemma 2.6.3. \square

2.6.7 E -facets. — From now on we also assume that p is odd. We aim at finding a polysimplicial G -equivariant structure on \mathcal{B} such that e_x only depends on the facet it belongs to. Simple examples show that the usual Bruhat–Tits structure on \mathcal{B} will not work. Instead, we will consider the intersection with \mathcal{B} of the Bruhat–Tits structure on $\mathcal{B}(\mathbf{G}, E)$ for a suitable tamely ramified extension E over F .

Lemma. — Assume p is odd. There is a tamely ramified Galois field extension E of F that splits a maximal F -torus of \mathbf{G}_ι (for any $\iota \in I$) and such that $\{r_0/2, \dots, r_{d-1}/2\} \subset v(E^\times)$, where v is the unique valuation on E that extends the normalized valuation of F .

Proof. Let E be a tamely ramified splitting field of some maximal F -torus \mathbf{S} in \mathbf{G}_ι . Lemma 2.3.3 shows that for $0 \leq i \leq d-1$, the real number r_i is a jump of the filtration on E^\times , hence belongs to $v(E^\times) = \frac{1}{e}\mathbb{Z}$ with e the ramification index of E/F . After replacing E by a quadratic ramified extension if necessary, we obtain that $r_i/2 \in v(E^\times)$ for each i , as desired. Then the Galois closure of this E meets the requirements of the lemma. \square

Let E be as in the above lemma. Recall that the reduced Bruhat–Tits building $\mathcal{B}^{\text{red}}(\mathbf{G}, E)$ of the reductive group \mathbf{G}_E over E carries a polysimplicial structure. The inverse image in $\mathcal{B}(\mathbf{G}, E)$ of a polysimplex of $\mathcal{B}^{\text{red}}(\mathbf{G}, E)$ will be called a facet of $\mathcal{B}(\mathbf{G}, E)$. Recall also, e.g., from [Lan00, Thm 2.1.1], that there is a canonical embedding $\mathcal{B}(\mathbf{G}, F) \hookrightarrow \mathcal{B}(\mathbf{G}, E)$, so we may and will identify \mathcal{B} with a subset of $\mathcal{B}(\mathbf{G}, E)$. Since \mathbf{G} is defined over F , the building $\mathcal{B}(\mathbf{G}, E)$ is equipped with an action of $\text{Gal}(E/F)$, and, by a result of Rousseau, see [Pra01], we have $\mathcal{B} = \mathcal{B}(\mathbf{G}, E)^{\text{Gal}(E/F)}$, because E/F is tamely ramified. The intersection of a facet of $\mathcal{B}(\mathbf{G}, E)$ with \mathcal{B} will be called an “ E -facet of \mathcal{B} ”. We thus obtain a partition of \mathcal{B} into “ E -facets”, such that the closure of an E -facet is a union of E -facets. We will denote by $\mathcal{F}_E(x)$ the E -facet of \mathcal{B} that contains $x \in \mathcal{B}$.

2.6.8 Proposition. — Let E be as in the previous lemma. Then we have:

i) $\forall \iota \in I, \forall x \in \mathcal{B}, x \in \mathcal{B}_\iota \Rightarrow \mathcal{F}_E(x) \subseteq \mathcal{B}_\iota$

ii) $\forall \iota \in I, \forall x, x' \in \mathcal{B}_\iota, \mathcal{F}_E(x) = \mathcal{F}_E(x') \Rightarrow (e_{\iota,x} = e_{\iota,x'} \text{ and } K_{\iota,x}^\dagger = K_{\iota,x'}^\dagger)$

iii) $\forall x, x' \in \mathcal{B}, \mathcal{F}_E(x) = \mathcal{F}_E(x') \Rightarrow e_x = e_{x'}$.

Proof. i) Denote by $\tilde{\mathcal{F}}_E(x)$ the facet of $\mathcal{B}(\mathbf{G}, E)$ that contains x . Since $(\mathbf{G}_\iota)_E$ is Levi subgroup of \mathbf{G}_E , the facet $\tilde{\mathcal{F}}_E(x)$ is contained in the image $\mathcal{B}_\iota(E)$ of any admissible embedding $\mathcal{B}(\mathbf{G}_\iota, E) \hookrightarrow \mathcal{B}(\mathbf{G}, E)$. Therefore $\mathcal{F}_E(x) = \tilde{\mathcal{F}}_E(x) \cap \mathcal{B} \subset \mathcal{B}_\iota(E)^{\text{Gal}(E/F)} = \mathcal{B}_\iota$, as desired.

ii) By [Vig97, Prop 1.1], the group $\mathbf{G}(E)_{x,r+}$ only depends on the facet containing x provided that $r \in v(E^\times)$ (in *loc.cit.* the valuation is normalized by $v(E^\times) = \mathbb{Z}$). Recall also from [Yu01, §2] that $G_{x,r+} = G \cap \mathbf{G}(E)_{x,r+}$. It follows that for each $i = 0, \dots, d$ the group $G_{\iota,x,(r_{i-1}/2)+}^i = G_\iota^i \cap \mathbf{G}(E)_{x,(r_{i-1}/2)+}$ only depends on the E -facet $\mathcal{F}_E(x)$. Therefore we obtain $K_{\iota,x}^+ = K_{\iota,x'}^+$. A similar argument shows that $K_{\iota,x}^\dagger = K_{\iota,x'}^\dagger$. Now Lemma 2.5.12 implies that $\check{\phi}_{\iota,x}^+ = \check{\phi}_{\iota,x'}^+$ hence also $e_{\iota,x} = e_{\iota,x'}$.

iii) By ii), it suffices to show that $I_x = I_{x'}$, which follows from i). \square

2.6.9 Proposition. — *Let E be an extension of F as in Lemma 2.6.7 and let $x, x' \in \mathcal{B}$. Suppose there exists an E -facet of \mathcal{B} whose closure contains both x and x' . Then for any $x'' \in]x, x'[$ we have $e_x e_{x'} = e_{x''} = e_{x'} e_x$.*

Proof. Note first that in this situation we have $x, x' \in \overline{\mathcal{F}_E(x'')}$.

We claim that $I_{x,x'} = I_{x''}$. Indeed, the inclusion $I_{x,x'} = I_x \cap I_{x'} \subseteq I_{x''}$ follows from the convexity of \mathcal{B}_ι in \mathcal{B} , while the other inclusion follows from \mathcal{B}_ι being a closed subset of \mathcal{B} together with $\mathcal{F}_E(x'') \subseteq \mathcal{B}_\iota$ for any $\iota \in I_{x''}$ by Proposition 2.6.8.i).

Now, let us fix $\iota \in I_{x,x'}$. By [Vig97, Prop 1.1], the groups $\mathbf{G}_\iota^i(E)_{x,r+}$ and $\mathbf{G}_\iota^i(E)_{x',r+}$ are contained in $\mathbf{G}_\iota^i(E)_{x'',r+}$ for each i , provided that $r \in v(E^\times)$. By taking Galois-fixed elements, it follows that $G_{\iota,x,(r_{i-1}/2)+}^i$ and $G_{\iota,x',(r_{i-1}/2)+}^i$ are contained in $G_{\iota,x'',(r_{i-1}/2)+}^i$. We infer that $K_{\iota,x}^+ \subseteq K_{\iota,x''}^+$ and $K_{\iota,x'}^+ \subseteq K_{\iota,x''}^+$. Lemma 2.5.12 then implies that $e_{\iota,x} e_{\iota,x''} = e_{\iota,x''}$ and $e_{\iota,x''} e_{\iota,x'} = e_{\iota,x''}$. We conclude thanks to Lemma 2.6.6 that $e_{\iota,x} e_{\iota,x'} = e_{\iota,x} e_{\iota,x''} e_{\iota,x'} = e_{\iota,x''}$. Using (2.6.5) we thus obtain the formula

$$e_x e_{x'} = \sum_{\iota \in I_{x''}/\sim_{x,x'}} e_{\iota,x''}, \text{ to compare with } e_{x''} = \sum_{\iota \in I_{x''}/\sim_{x''}} e_{\iota,x''}.$$

For $\iota_1, \iota_2 \in I_{x''}$, Lemma 2.6.1 tells us that

$$\begin{aligned} \iota_1 \sim_{x''} \iota_2 &\Leftrightarrow e_{\iota_1,x''} = e_{\iota_2,x''} \\ \iota_1 \sim_{x,x'} \iota_2 &\Leftrightarrow (e_{\iota_1,x} = e_{\iota_2,x} \text{ and } e_{\iota_1,x'} = e_{\iota_2,x'}) \end{aligned}$$

The equality $e_{\iota,x} e_{\iota,x'} = e_{\iota,x''}$ proved just above shows that $\iota_1 \sim_{x,x'} \iota_2 \Rightarrow \iota_1 \sim_{x''} \iota_2$. On the other hand, Lemma 2.6.1 also shows that

$$\begin{aligned} \iota_1 \sim_{x''} \iota_2 &\Leftrightarrow e_{\iota_1,x''} e_{\iota_2,x''} \neq 0 \\ \iota_1 \sim_{x,x'} \iota_2 &\Leftrightarrow (e_{\iota_1,x} e_{\iota_2,x} \neq 0 \text{ and } e_{\iota_1,x'} e_{\iota_2,x'} \neq 0) \end{aligned}$$

This time, the equalities $e_{\iota,x} e_{\iota,x'} = e_{\iota,x''} = e_{\iota,x'} e_{\iota,x}$ show that $\iota_1 \sim_{x''} \iota_2 \Rightarrow \iota_1 \sim_{x,x'} \iota_2$. \square

2.7 The category $\text{Rep}_R^{\phi,I}(G)$

We now construct the category attached to a wild inertia parameter $\phi : P_F \longrightarrow {}^L\mathbf{G}$ and a G -conjugacy class $I \subset I_\phi$ of F -rational embeddings $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$. If V is any smooth RG -module, it has an action of the Hecke algebra $\mathcal{H}_R(G)$ and in particular the idempotents $e_x = e_{\phi,I,x}$ act on it. This subsection is mainly devoted to the proof of the following theorem.

2.7.1 Theorem. — *The subcategory $\text{Rep}_R^{\phi,I}(G)$ of $\text{Rep}_R(G)$ defined by*

$$\text{Rep}_R^{\phi,I}(G) := \left\{ V \in \text{Rep}_R(G), V = \sum_{x \in \mathcal{B}} e_{\phi,I,x} V \right\}$$

is a Serre subcategory of $\text{Rep}_R(G)$, stable under arbitrary colimits, and generated by the following compact projective object of $\text{Rep}_R(G)$

$$P^{\phi,I} := \bigoplus_{x \in \Delta_0} \bigoplus_{\iota \in I_x / \sim_x} \text{ind}_{K_{\iota,x}^+}^G(\check{\phi}_{\iota,x}^+),$$

where Δ_0 denotes the set of e -vertices of a 1-chamber Δ of \mathcal{B} (a notion that will be introduced in 2.7.3 below). Moreover, any object $V \in \text{Rep}_R^{\phi,I}(G)$ is functorially an extension

$$(2.7.2) \quad V^{\phi,I} \hookrightarrow V \twoheadrightarrow V_{\phi,I}$$

where $V^{\phi,I} \in \text{Rep}_R^{\phi,I}(G)$ and $V_{\phi,I}$ has no subquotient that belongs to $\text{Rep}_R^{\phi,I}(G)$.

The strategy is to put ourselves in a position where we can apply [MS10, Thm 3.1], or at least closely follow its proof. This reference is concerned with systems of idempotents associated to vertices (more generally to polysimplices) in the reduced building $\mathcal{B}' := \mathcal{B}(\mathbf{G}_{\text{ad}}, F)$, while we have constructed idempotents associated to points of \mathcal{B} . However, our idempotent e_x only depends on the image of x in \mathcal{B}' , so that we actually have idempotents associated to points of \mathcal{B}' . Unfortunately, these idempotents are not constant on F -facets, but only on E -facets for some Galois extension E of F as in 2.6.7.

2.7.3 The e -subdivision of \mathcal{B}' . — Fix an integer $e \geq 1$. We define a subdivision of the polysimplicial structure on \mathcal{B}' in the following way.

Start with an apartment A of \mathcal{B}' and define an e -wall to be an affine hyperplane of the form $\varphi^{-1}(\frac{k}{e}t_1 + \frac{e-k}{e}t_2)$ where φ is an affine root on A , $t_1, t_2 \in \mathbb{R}$ are such that $\varphi^{-1}(t_1)$ and $\varphi^{-1}(t_2)$ are walls of A , and k is an integer between 0 and e . In particular, 1-walls are the usual walls and are also e -walls for any $e \geq 1$, and moreover any e -wall is parallel to some 1-wall. We thus get an enlarged collection of hyperplanes, leading to a refined partition of A into facets, that we call e -facets. We note that if o is a special point of A , then the e -walls of A are the images of the walls by the homothety of ratio $1/e$ centered at o . Indeed, for any affine root φ on A , it follows from [BT72, (6.2.16)] that the set of all $t \in \mathbb{R}$ such that $\varphi^{-1}(t + \varphi(o))$ is a wall is a discrete subgroup of \mathbb{R} . As a consequence, the e -facets are the images of the usual facets by the same homothety.

If A' is another apartment, we define e -walls and e -facets in the same way. Then for any $g \in G$, the action of g on \mathcal{B}' takes an e -wall of A to an e -wall of gA . In particular, if F is an

e -facet of A which intersects A' , then $F \subset A'$ and F is an e -facet of A' . Indeed, there is some $g \in G$ with $A' = gA$ and such that g fixes the 1-facet $F_1 \in \mathcal{B}'$ that contains F . This allows to define unambiguously the e -facets of \mathcal{B}' , and we get a polysimplicial structure on \mathcal{B}' which is preserved by the action of G . We will call it the e -subdivision of \mathcal{B}' .

Lemma. — *Let E be a tamely ramified Galois field extension of F with ramification index $e = e(E/F)$. Suppose that \mathbf{G} is split over E and quasi-split over the maximal unramified subextension E_0 of E . Then the e -subdivision of \mathcal{B}' refines the partition of \mathcal{B}' into E -facets, i.e., any E -facet is a union of e -facets.*

Proof. Given a maximal F -split torus \mathbf{T} of \mathbf{G} , [BT84, Cor. 5.1.12] ensures that we can find a maximal E_0 -split F -torus \mathbf{T}_0 that contains \mathbf{T} . Then the centralizer $\mathbf{S} = C_{\mathbf{G}}(\mathbf{T}_0)$ of \mathbf{T}_0 is a maximal F -torus of \mathbf{G} , and is split over E . In this situation there are inclusions of apartments $A = A(\mathbf{G}, \mathbf{T}, F) \subset A_0 = A(\mathbf{G}, \mathbf{T}_0, E_0) \subset A_S = A(\mathbf{G}, \mathbf{S}, E)$ and each subspace is obtained by taking suitable Galois invariants. By [BT84, Thm 5.1.20 iii)], the walls of the apartment A are exactly the non-trivial intersections of A with the walls of the apartment A_0 . Moreover, by [BT84, 4.2.4] each wall of A_0 is the intersection of A_0 with a wall of A_S . Conversely, the intersection of a wall of A_S with A_0 , when non-trivial, may not be a wall of A_0 but, at least, is parallel to a wall of A . More precisely, fix an origin o which is a special point in A_0 (e.g. that comes from a Chevalley–Steinberg system as in [BT84, 4.2.3]) and let a be a non-divisible root of \mathbf{T}_0 in \mathbf{G} , and let $E_0 \subset E_a \subset E$ be the associated extension (denoted by L_a in *loc. cit.*). Denote by $\Gamma_a \subset \mathbb{R}$ the set of real numbers v such that $\{x \in A_0, a(x) = v\}$ is a wall of A_0 . Then by [BT84, 4.2.21] we have $\Gamma_a = v(E_a^\times)$ (the valuation lattice of E_a) if $2a$ is not a root, and $\Gamma_a = \frac{1}{2}v(E_a^\times)$ if $2a$ is a root. On the other hand, let $\Gamma_{a,E} \subset \mathbb{R}$ be the set of real numbers v such that $\{x \in A_0, a(x) = v\}$ is the intersection of A_0 with a wall of A_S . If $2a$ is not a root and $v \in \Gamma_{a,E}$, then there is a root α of \mathbf{S} in \mathbf{G} that restricts to a and such that $\{x \in A_S, \alpha(x) = v\}$ is a wall of A_S , hence $v \in v(E^\times)$. If $2a$ is a root, then either there is α as above and then $v \in v(E^\times)$, or there are α, α' as above with $\alpha + \alpha'$ a root, and $\{x \in A_S, (\alpha + \alpha')(x) = 2v\}$ is a wall of A_S , in which case $v \in \frac{1}{2}v(E^\times)$. It follows that we have $\Gamma_{a,E} = v(E^\times)$ if $2a$ is not a root, and $\Gamma_{a,E} = \frac{1}{2}v(E^\times)$ if $2a$ is a root. In any case, for all non-divisible roots a of \mathbf{T}_0 we have $\Gamma_{a,E} = \frac{1}{e(E/E_a)}\Gamma_a$.

Since $e = e(E/E_0)$ is a common multiple of all $e(E/E_\alpha)$, the above discussion shows that the e -subdivision of the polysimplicial structure on A_0 refines the one that comes from A_S . Since the polysimplicial structure on A_0 induces the one on A (again by Thm 5.1.20 iii) of [BT84]), it follows that the e -subdivision of the polysimplicial structure on A refines its partition into E -facets. \square

2.7.4 From now on, we pick an integer e that is divisible by the ramification index of a tamely ramified field extension E of F that fulfills the requirements of both Lemmas 2.6.7 and 2.7.3. Then, for each $\iota \in I$, the image \mathcal{B}'_ι of \mathcal{B}_ι in \mathcal{B}' is a union of e -facets.

For $x \in \mathcal{B}'$ we denote by $\mathcal{F}_e(x)$ the unique e -facet of \mathcal{B}' that contains x . Further, we denote by $\mathcal{B}'_{\bullet/e}$ the partially ordered set of all e -facets, with the order given by $\mathcal{F}' \preceq \mathcal{F} \Leftrightarrow \mathcal{F}' \subseteq \overline{\mathcal{F}}$. We will also write $\mathcal{B}'_{d/e}$ for the set of d -dimensional e -facets. For $d = 0$ we also speak of “ e -vertices”. A family x_1, \dots, x_r of e -vertices are called “adjacent” if there exists an e -facet whose closure contains all these e -vertices. Then there is a unique e -facet $\mathcal{F}_e(x_1, \dots, x_r)$ with this property and which is minimal for the order defined above.

If x, x' are two points in \mathcal{B} , they are contained in a common apartment A . The intersection of all the half spaces associated to walls of A that contain x and x' is known to be independent of the choice of A . It is called the “combinatorial convex hull” of x and x' and we will denote it by $\mathcal{H}_1(x, x')$. It is a union of facets. Similarly we denote by $\mathcal{H}_e(x, x')$ the intersection of all e -half spaces (corresponding to e -walls) of A that contain x and x' . This is again independent of A and a union of e -facets. Obviously $[x, x'] \subset \mathcal{H}_e(x, x') \subset \mathcal{H}_1(x, x')$.

2.7.5 Lemma. — *Let the integer e be as in 2.7.4.*

Then the idempotents $(e_x)_{x \in \mathcal{B}'}$ have the following properties.

- i) *for all $x, x' \in \mathcal{B}'$ we have $\mathcal{F}_e(x) = \mathcal{F}_e(x') \Rightarrow e_x = e_{x'}$.*
- ii) *If x, x' are adjacent e -vertices and $x'' \in]x, x'[$, then $e_x e_{x'} = e_{x'} e_x = e_{x''}$.*
- iii) *If x, x', x'' are three e -vertices with $x' \in \mathcal{H}_e(x, x'')$ and x' adjacent to x , then $e_x e_{x'} e_{x''} = e_x e_{x''}$.*

In particular the system $(e_x)_{x \in \mathcal{B}'_{0/e}}$ is consistent in the sense of [MS10, Def. 2.1].

Proof. Thanks to Lemma 2.7.3, Part i) follows from Proposition 2.6.8 iii), and ii) follows from Proposition 2.6.9. To prove statement iii), let $y \in]x, x''[$ be sufficiently closed to x so that $x \in \overline{\mathcal{F}_e(y)}$. Then, $\mathcal{F}_e(y)$ is the unique e -facet that is maximal among those e -facets \mathcal{F} with $\overline{\mathcal{F}} \subset \mathcal{H}_e(x, x'')$ and $x \in \overline{\mathcal{F}}$. Indeed, this is proved as Lemma 2.9 of [MS10], since the geometric properties of the polysimplicial structure of \mathcal{B}' used in the proof of that lemma are satisfied by its e -subdivision. In particular, we have $x' \in \overline{\mathcal{F}_e(y)}$, and we may choose $y' \in \mathcal{F}_e(y)$ such that $y \in]x', y'[$. By Part i) we have $e_y = e_{y'}$. Using $e_y = e_{y'}$ and Propositions 2.6.6 and 2.6.9 repeatedly, we obtain $e_x e_{x''} = e_x e_y e_{x''} = e_x e_{x'} e_{y'} e_{x''} = e_x e_{x'} e_y e_{x''} = e_{x'} e_x e_y e_{x''} = e_{x'} e_x e_{x''} = e_x e_{x'} e_{x''}$. \square

Based on Part i) of the above lemma, we can use the following notation.

2.7.6 Notation. — *If \mathcal{F} is an e -facet of \mathcal{B}' , we set $e_{\mathcal{F}} := e_x$ for any $x \in \mathcal{F}$.*

By ii) of the last lemma, we have $e_{\mathcal{F}} e_{\mathcal{F}'} = e_{\mathcal{F}}$ for any two e -facets $\mathcal{F}, \mathcal{F}'$ such that $\mathcal{F}' \subset \overline{\mathcal{F}}$.

We now check that the proof of Theorem 2.4 of [MS10] can be adapted to our setting. Let $V \in \text{Rep}_R(G)$ be a smooth RG -module. It defines a coefficient system $\mathcal{F} \mapsto \mathcal{V}(\mathcal{F}) := e_{\mathcal{F}} V$ over $\mathcal{B}'_{\bullet/e}$, in which the transition maps $e_{\mathcal{F}} V \rightarrow e_{\mathcal{F}'} V$ for $\mathcal{F}' \subset \overline{\mathcal{F}}$ are inclusions, thanks to the above identities $e_{\mathcal{F}} e_{\mathcal{F}'} = e_{\mathcal{F}}$. After choosing an orientation of $\mathcal{B}'_{\bullet/e}$ we may form the cellular chain complex $\mathcal{C}(\mathcal{B}'_{\bullet/e}, \mathcal{V})$, whose homology we denote by $H_*(\mathcal{B}'_{\bullet/e}, \mathcal{V})$. More generally, for any polysimplicial subcomplex Σ of $\mathcal{B}'_{\bullet/e}$ we have a chain complex and its homology $H_*(\Sigma, \mathcal{V})$. We refer e.g. to [MS10, §1.1.2] for a brief introduction to these cellular chain complexes.

2.7.7 Lemma. (cf. [MS10, Theorem 2.4]) — *For any convex polysimplicial subcomplex Σ of $\mathcal{B}'_{\bullet/e}$, we have $H_0(\Sigma, \mathcal{V}) = \sum_{x \in \Sigma_0} e_x V$ and $H_n(\Sigma, \mathcal{V}) = 0$ for $n > 0$.*

Proof. We review the different steps of Meyer and Solleveld’s proof of Theorem 2.4 in [MS10].

Step 1. Prove it when Σ is a polysimplex. The argument below Lemma 2.18 of *loc. cit.* relies directly on Properties i), ii) and iii) of Lemma 2.7.5 and works without any change.

Step. 2. Divide and conquer method: Suppose Σ is *finite* and is the union of two convex subcomplexes Σ_+ and Σ_- with convex intersection Σ_0 . Then if the statement holds for Σ_+ , Σ_- and Σ_0 , it holds for Σ . This reduction step follows from Theorem 2.12 of *loc. cit.*, which asserts that the distribution $e_\Sigma := \sum_{\mathcal{F} \subset \Sigma} (-1)^{\dim(\mathcal{F})} e_{\mathcal{F}}$ is an idempotent such that $e_\Sigma e_{\mathcal{F}} = e_{\mathcal{F}} e_\Sigma = e_{\mathcal{F}}$ for all $\mathcal{F} \subset \Sigma$. This theorem in turn follows from Lemmas 2.8 and 2.9 and Proposition 2.2 of *loc. cit.* But, provided Properties i), ii) and iii) of Lemma 2.7.5, all these statements are concerned with the geometry of combinatorial convex hulls in an apartment, hence they still hold for any subdivision as in our case.

Step. 3. Prove that if Σ is *finite* and not a polysimplex, then it can be split as in Step 2. Here the argument has to be complemented a bit. Suppose first that Σ is contained in an apartment. Then there is an e -wall whose two associated open half-spaces intersect Σ non trivially. Simply take Σ_\pm to be the intersection with the closed half spaces, and Σ_0 the intersection with the wall. Now suppose that Σ is not contained in a single apartment. Then we can find a 1-chamber Δ whose closure intersects Σ non-trivially but does not contain it. Pick an apartment A that contains Δ and a wall of A that supports a face of Δ and intersects Σ non-trivially. It corresponds to some affine root a and we can use the retraction on A centered at Δ exactly as on p.140 of *loc.cit.*

Step. 4. The statement is now known when Σ is finite. It follows in the case that Σ is infinite by writing Σ as the union of an increasing sequence of finite convex subcomplexes Σ_n , which can always be done. Indeed, the chain complex Σ is the direct limit of the chain complexes of the Σ_n . \square

2.7.8 Proof of Theorem 2.7.1. — If $x \in \mathcal{B}'$, there is an e -vertex y that lies in the closure of the e -facet $\mathcal{F}_e(x)$. Then it follows from Proposition 2.6.9 that $e_x = e_y e_x$. Thus we see that $V \in \text{Rep}_R^{\phi, I}(G)$ if and only if $V = \sum_{x \in \mathcal{B}_{0/e}} e_{\phi, I, x} V$. Therefore, thanks to the case $\Sigma = \mathcal{B}'_{/e}$ of the previous lemma, the proof of Theorem 3.1 of [MS10] adapts verbatim to show that the category $\text{Rep}_R^{\phi, I}(G)$ is a Serre category that is stable under arbitrary colimits and generated as claimed in the theorem. Further let V be any smooth RG -module and put $V^{\phi, I} := \sum_{x \in \mathcal{B}} e_x V$. We certainly have $V^{\phi, I} \in \text{Rep}_R^{\phi, I}(G)$, and we see that the quotient $V_{\phi, I} := V/V^{\phi, I}$ is killed by all e_x so that no non-zero subquotient of $V_{\phi, I}$ belongs to $\text{Rep}_R^{\phi, I}(G)$.

2.7.9 Remark. — Let $(\Sigma_n)_{n \in \mathbb{N}}$ be an increasing sequence of convex polysimplicial subcomplexes of \mathcal{B}' such that $\mathcal{B}' = \bigcup_n \Sigma_n$. We have already recalled that $e_{\Sigma_n} := \sum_{\mathcal{F} \subset \Sigma_n} (-1)^{\dim(\mathcal{F})} e_{\mathcal{F}}$ is an idempotent such that $e_{\Sigma_n} e_{\mathcal{F}} = e_{\mathcal{F}} e_{\Sigma_n} = e_{\mathcal{F}}$ for all $\mathcal{F} \subset \Sigma_n$. It follows that $e_{\Sigma_{n-1}} e_{\Sigma_n} = e_{\Sigma_n}$ and that for all $V \in \text{Rep}_R(G)$ we have

$$\sum_{x \in \mathcal{B}} e_x V = \bigcup_n e_{\Sigma_n} V.$$

2.8 Some properties of $\text{Rep}_R^{\phi, I}(G)$

2.8.1 The direct factor problem. — We strongly believe that the category $\text{Rep}_R^{\phi, I}(G)$ is actually a direct factor of $\text{Rep}_R(G)$, but this does not follow formally from the work of Meyer–Solleveld, nor from the one of Yu. The problem is to show that the extension (2.7.2) splits, and more precisely that the subspace $\bigcap_x \ker(e_x|V)$ of V maps onto $V_{\phi, I}$.

What is missing is the existence of injective cogenerators in $\text{Rep}_R^{\phi,I}(G)$, or equivalently, of sufficiently many projective representations killed by all $e_{\phi,I,x}$. It seems that in order to bypass these problems, one needs some exhaustion result, e.g., as the ones proved by Kim [Kim07] and the second-named author [Fin21], or the one provided by the Bushnell–Kutzko–Stevens type theory. However, if one restricts attention to admissible objects over a complete local ring, then a duality trick implies the desired splitting.

Proposition. — *Let \mathcal{R} be a complete local commutative R -algebra, and let V be an admissible smooth $\mathcal{R}G$ -module (meaning that for any open, compact subgroup H of G , the \mathcal{R} -module V^H is noetherian). Then the extension (2.7.2) splits. In other words, the admissible category $\text{Adm}_{\mathcal{R}}^{\phi,I}(G)$ is a direct factor of $\text{Adm}_{\mathcal{R}}(G)$.*

Proof. Let \mathcal{E} be a Matlis module over \mathcal{R} (i.e., an injective hull of the residue field), and extend the Matlis duality functor to smooth $\mathcal{R}G$ -modules by putting $V^* := \text{Hom}_{\mathcal{R}}(V, \mathcal{E})^\infty$. Since p is invertible in \mathcal{R} , we have $(V^*)^H = \text{Hom}_{\mathcal{R}}(V^H, \mathcal{E})$ for any open pro- p -subgroup. Therefore, the usual Matlis duality theorem for noetherian \mathcal{R} -modules implies that for an admissible $\mathcal{R}G$ -module, the canonical map $V \rightarrow V^{**}$ is an isomorphism, and induces isomorphisms $W \mapsto (W^\perp)^\perp$ for each $\mathcal{R}G$ -submodule W of V .

Now, let e_x^* be the image of e_x by the anti-involution $g \mapsto g^{-1}$ on $\mathcal{H}_R(G)$. Note that the system of idempotents $(e_x^*)_{x \in \mathcal{B}_{0/e}}$ is the one attached to the pair $(\bar{\phi}, I)$ where $\bar{\cdot}$ denotes the automorphism of ${}^L G$ induced by complex conjugation. Let us put $eV := \sum_{x \in \mathcal{B}} e_x V$ and $e^*V^* := \sum_{x \in \mathcal{B}} e_x^* V^*$. Then we see that $(e^*V^*)^\perp = \bigcap_{x \in \mathcal{B}} \ker(e_x | V)$ and $(eV)^\perp = \bigcap_{x \in \mathcal{B}} \ker(e_x^* | V^*)$. By biduality we have $(\bigcap_{x \in \mathcal{B}} \ker(e_x | V))^\perp = e^*V^*$ and it follows that $V = eV \oplus \bigcap_{x \in \mathcal{B}} \ker(e_x | V)$ as desired. \square

2.8.2 Proposition. (Disjonction) — *Let (ϕ', I') and (ϕ, I) be two distinct pairs as in 2.7.1. Then the categories $\text{Rep}_R^{\phi', I'}(G)$ and $\text{Rep}_R^{\phi, I}(G)$ are orthogonal in the sense that for all objects $V \in \text{Rep}_R^{\phi, I}(G)$ and $V' \in \text{Rep}_R^{\phi', I'}(G')$ we have $\text{Ext}_{RG}^*(V, V') = \text{Ext}_{RG}^*(V', V) = \{0\}$.*

Proof. By (2.5.3) we have $e_{\phi', I', x'} e_{\phi, I, x} = 0$ for all $x', x \in \mathcal{B}$. It follows that $e_{\phi', I', x'} V = 0$ for all $V \in \text{Rep}_R^{\phi, I}(G)$, hence also $\text{Hom}_{RG}(V', V) = 0$ for all $V' \in \text{Rep}_R^{\phi', I'}(G')$. Using projective resolutions inside $\text{Rep}_R^{\phi', I'}(G')$, we also obtain that $\text{Ext}_{RG}^*(V', V) = 0$. \square

2.8.3 The “essentially tame” subcategory. — We introduce the full subcategory

$$\text{Rep}_R^{\text{et}}(G) := \left\{ V \in \text{Rep}_R(G), V = \sum_{\phi, I, x} e_{\phi, I, x} V \right\}.$$

Here “et” stands for “essentially tame” in order to stick to Bushnell and Henniart’s terminology, although we fear that this is a bit misleading. Morally (and under the hypotheses (H1) and (H2)), this subcategory should capture all the representations associated to Langlands parameters that are trivial on the derived subgroup $[P_F, P_F]$ of the wild inertia subgroup P_F . By construction, this is a Serre subcategory generated by projective objects and closed under arbitrary colimits, and Proposition 2.8.2 tells us that it decomposes as a direct product

$$\text{Rep}_R^{\text{et}}(G) = \prod_{(\phi, I)} \text{Rep}_R^{\phi, I}(G).$$

As mentioned above, we don't know in general whether it is a direct factor subcategory, but we will prove that under a further hypothesis it is the entire category. Recall that our construction of $\text{Rep}_R^{\phi, I}(G)$ applies to any ϕ such that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$, under the hypothesis that p is odd, and not a torsion prime of \mathbf{G} , nor of $\hat{\mathbf{G}}$. With a further (and much stronger) hypothesis, one can actually ensure that *all* centralizers $C_{\hat{\mathbf{G}}}(\phi)$ for $\phi \in \Phi(P_F, \mathbf{G})$ are Levi subgroups. Indeed, by Lemma 2.2.9 iv), this is the case whenever p does not divide the order of the absolute Weyl group of \mathbf{G} .

Under this hypothesis, it is thus natural to expect that all representations are essentially tame in the sense introduced above. This is indeed a consequence of the exhaustion results in [Fin21] and [Fin22].

2.8.4 Theorem. — *Suppose that p does not divide the order of the absolute Weyl group of \mathbf{G} . Then $\text{Rep}_R^{\text{et}}(G) = \text{Rep}_R(G)$.*

Proof. In order to prove the theorem, we have to show that for every $V \in \text{Rep}_R(G)$ there exists a wild inertia parameter ϕ , an embedding $\iota \in I_\phi$ and $y \in \mathcal{B}_\iota$ such that $e_{\phi, \iota, y} V \neq 0$. We will deduce this from a similar result obtained by [Fin21] and [Fin22] when proving that the types constructed by Kim and Yu produce types for all Bernstein blocks. We first recall the relevant result translated to our setting. Observe that in order to apply Yu's construction described in 2.4.7, all that is needed is a triple $(\vec{\mathbf{G}}, \vec{\psi}, x)$, that we will refer to as a “truncated Yu datum”, in which $\vec{\mathbf{G}} = (\mathbf{G}^0 \subset \dots \subset \mathbf{G}^d)$ is a tame twisted Levi sequence, $x \in \mathcal{B}$ lies in the image of an admissible embedding $\mathcal{B}(\mathbf{G}^0, F) \hookrightarrow \mathcal{B}$, and $\vec{\psi} = (\psi_i)_{i=0, \dots, d}$ is a collection of characters⁵ $\psi_i : \mathbf{G}^i(F) \rightarrow \mathbb{C}^\times$ such that

- the depths of $\psi_0, \dots, \psi_{d-1}$ form an increasing sequence $r_{-1} := 0 < r_0 < \dots < r_{d-1}$,
- either $\psi_d = 1$ (we then put $r_d = r_{d-1}$) or ψ_d has depth $r_d > r_{d-1}$.

To such a truncated Yu datum, the procedure of 2.4.7 attaches a pair $(K_x^+, \hat{\psi}_x)$ consisting of an open pro- p -subgroup and a character of this subgroup. This pair defines in turn an idempotent $e(\vec{\mathbf{G}}, \vec{\psi}, x)$ in $\mathcal{H}_R(G)$, supported on K_x^+ .

Analogous to the discussion at the beginning of the proof of [Fin22, Theorem 4.1], using the observation that the images of characters of pro- p groups factor through $\mu_p^\infty \rightarrow R^\times$ and using the divisibility of μ_p^∞ to extend characters, we may apply the arguments of the proofs of [Fin21, Theorem 6.1] (whose weaker form, the containment of a truncated datum in the notation of *loc. cit.*, suffices here) and [Fin21, Lemma 7.6] together with [Fin21, Lemma 7.3] to obtain the following result: *For any RG -module V , there is a normalized, generic truncated Yu datum $(\vec{\mathbf{G}}, \vec{\psi}, x)$ such that $e(\vec{\mathbf{G}}, \vec{\psi}, x)V \neq 0$.*

Here, a (truncated) Yu datum is called *generic* if for all $i < d$, the restriction $(\psi_i)|_{G_{x, r_i}^i}$ is represented by a \mathbf{G}^{i+1} -generic element (as in 2.5.7), and is called *normalized* if ψ_i is trivial on $(\mathbf{G}_i)_{\text{sc}}(F)$ for all i . With this result in hand, we are left to prove that for any normalized, generic, truncated Yu datum there is a wild inertia parameter ϕ , an embedding $\iota \in I_\phi$ and a point $y \in \mathcal{B}_\iota$ such that $e_{\phi, \iota, y} = e(\vec{\mathbf{G}}, \vec{\psi}, x)$. This follows from the next lemma. \square

⁵In the literature the standard notation for these characters is ϕ_i but in this paper the letter ϕ has already been dedicated to parameters.

2.8.5 Lemma. — Here we only assume that p is odd and not a torsion prime of $\hat{\mathbf{G}}$ (weakening a bit our usual assumptions). Let $(\vec{\mathbf{G}}, \vec{\psi}, x)$ be a normalized, generic truncated Yu datum. Then there is a pair (ϕ, ι) consisting of a wild inertia parameter ϕ of \mathbf{G} and an F -rational embedding $\iota \in I_\phi$ such that $\mathbf{G}_\iota^0 = \mathbf{G}^0$, $K_{\iota, x}^+ = K_x^+$, and $\check{\phi}_{\iota, x}^+ = \check{\psi}_x$.

Proof. The character $\check{\varphi}_0 := \prod_{k=0}^d (\psi_k)|_{\mathbf{G}^0}$ of $\mathbf{G}^0(F)$ is trivial on $(\mathbf{G}^0)_{\text{sc}}(F)$ since the datum is normalized. By Remark 2.4.2, $\check{\varphi}_0$ comes from some element $\hat{\varphi}_0 \in Z^1(W_F, Z(\hat{\mathbf{G}}^0))$.

Let us choose a tamely ramified maximal torus \mathbf{S} in \mathbf{G}^0 . There is a canonical W_F -equivariant embedding $Z(\hat{\mathbf{G}}^0) \hookrightarrow \hat{\mathbf{S}}$ that allows us to pushforward $\hat{\varphi}_0$ into $Z^1(W_F, \hat{\mathbf{S}})$, giving the Langlands parameter $\varphi_0 := \hat{\varphi}_0 \rtimes \text{Id}_{W_F} : W_F \longrightarrow {}^L\mathbf{S}$ of the character $(\check{\varphi}_0)|_{\mathbf{S}(F)}$. Consider the W_F -stable conjugacy class of embeddings $\hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$ that is dual to $\mathbf{S}_{\overline{F}} \rightarrow \mathbf{G}_{\overline{F}}$. Any such embedding $\kappa : \hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$ can be extended to a tamely ramified L -embedding ${}^L\kappa : {}^L\mathbf{S} \hookrightarrow {}^L\mathbf{G}$ (see [Kal19, Lemma 5.2.6]). We put $\varphi := {}^L\kappa \circ \varphi_0 : W_F \longrightarrow {}^L\mathbf{G}$ and $\phi := \varphi|_{P_F}$.

By construction, $\phi(P_F)$ is contained in $\kappa(\hat{\mathbf{S}})$, hence it is abelian. Since p is not a torsion prime for $\hat{\mathbf{G}}$, the centralizer $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$ by Lemma 2.2.9 iii). As usual, we write $\hat{\mathbf{S}}_\phi := C_{\hat{\mathbf{G}}}(\phi)_{\text{ab}}$. From φ , we get an action of W_F on $\hat{\mathbf{S}}_\phi$, and the embedding $\kappa : \hat{\mathbf{S}} \hookrightarrow C_{\hat{\mathbf{G}}}(\phi)$ induces a W_F -equivariant epimorphism of tori $\hat{\mathbf{S}} \twoheadrightarrow \hat{\mathbf{S}}_\phi$. The dual of this epimorphism then provides an F -rational embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{S} \hookrightarrow \mathbf{G}$, whose mere existence implies that ϕ is a wild inertia parameter of \mathbf{G} , by Proposition 2.2.3 ii).

What we have done so far is to associate a pair (ϕ, ι) to any normalized truncated Yu datum $(\vec{\mathbf{G}}, \vec{\psi}, x)$. We now use that our given truncated Yu datum is also generic, and we choose a tame extension E that splits \mathbf{S} . Then we claim that for any $r \in \mathbb{R}$ such that $r_{i-1} < r \leq r_i$ we have

$$\{\alpha \in \Sigma(\mathbf{S}, \mathbf{G}), \check{\varphi}_0(N_{E/F}(\alpha^\vee(E_r^\times))) = \{1\}\} = \Sigma(\mathbf{S}, \mathbf{G}^i).$$

Indeed, the case $i = 0$ is addressed in the proof of Lemma 3.6.9 of [Kal19] and the same proof applies to $i < d$. As in the proof of Lemma 2.3.3, it follows that the identification between $\Sigma(\mathbf{S}, \mathbf{G})$ and $\Sigma(\hat{\mathbf{S}}, \hat{\mathbf{G}})^\vee$ identifies $\Sigma(\mathbf{S}, \mathbf{G}^i)$ and $\Sigma(\hat{\mathbf{S}}, C_{\hat{\mathbf{G}}}(\phi(I_F^r)))^\vee$ for $r_{i-1} < r \leq r_i$. In other words we have $\mathbf{G}^i = C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi, r}))$ and, in particular $\mathbf{G}^i = \mathbf{G}_\iota^i$. It also follows that $K_{\iota, x}^+ = K_x^+$ for any point $x \in \mathcal{B}_\iota$.

Finally, for each i , the character $\check{\varphi}_i := \prod_{k=i}^d (\psi_k)|_{\mathbf{G}_i(F)}$ of $\mathbf{G}_i(F)$ is also trivial on $(\mathbf{G}_i)_{\text{sc}}(F)$ for all i , hence comes from some element $\hat{\varphi}_i \in Z^1(W_F, Z(\hat{\mathbf{G}}^i))$ and, by construction, we have $(\hat{\varphi}_0)|_{I_F^{r_i}} = (\hat{\varphi}_i)|_{I_F^{r_i}}$ in $H^1(I_F^{r_i}, Z(\hat{\mathbf{G}}_0))$. We find ourselves in the setting of 2.4.5 and we infer that $\check{\phi}_{\iota, x}^+ = \check{\psi}_x$. \square

2.8.6 Compatibility with isogenies. — To any isogeny $f : \mathbf{G}' \longrightarrow \mathbf{G}$ is associated a canonical conjugacy class of dual isogenies $\hat{f} : \hat{\mathbf{G}} \longrightarrow \hat{\mathbf{G}}'$. Moreover, if f is defined over F , then any such dual isogeny can be extended to a morphism of L -groups ${}^L\mathbf{G} \longrightarrow {}^L\mathbf{G}'$. We thus get a well defined transfer map $f^* : \Phi(P_F, \mathbf{G}) \mapsto \Phi(P_F, \mathbf{G}')$. Note that our hypothesis (H1) and (H2) hold for \mathbf{G}' since they are assumed to hold for \mathbf{G} . If we fix a dual isogeny \hat{f} and a morphism $\hat{\phi} : P_F \longrightarrow \hat{\mathbf{G}}$ such that $\phi = \hat{\phi} \times \text{Id}$ is a wild inertia parameter, then we obtain an isogeny of Levi subgroups $C_{\hat{\mathbf{G}}}(\phi) = C_{\hat{\mathbf{G}}}(\hat{\phi}) \longrightarrow C_{\hat{\mathbf{G}}'}(\hat{f} \circ \hat{\phi})$ with kernel $\ker(\hat{f})$, which dually provides a conjugacy class of F -rational isogenies $\mathbf{G}_{f^*\phi} \longrightarrow \mathbf{G}_\phi$ with kernel $\ker(f)$ together with an F -rational isogeny $\mathbf{S}_{f^*\phi} \longrightarrow \mathbf{S}_\phi$. Now any $\iota \in I_\phi$ induces an isomorphism $\mathbf{S}_\phi \xrightarrow{\sim} Z(\mathbf{G}_\iota)^\circ$, and the discussion in 2.1.3 shows that this isomorphism lifts uniquely to an

isomorphism $\mathbf{S}_{f^*\phi} \xrightarrow{\sim} Z(f^{-1}(\mathbf{G}_\iota))^\circ$, thus providing an element $f^*\iota \in I_{f^*\phi}$. In this way we obtain a bijection $I_\phi = I_{f^*\phi}$ that respects F -rationality, but the G' -conjugacy is a priori coarser than the G -conjugacy.

Proposition. — *In this setting, the pull-back functor $f^* : \text{Rep}_R(G) \rightarrow \text{Rep}_R(G')$ takes $\text{Rep}_R^{\phi,I}(G)$ into $\prod_{I' \subseteq f^*I} \text{Rep}_R^{f^*\phi, I'}(G')$.*

Proof. Let us identify the Bruhat–Tits buildings of \mathbf{G} and \mathbf{G}' . Then, from the definitions we see that $\mathcal{B}_\iota = \mathcal{B}_{f^*\iota}$ for any $\iota \in I_\phi$. Moreover, for $x \in \mathcal{B}_\iota$, we have a surjection $K_{f^*\phi, f^*\iota, x}^+ \rightarrow K_{\phi, \iota, x}^+$ and the construction of the characters shows that $(f^*\phi)_{f^*\iota, x}^+$ is the pull back of $\check{\phi}_{\iota, x}^+$. The claim then follows from the definition of the categories under consideration. \square

2.8.7 Compatibility with parabolic induction. — Suppose that ϕ comes from an F -Levi subgroup \mathbf{M} of \mathbf{G} , in the sense that there is a dual embedding ${}^L\mathbf{M} \rightarrow {}^L\mathbf{G}$ and a factorization $\phi : P_F \xrightarrow{\phi_M} {}^L\mathbf{M} \rightarrow {}^L\mathbf{G}$, where ϕ_M is a wild inertia parameter of \mathbf{M} (recall that this means that ϕ_M admits an extension to a relevant Langlands parameter $W'_F \rightarrow {}^L\mathbf{M}$).

In this context we have $\hat{\mathbf{M}}_{\phi_M} = \hat{\mathbf{M}} \cap \hat{\mathbf{G}}_\phi$, whence a W_F -equivariant surjection $\hat{\mathbf{S}}_{\phi_M} \rightarrow \hat{\mathbf{S}}_\phi$, which on the dual side induces an injection of F -tori $\mathbf{S}_\phi \hookrightarrow \mathbf{S}_{\phi_M}$. It follows that any F -rational Levi-center-embedding $\iota_M : \mathbf{S}_{\phi_M} \hookrightarrow \mathbf{M}$ in the set I_{ϕ_M} induces an embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in the set I_ϕ . Obviously, M -conjugate embeddings lead to G -conjugate embeddings, so that any choice of an M -conjugacy class $I_M \subset I_{\phi_M}$ points to a G -conjugacy class $I \subset I_\phi$. We will use the notation $(\phi_M, I_M) \mapsto (\phi, I)$ to express the fact that ϕ comes from ϕ_M and I from I_M .

Theorem. — *Let \mathbf{P} be a parabolic F -subgroup of \mathbf{G} with Levi component \mathbf{M} , and denote by i_P the associated (not normalized) parabolic induction functor, and by r_P the corresponding Jacquet functor.*

i) If the pair (ϕ, I) comes from the pair (ϕ_M, I_M) , then we have

$$i_P \left(\text{Rep}_R^{\phi_M, I_M}(M) \right) \subseteq \text{Rep}_R^{\phi, I}(G).$$

ii) Assuming that p does not divide the order of the absolute Weyl group of \mathbf{G} , we have

$$r_P(\text{Rep}_R^{\phi, I}(G)) \subseteq \prod_{(\phi_M, I_M) \mapsto (\phi, I)} \text{Rep}_R^{\phi_M, I_M}(M).$$

iii) Assuming further that $C_{\hat{\mathbf{G}}}(\phi) \subseteq \hat{\mathbf{M}}$, then i_P induces an equivalence of categories

$$\text{Rep}_R^{\phi_M, I_M}(M) \xrightarrow{\sim} \text{Rep}_R^{\phi, I}(G)$$

with quasi inverse the composition of r_P and the projection onto the (ϕ_M, I_M) -factor.

2.8.8 We start with some preliminary observations to prepare for proving the theorem. First note that the equality $\hat{\mathbf{M}}_{\phi_M} = \hat{\mathbf{M}} \cap \hat{\mathbf{G}}_\phi$ provides two surjective maps $\hat{\mathbf{S}}_{\phi_M} \rightarrow \hat{\mathbf{S}}_\phi$ and $\hat{\mathbf{S}}_{\phi_M} \rightarrow \hat{\mathbf{M}}_{\text{ab}}$ whose product $\hat{\mathbf{S}}_{\phi_M} \rightarrow \hat{\mathbf{S}}_\phi \times \hat{\mathbf{M}}_{\text{ab}}$ has finite kernel (since $Z(\hat{\mathbf{M}}_{\phi_M})^\circ = Z(\hat{\mathbf{M}})^\circ Z(\hat{\mathbf{G}}_\phi)^\circ$). Dually, this provides two inclusions $\mathbf{S}_\phi \subset \mathbf{S}_{\phi_M}$ and $Z(\mathbf{M})^\circ \subset \mathbf{S}_{\phi_M}$ such that $\mathbf{S}_{\phi_M} = \mathbf{S}_\phi Z(\mathbf{M})^\circ$.

In particular, if $\iota_M \in I_M$ induces $\iota \in I_\phi$ through the first embedding, we have $\iota_M(\mathbf{S}_{\phi_M}) = \iota(\mathbf{S}_\phi)Z(\mathbf{M})^\circ$ and therefore $\mathbf{M}_{\iota_M} = \mathbf{M} \cap \mathbf{G}_\iota$.

Denote by \mathcal{B}_M the image of any admissible embedding $\mathcal{B}(\mathbf{M}, F) \hookrightarrow \mathcal{B}(\mathbf{G}, F)$. Then we have $\mathcal{B}_{\iota_M} = \mathcal{B}_\iota \cap \mathcal{B}_M$. Now, given a point $x \in \mathcal{B}_{\iota_M}$, the constructions of the previous subsections provide us with two idempotents $e_{\iota, x} \in \mathcal{H}_R(G)$ and $e_{\iota_M, x} \in \mathcal{H}_R(M)$ respectively associated to pairs $(K_{\iota, x}^+, \check{\phi}_{\iota, x}^+)$ and $(K_{\iota_M, x}^+, \check{\phi}_{\iota_M, x}^+)$ consisting of an open pro- p -group of G , resp., M , and a character of that group. In order to compare these two idempotents, it is useful to enlarge the ring $\mathcal{H}_R(G)$ to the ring RG consisting of all compactly supported distributions on G (not necessarily locally constant). We will use the analogous notation for any open, compact subgroup of G as introduced in Section 1.1. In particular, we may regard $e_{\iota, x}$ and $e_{\iota_M, x}$ as idempotents in the ring RG_x of all distributions on the stabilizer G_x of x . This ring also contains the averaging idempotents e_{U_x} , $e_{U_{x,0+}}$, $e_{U_{\iota, x}^\dagger}$ and $e_{U_{\iota, x}^+}$ of the closed pro- p -subgroups $U_x := U \cap G_x$, $U_{x,0+} := U \cap G_{x,0+}$, $U_{\iota, x}^\dagger := U \cap K_{\iota, x}^\dagger$ and $U_{\iota, x}^+ := U \cap K_{\iota, x}^+$ of G_x , where \mathbf{U} denotes the unipotent radical of \mathbf{P} . Similarly, we have idempotents $e_{\bar{U}_x}$, $e_{\bar{U}_{x,0+}}$, $e_{\bar{U}_{\iota, x}^\dagger}$ and $e_{\bar{U}_{\iota, x}^+}$, where $\bar{\mathbf{U}}$ is the unipotent radical of the opposite parabolic subgroup $\bar{\mathbf{P}}$ of \mathbf{P} with respect to \mathbf{M} .

The following lemma contains the main technical points of the proof of the above theorem.

2.8.9 Lemma. — *For any $x \in \mathcal{B}_{\iota_M}$ and with the foregoing notation, we have:*

- i) $e_{\iota, x} = e_{U_{\iota, x}^+} e_{\iota_M, x} e_{\bar{U}_{\iota, x}^+} = e_{U_{\iota, x}^\dagger} e_{\bar{U}_{\iota, x}^\dagger} e_{\iota_M, x} = e_{\bar{U}_{\iota, x}^\dagger} e_{\iota_M, x} e_{U_{\iota, x}^+}$ in $RK_{\iota, x}^+$.
- ii) $(e_{U_{\iota, x}^\dagger} e_{\bar{U}_{\iota, x}^\dagger} e_{U_{\iota, x}^+}) e_{\iota, x} = |U_{\iota, x}^\dagger / U_{\iota, x}^+|^{-1} e_{U_{\iota, x}^\dagger} e_{\iota, x}$
- iii) $e_{\iota, x} \in RK_{\iota, x}^\dagger e_{U_{\iota, x}^\dagger} e_{\iota_M, x} e_{\bar{U}_{\iota, x}^\dagger} RK_{\iota, x}^\dagger$
- iv) $e_{U_{x,0+}} e_{\bar{U}_x} e_{\iota_M, x} \in RG_x e_{U_x} e_{\bar{U}_x} e_{\iota_M, x}$ and $e_{\iota_M, x} e_{U_x} e_{\bar{U}_{x,0+}} \in e_{\iota_M, x} e_{U_x} e_{\bar{U}_x} RG_x$.
- v) $e_{\iota, x} e_{U_x} e_{\iota, x} = |U_x / U_{\iota, x}|^{-1} e_{U_{\iota, x}} e_{\iota, x}$ and $e_{\iota, x} e_{U_{x,0+}} e_{\iota, x} = |U_{x,0+} / U_{\iota, x}^\dagger|^{-1} e_{U_{\iota, x}^\dagger} e_{\iota, x}$.
- vi) If $C_{\hat{\mathbf{G}}}(\phi) \subseteq \hat{\mathbf{M}}$, then $U_{\iota, x} = U_{\iota, x}^\dagger$ and $\bar{U}_{\iota, x} = \bar{U}_{\iota, x}^\dagger$.

Proof. i) Since $\iota(\mathbf{S}_\phi) \subset \iota_M(\mathbf{S}_{\phi_M}) \subset \mathbf{M}$, we may apply 2.4.9 on the Iwahori factorization of $(K_{\iota, x}^+, \check{\phi}_{\iota, x}^+)$. Denoting by $e_{\iota, x}^M$ the idempotent of RG_x associated to the character $(\check{\phi}_{\iota, x}^+)|_{M \cap K_{\iota, x}^+}$, this lemma implies the following decomposition in RG_x (actually in $RK_{\iota, x}^+$):

$$e_{\iota, x} = e_{U_{\iota, x}^+} e_{\iota, x}^M e_{\bar{U}_{\iota, x}^+} = e_{U_{\iota, x}^\dagger} e_{\bar{U}_{\iota, x}^\dagger} e_{\iota, x}^M = e_{\bar{U}_{\iota, x}^\dagger} e_{\iota, x}^M e_{U_{\iota, x}^+}.$$

So it remains to see that $e_{\iota_M, x} = e_{\iota, x}^M$. Equivalently, we need to show that

$$M \cap K_{\iota, x}^+ = K_{\iota_M, x}^+ \text{ and } (\check{\phi}_{\iota, x}^+)|_{M \cap K_{\iota, x}^+} = \check{\phi}_{\iota_M, x}^+.$$

Let us first check the equality of groups. We have seen in 2.8.8 that $\iota_M(\mathbf{S}_{\phi_M}) = \iota(\mathbf{S}_\phi)Z(\mathbf{M})^\circ$ and therefore $\mathbf{M}_{\iota_M} = \mathbf{M} \cap \mathbf{G}_\iota$. Similarly, for any $r > 0$ we have $\iota_M(\mathbf{S}_{\phi_M, r}) = \iota(\mathbf{S}_{\phi, r})Z(\mathbf{M})^\circ$ and therefore $C_{\mathbf{M}}(\iota_M(\mathbf{S}_{\phi_M, r})) = \mathbf{M} \cap C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi, r}))$. It follows in particular that the set of jumps $r'_{-1} = 0 < r'_0 < \dots < r'_{d'-1}$ of the decreasing filtration $(\mathbf{S}_{\phi_M, r})_r$ is a subset of the set of jumps $r_{-1} = 0 < r_0 < \dots < r_{d-1}$ of the filtration $(\mathbf{S}_{\phi, r})_r$. For $0 \leq j < d'$ write i_j for the unique integer between 0 and $d-1$ such that $r'_j = r_{i_j}$, and put $i_{-1} = -1$. Then, with the notation of 2.4.7, we

have $M \cap G_{\ell_M, x, r+}^i = M_{\ell_M, x, r+}^j$ for all $i = 0, \dots, d$ and j such that $i_{j-1} < i \leq i_j$, and in particular we see that $M \cap G_{\ell_M, x, (r_{i-1}/2)+}^i \subseteq M_{\ell_M, x, (r'_{j-1}/2)+}^j$ with equality if $i = i_{j-1} + 1$. This implies the second equality in:

$$\begin{aligned} M \cap K_{\ell_M, x}^+ &= (M \cap G_{\ell_M, x, 0+}^0)(M \cap G_{\ell_M, x, (r_0/2)+}^1) \cdots (M \cap G_{\ell_M, x, (r_{d-1}/2)+}^d) \\ &= M_{\ell_M, 0+}^0 M_{\ell_M, (r'_0/2)+}^1 \cdots M_{\ell_M, (r'_{d-1}/2)+}^d = K_{\ell_M, x}^+ \end{aligned}$$

The first equality follows from the Iwahori decomposition of each $G_{\ell_M, x, (r_{i-1}/2)+}^i$ with respect to the pair $(\bar{\mathbf{P}}, \mathbf{P})$, as in the proof of Lemma 2.4.9.

Let us now turn to characters. By definition, $\check{\phi}_{\ell_M, x}^+$ is a product $\prod_{j=0}^{d'} (\psi_{M, j}^+) |_{K_{\ell_M, x}^+}$ with $\psi_{M, j}^+$ a certain character of $M_{\ell_M, x, 0+}^j M_{x, (r'_j/2)+}$, while $(\check{\phi}_{\ell_M, x}^+) |_{K_{\ell_M, x}^+}$ is a product $\prod_{i=0}^d (\psi_i^+) |_{K_{\ell_M, x}^+}$ with ψ_i^+ a certain character of $G_{\ell_M, x, 0+}^i G_{x, (r_i/2)+}$. Note that if $i_{j-1} < i \leq i_j$, we have

$$M \cap G_{\ell_M, x, 0+}^i G_{x, (r_i/2)+} = (M \cap G_{\ell_M, x, 0+}^i) M_{x, (r_i/2)+} = M_{\ell_M, x, 0+}^j M_{x, (r_i/2)+} \supseteq M_{\ell_M, x, 0+}^j M_{x, (r'_j/2)+}.$$

Therefore, it will suffice to prove that $\psi_{M, j}^+ = \prod_{i=i_{j-1}+1}^{i_j} (\psi_i^+) |_{M_{\ell_M, x, 0+}^j M_{x, (r'_j/2)+}}$. Recall that $(\psi_i^+) |_{G_{\ell_M, x, 0+}^i}$ is the restriction of a character $\check{\varphi}_i \check{\varphi}_{i+1}^{-1}$ of $G_{\ell_M}^i$ that depends on the choice of $\hat{\varphi}_i \in H^1(W_F, Z(\hat{\mathbf{G}}_{\phi, r_{i-1}+}))$ extending $\hat{\phi} |_{I_F^{r_{i-1}+}}$ and the choice of $\hat{\varphi}_{i+1} \in H^1(W_F, Z(\hat{\mathbf{G}}_{\phi, r_i+}))$ extending $\hat{\phi} |_{I_F^{r_i+}}$. Similarly, $(\psi_{M, j}^+) |_{M_{\ell_M, x, 0+}^j}$ is the restriction of a character $\check{\varphi}_{M, j} \check{\varphi}_{M, j+1}^{-1}$ that depends on the choice of $\hat{\varphi}_{M, j} \in H^1(W_F, Z(\hat{\mathbf{M}}_{\phi_M, r'_{j-1}+}))$ extending $\hat{\phi}_M |_{I_F^{r'_{j-1}+}}$ and the choice of $\hat{\varphi}_{M, j+1} \in H^1(W_F, Z(\hat{\mathbf{M}}_{\phi_M, r'_j+}))$ extending $\hat{\phi}_M |_{I_F^{r'_j+}}$. By Lemma 2.4.8, these choices eventually do not matter, in the sense that $\prod_{j=0}^{d'} (\psi_{M, j}^+) |_{K_{\ell_M, x}^+}$ and $\prod_{i=0}^d (\psi_i^+) |_{K_{\ell_M, x}^+}$ do not depend on them, so we may choose $\hat{\varphi}_{M, k}$ to be the composition of $\hat{\varphi}_{i_{k-1}+1}$ with the inclusion $Z(\hat{\mathbf{G}}_{\phi, r_{i_{k-1}+}}) \subset Z(\hat{\mathbf{M}}_{\phi_M, r'_{k-1}+})$ for all $1 \leq k \leq d'$. In this way, we ensure that the characters $\psi_{M, j}^+$ and $\prod_{i=i_{j-1}+1}^{i_j} \psi_i^+$ coincide on $M_{\ell_M, x, 0+}^j$. It then remains to see that they also coincide on $M_{x, (r'_j/2)+}$, by checking that Yu's extension procedures over M and G are compatible. For G and index i , this procedure rests on the decomposition $\mathfrak{g}_{x, (r_i/2)+:r_i+} = \mathfrak{g}_{\ell_M, x, (r_i/2)+:r_i+}^i \oplus \mathfrak{n}_{\ell_M, x, (r_i/2)+:r_i+}^i$. The key point is then that whenever $i_{j-1} < i \leq i_j$, intersecting this decomposition with \mathfrak{m} (which amounts to taking the weight-0 part of the action of the maximal split central torus of M) yields back the corresponding decomposition $\mathfrak{m}_{x, (r_i/2)+:r_i+} = \mathfrak{m}_{\ell_M, x, (r_i/2)+:r_i+}^j \oplus \mathfrak{n}_{\ell_M, x, (r_i/2)+:r_i+}^{M, j}$.

ii) and iii) Thanks to the Heisenberg property of Proposition 2.5.13, the computation of [Dat09, §5.28] shows⁶ that for each character $\chi : U_{\ell_M, x}^\dagger / U_{\ell_M, x}^+ \rightarrow R^\times$, and denoting by e_χ the associated idempotent in $RU_{\ell_M, x}^\dagger$, we have

$$|U_{\ell_M, x}^\dagger / U_{\ell_M, x}^+| e_\chi e_{\bar{U}_{\ell_M, x}^\dagger} e_\chi e_{\ell_M, x} = e_\chi e_{\ell_M, x}.$$

Taking $\chi = 1$, we obtain ii). Summing over all χ 's we get

$$|U_{\ell_M, x}^\dagger / U_{\ell_M, x}^+| \sum_{\chi: U_{\ell_M, x}^\dagger / U_{\ell_M, x}^+ \rightarrow R^\times} e_\chi e_{\bar{U}_{\ell_M, x}^\dagger} e_\chi e_{\ell_M, x} = e_{\ell_M, x},$$

⁶Indeed, we specialize the notation of *loc. cit.* as follows: $G^\dagger := K_{\ell_M, x}^\dagger$, $G^* := K_{\ell_M, x}^+$, $U^\dagger := U_{\ell_M, x}^\dagger$, $U^* := U_{\ell_M, x}^+$, and $\theta := \check{\phi}_{\ell_M, y}^+$.

showing in particular that $e_{\iota,x} \in RK_{\iota,x}^\dagger e_{\bar{U}_{\iota,x}^\dagger} RK_{\iota,x}^\dagger$. Similarly, we have $e_{\iota,x} \in RK_{\iota,x}^\dagger e_{U_{\iota,x}^\dagger} RK_{\iota,x}^\dagger$, hence also $e_{\iota,x} \in RK_{\iota,x}^\dagger e_{U_{\iota,x}^\dagger} RK_{\iota,x}^\dagger e_{\bar{U}_{\iota,x}^\dagger} RK_{\iota,x}^\dagger$. But $K_{\iota,x}^\dagger$ also has the Iwahori decomposition property with respect to $(\mathbf{P}, \bar{\mathbf{P}})$, so that $RK_{\iota,x}^\dagger e_{U_{\iota,x}^\dagger} RK_{\iota,x}^\dagger e_{\bar{U}_{\iota,x}^\dagger} RK_{\iota,x}^\dagger = RK_{\iota,x}^\dagger e_{U_{\iota,x}^\dagger} e_{\bar{U}_{\iota,x}^\dagger} RK_{\iota,x}^\dagger$. Finally, recall that $e_{\iota,x}$ is central in $RK_{\iota,x}^\dagger$ and observe that $e_{U_{\iota,x}^\dagger} e_{\iota,x} e_{\bar{U}_{\iota,x}^\dagger} = e_{U_{\iota,x}^\dagger} e_{\iota_M,x} e_{\bar{U}_{\iota,x}^\dagger}$.

iv) This is Proposition 9.3 of [Dat09]. However, since the proof there leaves many details to the reader, we supply these details here. That proposition is ultimately a consequence of Corollary 5.10 of *loc. cit.* applied to the following set up:

- $\underline{G} := G_{x,0}$ (and therefore $\underline{G}^\dagger = G_{x,0+}$, see below),
- $\underline{G}' := K_{\iota,x}^\circ$ (and therefore $\underline{G}'^\dagger = K_{\iota,x}^\dagger$ and $\underline{M}' = K_{\iota_M,x}^\dagger$),
- $\varepsilon' := e_{\iota_M,x}$ and $\tilde{\varepsilon}' := e_{\iota,x}$.

In order to apply Corollary 5.10 of *loc. cit.*, there are a number of properties to check. First of all, both \underline{G} and \underline{G}' need to be groups of integral points of a connected smooth model of \mathbf{G} over \mathcal{O}_F . Here, by definition $G_{x,0}$ is the parahoric subgroup of G_x and the desired model was constructed by Bruhat–Tits, while the desired model for $K_{\iota,x}^\circ$ was constructed by Yu in [Yu15, Prop 10.2]. Then, the groups \underline{G}^\dagger , resp., \underline{G}'^\dagger should be the integral points of the dilatation of the unipotent radical of the special fiber of \underline{G} , resp., \underline{G}' , or more explicitly, the pro- p -radical of \underline{G} , resp., \underline{G}' . Hence they are indeed given by $G_{x,0+}$, resp., $K_{\iota,x}^\dagger$. Note that the hypothesis $\underline{G}' \cap \underline{G}^\dagger \supseteq \underline{G}'^\dagger$ of *loc. cit.* is indeed satisfied (and even equality holds). Next, the parabolic subgroups \mathbf{P} and $\bar{\mathbf{P}}$ should be \underline{G}' -admissible in the sense of [Dat09, §5.1]. For this, it is enough to see that the maximal split central torus \mathbf{A}_M of \mathbf{M} extends to a subtorus of Yu’s model of $K_{\iota,x}^\circ$. But the latter contains the Bruhat–Tits model of the parahoric subgroup $G_{\iota,x,0}^0$ of G_ι by [Yu15, Prop 10.4 iv)]. Hence any maximal split torus \mathbf{S} of \mathbf{G}_ι whose apartment in \mathcal{B}_ι contains x extends to a split torus of Yu’s model. Since $x \in \mathcal{B}_{\iota_M} = \mathcal{B}(\mathbf{M}_{\iota_M})$ and $\mathbf{M}_{\iota_M} = \mathbf{M} \cap \mathbf{G}_\iota$ is an F -Levi subgroup of \mathbf{G}_ι , we may choose such an \mathbf{S} that is contained in \mathbf{M}_{ι_M} . But \mathbf{A}_M is contained in the center of \mathbf{M}_{ι_M} , hence also in \mathbf{S} , and we see that \mathbf{A}_M extends to a split subtorus of Yu’s model. There are three further requirements on the idempotents ε' and $\tilde{\varepsilon}'$. First, ε' should be “essentially of depth zero” in the sense of [Dat09, Lemma 5.6]. Actually, we have checked this property in the proof of ii) above. Next, hypothesis i) of [Dat09, Cor 5.10] follows from i) of this lemma, while hypothesis ii) of *loc. cit.* follows from Proposition 2.5.2 ii) (which implies more generally that the centralizer of $e_{\iota,x}$ in G_x is $K_{\iota,x}$). We may now apply Corollary 5.10 of [Dat09], thus completing the proof of iii).

v) By Proposition 2.5.2 ii) and (2.4.12), we have $e_{\iota,x} u e_{\iota,x} = 0$ for all $u \in U_x \setminus U_{\iota,x}$ while, by Proposition 2.5.2 i), we have $e_{\iota,x} u e_{\iota,x} = u e_{\iota,x}$ if $u \in U_{\iota,x}$. Writing $e_{U_x} = |U_x/U_{\iota,x}|^{-1} \sum_{u \in U_x/U_{\iota,x}} u e_{U_{\iota,x}}$ we obtain the first identity. The second one is proved in the same way.

vi) The hypothesis $C_{\hat{\mathbf{G}}}(\phi) \subseteq \hat{\mathbf{M}}$ implies that $\mathbf{S}_\phi = \mathbf{S}_{\phi_M}$. Since $\iota_M(\mathbf{S}_{\phi_M}) \supseteq Z(\mathbf{M})^\circ$, we obtain that $\mathbf{G}_\iota \subseteq \mathbf{M}$ (and actually $\mathbf{G}_\iota = \mathbf{M}_{\iota_M}$). In particular, we have $\mathbf{U} \cap \mathbf{G}_\iota = \bar{\mathbf{U}} \cap \mathbf{G}_\iota = \{1\}$ and it follows that $U \cap K_{\iota,x} = U \cap K_{\iota,x}^\dagger$ and $\bar{U} \cap K_{\iota,x} = \bar{U} \cap K_{\iota,x}^\dagger$. \square

2.8.10 Lemma. — *With the notation of 2.8.8, there is a $G \times K_{\iota_M,x}$ -equivariant isomorphism*

$$i_P \left(\text{ind}_{K_{\iota_M,x}^+}^M \left(\check{\phi}_{\iota_M,x}^+ \right) \right) \simeq \text{ind}_{U_{\iota,x} K_{\iota,x}^+}^G \left(\tilde{\phi}_{\iota,x}^+ \right)$$

where $\check{\phi}_{\ell,x}^+$ is the unique character of $U_{\ell,x}K_{\ell,x}^+$ that extends $\check{\phi}_{\ell,x}^+$ and the trivial character of $U_{\ell,x}$.

Here the actions of $K_{\ell_M,x}$ come from the fact this group centralizes both the character $\check{\phi}_{\ell_M,x}^+$ of $K_{\ell_M,x}^+$ and the character $\check{\phi}_{\ell,x}^+$ of $U_{\ell,x}K_{\ell,x}^+$. Explicitly, and through the identifications

$$\text{ind}_{K_{\ell_M,x}^+}^M(\check{\phi}_{\ell_M,x}^+) = \mathcal{H}_R(M)e_{\ell_M,x} \text{ and } \text{ind}_{U_{\ell,x}K_{\ell,x}^+}^G(\check{\phi}_{\ell,x}^+) = \mathcal{H}_R(G)e_{U_{\ell,x}}e_{\ell,x}$$

these actions are given by multiplication on the right.

Proof. Thanks to (iv) of Lemma 2.8.9, [Dat09, Cor. 3.6(ii)] provides us with an $RG \otimes RK_{\ell_M,x}$ -linear isomorphism

$$\text{ind}_{G_x}^G(\mathcal{H}_R(G_x)e_{\bar{U}_{x,0+}}e_{U_x}e_{\ell_M,x}) \xrightarrow{\sim} i_P \left(\text{ind}_{K_{\ell_M,x}^+}^M(\check{\phi}_{\ell_M,x}^+) \right).$$

Note that in *loc. cit.*, the equivariance is for $RG \otimes_R (e_{\ell_M,x}RM_x e_{\ell_M,x})$, and here we compose the second action with the map $RK_{\ell_M,x} \xrightarrow{\times e_{\ell_M,x}} e_{\ell_M,x}RM_x e_{\ell_M,x}$. (Note also that replacing i_P by the normalized induction provides an isomorphic co-domain of the above isomorphism.)

It thus remains to find an $RG_x \otimes_R RK_{\ell_M,x}$ -linear isomorphism between

$$\mathcal{H}_R(G_x)e_{\bar{U}_{x,0+}}e_{U_x}e_{\ell_M,x} = \mathcal{H}_R(G_x)e_{\bar{U}_{x,0+}}e_{\ell_M,x}e_{U_x} = \mathcal{H}_R(G_x)e_{\bar{U}_{x,0+}}e_{\ell,x}e_{U_x},$$

where the first equality follows from $K_{\ell_M,x} \subset M_x$ normalizing U_x , and

$$\mathcal{H}_R(G_x)e_{U_{\ell,x}}e_{\ell,x} = \text{ind}_{U_{\ell,x}K_{\ell,x}^+}^{G_x}(\check{\phi}_{\ell,x}^+).$$

By Lemma 2.8.9 (v), the inclusions

$$\mathcal{H}_R(G_x)e_{\ell,x}e_{\bar{U}_{x,0+}}e_{\ell,x} \subseteq \mathcal{H}_R(G_x)e_{\bar{U}_{x,0+}}e_{\ell,x} \subseteq \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{\ell,x}$$

are equalities. For the analogous reason, i.e., because $e_{\ell,x}e_{U_{\ell,x}}$ is a scalar multiple of $e_{\ell,x}e_{U_x}e_{\ell,x}$, the maps

$$\mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{\ell,x}e_{U_{\ell,x}} \xrightarrow{\times e_{U_x}} \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{\ell,x}e_{U_x} \xrightarrow{\times e_{\ell,x}} \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{\ell,x}e_{U_x}e_{\ell,x}$$

are isomorphisms. Similarly, by Lemma 2.8.9 (ii), the inclusions

$$\mathcal{H}_R(G_x)e_{U_{\ell,x}^\dagger}e_{\bar{U}_{\ell,x}^\dagger}e_{U_{\ell,x}^\dagger}e_{\ell,x} \subseteq \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{U_{\ell,x}^\dagger}e_{\ell,x} \subseteq \mathcal{H}_R(G_x)e_{U_{\ell,x}^\dagger}e_{\ell,x}$$

are equalities. From the last equality, and Proposition 2.5.2(i), we obtain using $e_{U_{\ell,x}^\dagger}e_{U_{\ell,x}} = e_{U_{\ell,x}}$ that

$$\begin{aligned} \mathcal{H}_R(G_x)e_{U_{\ell,x}}e_{\ell,x} &= \mathcal{H}_R(G_x)e_{U_{\ell,x}^\dagger}e_{U_{\ell,x}}e_{\ell,x} = \mathcal{H}_R(G_x)e_{U_{\ell,x}^\dagger}e_{\ell,x}e_{U_{\ell,x}} = \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{U_{\ell,x}^\dagger}e_{\ell,x}e_{U_{\ell,x}} \\ &= \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{\ell,x}e_{U_{\ell,x}}. \end{aligned}$$

Combining all the above observations, we also have

$$\mathcal{H}_R(G_x)e_{\bar{U}_{x,0+}}e_{\ell,x}e_{U_x} = \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{\ell,x}e_{U_x} = \mathcal{H}_R(G_x)e_{\bar{U}_{\ell,x}^\dagger}e_{\ell,x}e_{U_x},$$

and we see that the right multiplication by e_{U_x} induces an isomorphism $\mathcal{H}_R(G_x)e_{U_{\ell,x}}e_{\ell,x} \xrightarrow{\sim} \mathcal{H}_R(G_x)e_{\bar{U}_{x,0+}}e_{\ell,x}e_{U_x}$, which is $G_x \times K_{\ell_M,x}$ -equivariant because $K_{\ell_M,x} \subset M_x$ normalizes U_x . \square

2.8.11 Proof of Theorem 2.8.7. — i) In view of the projective generator given in Theorem 2.7.1, it is sufficient to prove that for each $\iota_M \in I_M$ and $x \in \mathcal{B}_\iota \cap \mathcal{B}_M$ we have

$$i_P \left(\text{ind}_{K_{\iota_M, x}^+}^M \left(\check{\phi}_{\iota_M, x}^+ \right) \right) \in \text{Rep}_R^{\phi, I}(G).$$

This follows from Lemma 2.8.10.

ii) The hypothesis on p is inherited by Levi subgroups of \mathbf{G} so that, by Theorem 2.8.4, we have the two decompositions $\text{Rep}_R(G) = \prod \text{Rep}_R^{\phi, I}(G)$ and $\text{Rep}_R(M) = \prod \text{Rep}_R^{\phi_M, I_M}(M)$. Therefore ii) follows from i) by Frobenius reciprocity.

iii) The equivalence can be checked after adjoining a square root of p to the coefficient field, so we may assume without loss of generality that R is a $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra. Now it suffices to prove the stated claim for the normalized parabolic induction and Jacquet functor as this normalization preserves the categories and equivalence. So for the remainder of the proof we denote by $r_P = r_P^G$ and $i_P = i_P^G$ the normalized Jacquet functor and the normalized parabolic induction, and we write \tilde{r}_P for the composition of r_P with the projection on $\text{Rep}_R^{\phi_M, I_M}(M)$. We will first show that $\tilde{r}_P \circ i_P$ is isomorphic to the identity functor on $\text{Rep}_R^{\phi_M, I_M}(M)$ so that, in particular, i_P is fully faithful on $\text{Rep}_R^{\phi_M, I_M}(M)$.

To this aim, recall that Frobenius reciprocity is given by a natural transformation $r_P \circ i_P \rightarrow \text{Id}$ which is an epimorphism in the category of additive endofunctors of $\text{Rep}_R(M)$ and whose kernel is described by the Mackey formula as follows: there is a filtration indexed by double cosets $P\dot{w}P$ in $G \setminus P$ whose graded pieces are of the form $\mathcal{F}_{\dot{w}} := \text{Ad}_{\dot{w}} \circ i_{P \cap M^w}^M \circ r_{M \cap P^w}^M$. Here, we have chosen representatives \dot{w} in the rational normalizer $N_G(T)$ of a maximally split maximal torus \mathbf{T} of \mathbf{M} and w is the image of \dot{w} in the absolute Weyl group $W_{\mathbf{G}}(\mathbf{T})$. In this situation, $\mathbf{M} \cap \mathbf{P}^w$ is a parabolic F -subgroup of \mathbf{M} with Levi component $\mathbf{M} \cap \mathbf{M}^w$, while $\mathbf{P} \cap \mathbf{M}^w$ is a parabolic subgroup of \mathbf{M}^w with the same Levi component $\mathbf{M} \cap \mathbf{M}^w$. It then follows from the parts i) and ii) that

$$\mathcal{F}_{\dot{w}} \left(\text{Rep}_R^{\phi_M, I_M}(M) \right) \subseteq \prod_{(\phi_w, I_w) \mapsto (\phi_M, I_M)} \text{Ad}_{\dot{w}} \left(\text{Rep}_R^{\phi_w, I_w}(M^w) \right)$$

where the product is over pairs (ϕ_w, I_w) relative to $\mathbf{M} \cap \mathbf{M}^w$ that map to (ϕ_M, I_M) , and whose pushforward to \mathbf{M}^w we still denote by (ϕ_w, I_w) .

Let us draw the dual picture. We may assume that $\hat{\mathbf{M}}$ contains a reference maximal torus $\hat{\mathbf{T}}$ in $\hat{\mathbf{G}}$ (part of a W_F -stable pinning of $\hat{\mathbf{G}}$). We have a duality between \mathbf{T} and $\hat{\mathbf{T}}$ that exchanges roots and coroots. This induces an isomorphism $w \mapsto \hat{w}$, $W_{\mathbf{G}}(\mathbf{T}) \xrightarrow{\sim} W_{\hat{\mathbf{G}}}(\hat{\mathbf{T}})$. Let us choose a lift \hat{w} of w in the normalizer $N_{\hat{\mathbf{G}}}(\hat{\mathbf{T}})$. Then $\hat{\mathbf{M}}^{\hat{w}}$ is a Levi subgroup of $\hat{\mathbf{G}}$ that is dual to \mathbf{M}^w and $\text{Ad}_{\hat{w}^{-1}}$ is a dual isogeny (actually isomorphism) to $\text{Ad}_{\dot{w}}$. Therefore the last inclusion can be rewritten as

$$\mathcal{F}_{\dot{w}} \left(\text{Rep}_R^{\phi_M, I_M}(M) \right) \subseteq \prod_{(\phi_w, I_w) \mapsto (\phi_M, I_M)} \left(\text{Rep}_R^{\text{Ad}_{\hat{w}^{-1}}(\phi_w, I_w)}(M) \right)$$

with the same convention as above. Now let $\phi_w : P_F \rightarrow \hat{\mathbf{M}} \cap \hat{\mathbf{M}}^{\hat{w}}$ be a parameter for $\mathbf{M} \cap \mathbf{M}^w$ whose pushforward to $\hat{\mathbf{M}}$ represents ϕ_M . Assume that $\text{Ad}_{\hat{w}^{-1}}(\phi_w)$ also represents ϕ_M . Then there is some $\hat{m} \in \hat{\mathbf{M}}$ such that $\text{Ad}_{\hat{m}\hat{w}^{-1}}(\phi_w) = \phi_w$, i.e., $\hat{m}\hat{w}^{-1} \in C_{\hat{\mathbf{G}}}(\phi_w)$. By our assumption, this implies that $\hat{w} \in \hat{\mathbf{M}}$, hence $\dot{w} \in \mathbf{M}$, which contradicts the fact that $P\dot{w}P \neq P$. This means

that the projection of $\mathcal{F}_{\dot{w}} \left(\text{Rep}_R^{\phi_M, I_M}(M) \right)$ on $\text{Rep}_R^{\phi_M, I_M}(M)$ is zero, and finally we have proven that the natural transformation $r_P \circ i_P \longrightarrow \text{Id}$ induces an isomorphism $\tilde{r}_P \circ i_P \xrightarrow{\sim} \text{Id}_{\text{Rep}^{\phi_M, I_M}(M)}$.

Now, to conclude that \tilde{r}_P and i_P are quasi-inverse equivalences of categories, it suffices to prove that \tilde{r}_P is conservative on $\text{Rep}_R^{\phi, I}(G)$. So let V be a non-zero object of $\text{Rep}_R^{\phi, I}(G)$. By definition, there is a point $x \in \mathcal{B}_\iota$ such that $e_{\iota, x}V \neq 0$. Recall that, under the hypothesis $C_{\hat{\mathbf{G}}}(\phi) \subseteq \hat{\mathbf{M}}$, we have $\mathbf{G}_\iota \subseteq \mathbf{M}$, so that $\mathcal{B}_\iota \subseteq \mathcal{B}_M$ and x actually lies in \mathcal{B}_{ι_M} . We claim that $e_{U_x}e_{\bar{U}_{x,0^+}}e_{\iota_M, x}V \neq 0$. Indeed, by (i) and (v) of Lemma 2.8.9, we have $e_{\iota, x}e_{U_x}e_{\bar{U}_{x,0^+}}e_{\iota_M, x}e_{\iota, x} = e_{\iota, x}e_{U_x}e_{\iota_M, x}e_{\bar{U}_{x,0^+}}e_{\iota, x} = e_{\iota, x}e_{U_x}e_{\bar{U}_{x,0^+}}e_{\iota, x} = e_{U_{\iota, x}}e_{\bar{U}_{\iota, x}}e_{\iota, x}$, which is also equal to $e_{U_{\iota, x}}e_{\bar{U}_{\iota, x}}e_{\iota, x} = e_{U_{\iota, x}}e_{\bar{U}_{\iota, x}}e_{\iota, x} = e_{U_{\iota, x}}e_{\bar{U}_{\iota, x}}e_{\iota, x}$, by (vi) of the same lemma. So we deduce from Lemma 2.8.9(iii) that $e_{\iota, x} \in RK_{\iota, x}e_{U_x}e_{\bar{U}_{x,0^+}}e_{\iota_M, x}RK_{\iota, x}$ and our claim $e_{U_x}e_{\bar{U}_{x,0^+}}e_{\iota_M, x}V \neq 0$ follows from $e_{\iota, x}V \neq 0$. Using now part (iv) of Lemma 2.8.9, we may apply Prop 3.1 of [Dat09], which tells us that the natural map $V \longrightarrow r_P(V) = V_U$ induces an isomorphism $e_{U_x}e_{\bar{U}_{x,0^+}}e_{\iota_M, x}V \xrightarrow{\sim} e_{\iota_M, x}r_P(V)$. In particular, $e_{\iota_M, x}r_P(V)$ is non-zero, hence the projection of $r_P(V)$ on $\text{Rep}^{\phi_M, I_M}(M)$ is non-zero.

2.8.12 Proposition. — *In the setting of Theorem 2.8.7 iii), the adjoint pair of equivalences $\tilde{r}_P : \text{Rep}_R^{(\phi, I)}(G) \rightleftarrows \text{Rep}_R^{(\phi_M, I_M)}(M) : i_P$ is independent of the choice of \mathbf{P} , up to isomorphism.*

Proof. As in the proof of Theorem 2.8.7iii) we may adjoin a square root of p to our coefficient field and assume that \tilde{r}_P and i_P are normalized. Let \mathbf{Q} be another parabolic F -subgroup of \mathbf{G} with Levi component \mathbf{M} , yielding another adjoint pair of inverse equivalences (\tilde{r}_Q, i_Q) . By the geometric lemma, the composition $r_P \circ i_Q$ has a filtration indexed by double cosets $Q\dot{w}P$ in G with graded pieces of the form $\text{Ad}_{\dot{w}} \circ i_{P \cap M^w} \circ r_{M \cap Q^w}$. In particular, the graded piece associated to the coset QP is the identity endofunctor of $\text{Rep}_R(M)$. As in the proof of Theorem 2.8.7iii), only that graded piece survives when restricting the domain to $\text{Rep}_R^{(\phi_M, I_M)}(M)$ and projecting onto this factor. So we obtain an isomorphism $\tilde{r}_P \circ i_Q \xrightarrow{\sim} \text{Id}_{\text{Rep}_R^{(\phi_M, I_M)}(M)}$. By adjunction, we obtain a morphism $(i_Q)|_{\text{Rep}_R^{(\phi_M, I_M)}(M)} \longrightarrow (i_P)|_{\text{Rep}_R^{(\phi_M, I_M)}(M)}$ whose composition with \tilde{r}_P is an isomorphism. Since \tilde{r}_P is conservative (as we saw in the proof of Theorem 2.8.7iii)), that morphism is an isomorphism too. \square

3 Equivalences of categories and reduction to depth zero

Recall that we assume that p is odd and satisfies hypotheses (H1) and (H2), i.e., p is not a torsion prime for \mathbf{G} nor for $\hat{\mathbf{G}}$. To a pair (ϕ, I) consisting of a wild inertia parameter $\phi : P_F \longrightarrow \hat{\mathbf{G}}$ whose centralizer $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$, and a G -conjugacy class I of rational Levi-center-embeddings $\mathbf{S}_\phi \hookrightarrow \mathbf{G}$, we have associated in Section 2.7 a Serre subcategory $\text{Rep}_R^{(\phi, I)}(G)$ of $\text{Rep}_R(G)$, where R is any commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra. Picking $\iota \in I$, we can also view ϕ as a wild inertia parameter for \mathbf{G}_ι with the same associated torus \mathbf{S}_ϕ , for which the corresponding set of Levi-center-embeddings is just the singleton $\{\iota\}$, whence a Serre subcategory $\text{Rep}_R^\phi(G_\iota)$. The aim of this section is to construct some “natural” equivalences

$$\text{Rep}_R^\phi(G_\iota) \xrightarrow{\sim} \text{Rep}_R^{(\phi, I)}(G).$$

As explained in 2.2.6, $\text{Rep}_R^\phi(G_\iota)$ is a twist of the depth-0 subcategory $\text{Rep}_R^1(G_\iota)$ by any character $\tilde{\varphi} : G_\iota \rightarrow R^\times$ associated to an extension of ϕ to a 1-cocycle $\varphi : W_F \rightarrow Z(\hat{\mathbf{G}}_\iota) = Z(C_{\hat{\mathbf{G}}}(\phi))$. So, these equivalences of categories fall in the general paradigm of “reduction to depth 0” and, actually, what we are going to construct are equivalences of categories

$$\text{Rep}_R^1(G_\iota) \xrightarrow{\sim} \text{Rep}_R^{(\phi, I)}(G).$$

Unless specified otherwise, R denotes a commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R . However, our main results will need R to contain a 4th root of unity and a square root of p .

3.1 Main result

We fix a pair (ϕ, I) and an embedding $\iota \in I$ as above. Recall that \mathcal{B}'_ι denotes the image of the extended Bruhat–Tits building of \mathbf{G}_ι in the reduced Bruhat–Tits building \mathcal{B}' of \mathbf{G} , the group $G_{\iota, x}$ is the stabilizer of a point $x \in \mathcal{B}'_\iota$ in G_ι and we define the group $K_{\iota, x}$ to be $G_{\iota, x}K_{\iota, x}^\dagger$, where $K_{\iota, x}^\dagger := K_{\iota, \hat{x}}^\dagger$ denotes the group defined in 2.5.1 for \hat{x} any preimage of x in \mathcal{B}_ι . Similarly we write $K_{\iota, x}^+ := K_{\iota, \hat{x}}^+$ for \hat{x} any preimage of x in \mathcal{B}_ι .

3.1.1 Theorem. — *If R is a $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra, there exists an equivalence of categories*

$$\mathcal{I} : \text{Rep}_R^1(G_\iota) \xrightarrow{\sim} \text{Rep}_R^{(\phi, I)}(G)$$

and an explicitly constructed family $\{\mathcal{W}_{\iota, x}\}_{x \in \mathcal{B}'_\iota}$ of $RK_{\iota, x}$ -modules $\mathcal{W}_{\iota, x}$ such that, for any $x \in \mathcal{B}'_\iota$ and any R -representation ρ of $G_{\iota, x}$ that is trivial on $G_{\iota, x, 0+}$, there is an isomorphism of RG -modules

$$\mathcal{I}(\text{ind}_{G_{\iota, x}}^{G_\iota}(\rho)) \simeq \text{ind}_{K_{\iota, x}}^G(\mathcal{W}_{\iota, x} \otimes_R \rho)$$

through which \mathcal{I} induces an isomorphism of R -algebras

$$\text{End}_{RG_\iota}(\text{ind}_{G_{\iota, x}}^{G_\iota}(\rho)) \xrightarrow{\sim} \text{End}_{RG}(\text{ind}_{K_{\iota, x}}^G(\mathcal{W}_{\iota, x} \otimes_R \rho)).$$

The proof of this theorem will be the content of most of the remainder of this paper. We refer to Theorem 3.4.11 and Corollary 3.4.14 for more details on the $RK_{\iota, x}$ -modules $\mathcal{W}_{\iota, x}$, which require more preparation before being properly defined. Suffice it to say, they will be constructed via a suitable theory of Heisenberg–Weil representations over R that we introduce in 3.3.1. That description will allow us to deduce an isomorphism between positive-depth and depth-zero Hecke algebras attached to types, see 4.1, in particular Corollary 4.1.2, for details.

3.2 Localization on the buildings

In the remainder of the paper, the integer e is always supposed to satisfy the conditions of 2.7.4, and be such that barycenters of 1-facets are e -vertices (this ensures that stabilizers of e -facets in G are also fixators of these e -facets).

3.2.1 G -equivariant coefficient systems on \mathcal{B}' . — We define a category $[\mathcal{B}'_{\bullet/e}/G]$ as follows:

- Its set of objects is the set $\mathcal{B}'_{\bullet/e}$ of e -facets of the reduced building of G .
- For $\mathcal{F}, \mathcal{F}' \in \mathcal{B}'_{\bullet/e}$, we set $\text{Hom}(\mathcal{F}, \mathcal{F}') := \{g \in G, \overline{g\mathcal{F}} \supseteq \mathcal{F}'\}$, with composition given by multiplication in G .

A G -equivariant coefficient system of R -modules is a functor $\mathcal{V} : [\mathcal{B}'_{\bullet/e}/G] \rightarrow R\text{-Mod}$.

Concretely, it is given by a collection of R -modules $(\mathcal{V}_{\mathcal{F}})_{\mathcal{F} \in \mathcal{B}'_{\bullet/e}}$ with face maps $\mathcal{V}_{\mathcal{F}} \xrightarrow{\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'}} \mathcal{V}_{\mathcal{F}'}$ for each pair of facets such that $\overline{\mathcal{F}} \supseteq \mathcal{F}'$ (corresponding to $g = 1 \in \text{Hom}(\mathcal{F}, \mathcal{F}')$), and isomorphisms $g_{\mathcal{V}, \mathcal{F}} : \mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} \mathcal{V}_{g\mathcal{F}}$ subject to appropriate transitivity and compatibility relations. We might also simply write $\beta_{\mathcal{F}, \mathcal{F}'}$ for $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'}$ and $g_{\mathcal{F}}$ for $g_{\mathcal{V}, \mathcal{F}}$ if \mathcal{V} is clear from the context. In particular, each $\mathcal{V}_{\mathcal{F}}$ carries an R -linear action of the stabilizer $G_{\mathcal{F}}$ of \mathcal{F} in G . We say that the G -equivariant coefficient system is *smooth* if these actions are smooth. Smooth G -equivariant coefficient systems form an abelian category $\text{Coef}_R(\mathcal{B}'_{\bullet/e}/G)$ where the morphisms are given by natural transformations of functors.

3.2.2 Coefficient systems and representations. — There is a projection functor $[\mathcal{B}'_{\bullet/e}/G] \xrightarrow{\pi} [* / G]$ to the category $[* / G]$ with 1 object $*$ with set of endomorphisms G . Note that a representation V of G on an R -module is given by a functor $[* / G] \rightarrow R\text{-Mod}$. Composition with π then provides an exact functor

$$\pi^* : \text{Rep}_R(G) \rightarrow \text{Coef}_R(\mathcal{B}'_{\bullet/e}/G),$$

which takes V to the constant coefficient system $\mathcal{F} \mapsto V$ with G -equivariant structure given by the action of G on V . The functor π^* has a left adjoint that we denote by $\pi_!$. Explicitly we have

$$\begin{aligned} \pi_! : \text{Coef}_R(\mathcal{B}'_{\bullet/e}/G) &\rightarrow \text{Rep}_R(G) \\ \mathcal{V} &\mapsto \text{colim}_{\mathcal{B}'_{\bullet/e}} \mathcal{V} \end{aligned}$$

where the colimit is taken over the functor \mathcal{V} restricted to the category associated with the poset $\mathcal{B}'_{\bullet/e}$, and the action of G arises from its action on that poset and the G -equivariant structure on \mathcal{V} .

3.2.3 (ϕ, I) -coefficient systems. — Recall from Proposition 2.6.8 iii) that for $x \in \mathcal{B}$, the idempotent $e_{\phi, I, x}$ only depends on the e -facet $\mathcal{F} \in \mathcal{B}'$ that contains the image of x in \mathcal{B}' , so we may denote this idempotent by $e_{\phi, I, \mathcal{F}}$. We define:

$$\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G) := \begin{array}{l} \text{the full subcategory of } \text{Coef}_R(\mathcal{B}'_{\bullet/e}/G) \text{ that consists of all} \\ \text{smooth } G\text{-equivariant coefficient systems } \mathcal{V} \text{ such that, for} \\ \text{any facet } \mathcal{F}, \text{ we have } \mathcal{V}_{\mathcal{F}} = e_{\phi, I, \mathcal{F}} \mathcal{V}_{\mathcal{F}}. \end{array}$$

It follows from the definition of $\pi_!$ that

$$\pi_!(\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)) \subset \text{Rep}_R^{\phi, I}(G).$$

By Proposition 2.6.9, we have equalities $e_{\phi, I, \mathcal{F}} e_{\phi, I, \mathcal{F}'} = e_{\phi, I, \mathcal{F}}$ whenever $\overline{\mathcal{F}} \supseteq \mathcal{F}'$. Therefore, for any smooth G -equivariant coefficient system \mathcal{V} on $\mathcal{B}'_{\bullet/e}$, we have $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'}(e_{\phi, I, \mathcal{F}} \mathcal{V}_{\mathcal{F}}) \subseteq e_{\phi, I, \mathcal{F}'} \mathcal{V}_{\mathcal{F}'}$.

On the other hand, the fact that $e_{\phi,I,g\mathcal{F}} = ge_{\phi,I,\mathcal{F}}g^{-1}$ implies that $g_{\mathcal{V},\mathcal{F}}(e_{\phi,I,\mathcal{F}}\mathcal{V}_{\mathcal{F}}) = e_{\phi,I,g\mathcal{F}}\mathcal{V}_{g\mathcal{F}}$. We thus get a functor

$$\begin{aligned} (-)^{\phi,I} : \text{Coef}_R(\mathcal{B}'_{\bullet/e}/G) &\rightarrow \text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G) \\ \mathcal{V} &\mapsto \mathcal{V}^{\phi,I} : \mathcal{F} \mapsto e_{\phi,I,\mathcal{F}}\mathcal{V}_{\mathcal{F}} \end{aligned}$$

which is right adjoint to the inclusion functor $\text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G) \subset \text{Coef}_R(\mathcal{B}'_{\bullet/e}/G)$. Composing with π^* , we get a functor

$$(\pi^*)^{\phi,I} : \text{Rep}_R(G) \longrightarrow \text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)$$

that is right adjoint to $\pi_!$ restricted to $\text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)$. Explicitly, for $V \in \text{Rep}_R(G)$, the coefficient system $(\pi^*)^{\phi,I}(V)$ is given by the data

- $\forall \mathcal{F} \in \mathcal{B}'_{\bullet/e}$, $(\pi^*)^{\phi,I}(V)_{\mathcal{F}} = e_{\phi,I,\mathcal{F}}V$
- $\forall \mathcal{F}, \mathcal{F}'$ with $\overline{\mathcal{F}} \supseteq \mathcal{F}'$, the map $\beta_{\mathcal{F},\mathcal{F}'}$ is the inclusion $e_{\phi,I,\mathcal{F}}V = e_{\phi,I,\mathcal{F}}e_{\phi,I,\mathcal{F}'}V \subseteq e_{\phi,I,\mathcal{F}'}V$
- $\forall \mathcal{F} \in \mathcal{B}'_{\bullet/e}, \forall g \in G$, the map $g_{\mathcal{F}} : e_{\phi,I,\mathcal{F}}V \xrightarrow{g} e_{\phi,I,g\mathcal{F}}V = ge_{\phi,I,\mathcal{F}}g^{-1}V$ is given by the action of g .

We observe that $(\pi^*)^{\phi,I}$ factors through the following full subcategory:

$$\text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}} := \begin{array}{l} \text{the full subcategory of } \text{Coef}_R(\mathcal{B}'_{\bullet/e}/G) \text{ that consists of} \\ \text{all smooth } G\text{-equivariant coefficient systems } \mathcal{V} \text{ such that,} \\ \text{for any pair of facets } (\mathcal{F}, \mathcal{F}') \text{ satisfying } \overline{\mathcal{F}} \supseteq \mathcal{F}', \text{ the map} \\ \beta_{\mathcal{V},\mathcal{F},\mathcal{F}'} \text{ induces an isomorphism } \mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} e_{\phi,I,\mathcal{F}}\mathcal{V}_{\mathcal{F}'} \end{array}$$

We refer to coefficient systems in $\text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}}$ as *Cartesian (ϕ, I) -coefficient systems*.

Proposition. — *The above functors induce quasi-inverse equivalences of categories*

$$(3.2.4) \quad \pi_! : \text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}} \xrightleftharpoons{\sim} \text{Rep}_R^{\phi,I}(G) : (\pi^*)^{\phi,I}$$

Proof. This is Theorem (4.11) of [Lan21] applied to the system of idempotents $(e_{\phi,I,x})_{x \in \mathcal{B}'_{0/e}}$, except that our facet decomposition is the e -subdivision of the one used in *loc. cit.*, and in particular our vertices are not necessarily vertices in the context of *loc. cit.* Beyond the original techniques of [MS10] that we have already adapted to our context in the proof of Lemma 2.7.7, the main point of the argument of Lanard in [Lan21] is the construction of “local maps” in §3 there. This relies in turn on the notion of “admissible path” between vertices (§2 of *loc. cit.*), which only depends on the geometry of the facet decomposition of an apartment. But our facet decomposition is homothetic to that used in [Lan21], therefore the notion of admissible path works the same, and the argument goes through in our setting. \square

3.2.5 G_ι -equivariant coefficient systems on \mathcal{B}'_ι . — Fix $\iota \in I$. Recall from 2.7.4 that, due to our choice of integer e , the image \mathcal{B}'_ι of \mathcal{B}_ι in \mathcal{B}' is stable under the e -facet decomposition of \mathcal{B}' , in the sense that an e -facet of \mathcal{B}' is either contained in, or disjoint from, \mathcal{B}'_ι . We denote by $\mathcal{B}'_{\iota, \bullet/e}$ the set of e -facets of \mathcal{B}' contained in \mathcal{B}'_ι .⁷ Since the action of G_ι on \mathcal{B}'_ι preserves the e -facet decomposition, we may define a category $[\mathcal{B}'_{\iota, \bullet/e}/G_\iota]$ and an abelian R -linear category $\text{Coef}_R(\mathcal{B}'_{\iota, \bullet/e}/G_\iota)$ of smooth G_ι -equivariant coefficient systems on $\mathcal{B}'_{\iota, \bullet/e}$ as in 3.2.1. We then have a projection functor $[\mathcal{B}'_{\iota, \bullet/e}/G_\iota] \xrightarrow{\pi_\iota} [*/G_\iota]$ that induces a pair of adjoint functors

$$(\pi_{\iota,!}, \pi_\iota^*) : \text{Coef}_R(\mathcal{B}'_{\iota, \bullet/e}/G_\iota) \rightleftarrows \text{Rep}_R(G_\iota),$$

as in 3.2.2. Moreover, since $G_{\iota, x, 0+}$ only depends on the e -facet $\mathcal{F}_e(x)$ containing the point x , we may also consider the category of “depth 0” coefficient systems defined as

$$\begin{aligned} \text{Coef}_R^1(\mathcal{B}'_{\iota, \bullet/e}/G_\iota) &:= \text{the full subcategory of } \text{Coef}_R(\mathcal{B}'_{\iota, \bullet/e}/G_\iota) \text{ that consists of} \\ &\text{all smooth } G_\iota\text{-equivariant coefficient systems } \mathcal{V} \text{ such that,} \\ &\text{for any facet } \mathcal{F}, \text{ we have } \mathcal{V}_{\mathcal{F}} = e_{G_{\iota, \mathcal{F}, 0+}} \mathcal{V}_{\mathcal{F}}, \end{aligned}$$

where $e_{G_{\iota, \mathcal{F}, 0+}}$ is the idempotent averaging over $G_{\iota, \mathcal{F}, 0+}$ and thereby projecting onto the $G_{\iota, \mathcal{F}, 0+}$ -fixed vectors. We observe that $\pi_{\iota,!}(\text{Coef}_R^1(\mathcal{B}'_{\iota, \bullet/e}/G_\iota)) \subset \text{Rep}_R^1(G_\iota)$.

As in 3.2.3, the equalities $e_{\iota, \mathcal{F}, 0+} e_{\iota, \mathcal{F}', 0+} = e_{\iota, \mathcal{F}, 0+}$ whenever $\overline{\mathcal{F}} \supseteq \mathcal{F}'$ and $e_{\iota, g\mathcal{F}, 0+} = g e_{\iota, \mathcal{F}, 0+} g^{-1}$ whenever $g \in G_\iota$ allow us to construct a right adjoint $\mathcal{V} \mapsto \mathcal{V}^1$ to the inclusion of $\text{Coef}_R^1(\mathcal{B}'_{\iota, \bullet/e}/G_\iota)$ into $\text{Coef}_R(\mathcal{B}'_{\iota, \bullet/e}/G_\iota)$, by setting $\mathcal{V}_{\mathcal{F}}^1 := e_{\iota, \mathcal{F}, 0+} \mathcal{V}_{\mathcal{F}}$ for all $\mathcal{F} \in \mathcal{B}'_{\iota, \bullet/e}$. Precomposing with π_ι^* then provides us with a pair of adjoint functors

$$(\pi_{\iota,!}, (\pi_\iota^*)^1) : \text{Coef}_R^1(\mathcal{B}'_{\iota, \bullet/e}/G_\iota) \rightleftarrows \text{Rep}_R^1(G_\iota).$$

As in 3.2.3, $(\pi_\iota^*)^1$ factors through the following full subcategory

$$\begin{aligned} \text{Coef}_R^1(\mathcal{B}'_{\iota, \bullet/e}/G_\iota)^{\text{cart}} &:= \text{the full subcategory of } \text{Coef}_R(\mathcal{B}'_{\iota, \bullet/e}/G_\iota) \text{ that consists of} \\ &\text{all smooth } G_\iota\text{-equivariant coefficient systems } \mathcal{V} \text{ such that,} \\ &\text{for any pair of facets } (\mathcal{F}, \mathcal{F}') \text{ satisfying } \overline{\mathcal{F}} \supseteq \mathcal{F}', \text{ the map} \\ &\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'} \text{ induces an isomorphism } \mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} e_{G_{\iota, \mathcal{F}, 0+}} \mathcal{V}_{\mathcal{F}'}, \end{aligned}$$

to whose objects we refer to as *Cartesian depth-0* coefficient systems.

Proposition. — *The two above functors induce quasi-inverse equivalences of categories*

$$(3.2.6) \quad \pi_{\iota,!} : \text{Coef}_R^1(\mathcal{B}'_{\iota, \bullet/e}/G_\iota)^{\text{cart}} \xrightleftharpoons{\sim} \text{Rep}_R^1(G_\iota) : (\pi_\iota^*)^1$$

Proof. This again follows from [Lan21] with the same adaptation as in the proof of Proposition 3.2.2, namely one needs to define “local maps” using “admissible paths”. But for two e -facets $\mathcal{F}, \mathcal{F}'$ contained in \mathcal{B}'_ι , the combinatorial convex hull (for the e -facet decomposition) $\mathcal{H}_e(\mathcal{F}, \mathcal{F}')$ is contained in \mathcal{B}'_ι and, therefore, any admissible path from \mathcal{F} to \mathcal{F}' is contained in \mathcal{B}'_ι . \square

In view of the equivalences (3.2.4) and (3.2.6), our aim will now be to construct a pair of adjoint equivalences of categories:

$$(3.2.7) \quad \text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}} \xrightleftharpoons{\sim} \text{Coef}_R^1(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}},$$

which we discuss in the next subsection.

⁷Note that there is also an intrinsic notion of e -facet for \mathcal{B}'_ι , which is a priori coarser, and that we *do not* consider here. So, by “ e -facet” we always mean an e -facet for \mathcal{B}' .

3.3 Heisenberg–Weil coefficient systems

In order to construct an equivalence as in (3.2.7) we will use an auxiliary coefficient system that we call a *Heisenberg–Weil coefficient system* and introduce below. To define such a coefficient system, we will rely on some results that we will prove in Section 3.5. We will also state some related results in this subsection that we will prove in Sections 3.6 and 3.7. In Section 3.4 we then prove Theorem 3.1.1 assuming the results we stated in this subsection, Section 3.3. Since Sections 3.5, 3.6 and 3.7 do not rely on results from Sections 3.1, 3.2, 3.3 and 3.4, there is no circular reasoning. Presenting the results in this order allows readers interested in the results rather than the technical details of the proof to faster reach their goal and at the same time provides readers interested in all the details with a motivation for Sections 3.5, 3.6 and 3.7.

3.3.1 Heisenberg and Weil representations. — We will need the notion of Heisenberg representation, which we have to adapt to our setting of a general commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R . According to Proposition (2.6.8) ii), for $x \in \mathcal{B}_l$, the groups $K_{l,x}^\dagger$ and $K_{l,x}^+$ and the character $\check{\phi}_{l,x}^+$ depend only on the e -facet \mathcal{F} containing x . We may thus denote them by $K_{l,\mathcal{F}}^\dagger$, $K_{l,\mathcal{F}}^+$ and $\check{\phi}_{l,\mathcal{F}}^+$.

Definition.— For $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$, a Heisenberg representation $\eta_{l,\mathcal{F}}$ for the triple $(K_{l,\mathcal{F}}^\dagger, K_{l,\mathcal{F}}^+, \check{\phi}_{l,\mathcal{F}}^+)$ is a finitely generated, projective $RK_{l,\mathcal{F}}^\dagger$ -module that is $\check{\phi}_{l,\mathcal{F}}^+$ -isotypic when restricted to $RK_{l,\mathcal{F}}^+$ and that has $RK_{l,\mathcal{F}}^\dagger$ -endomorphism ring R .

We refer to Lemma 3.5.2 for a proof of existence, a discussion of uniqueness, and a proof of the main property of interest to us, which is that such a representation is a projective generator of the category $\mathcal{H}_R(K_{l,\mathcal{F}}^\dagger)_{e_{l,\mathcal{F}}} - \text{Mod}$ of all $\check{\phi}_{l,\mathcal{F}}^+$ -isotypic smooth $RK_{l,\mathcal{F}}^\dagger$ -modules. In particular, we have a pair of adjoint equivalences of categories:

$$(3.3.2) \quad \text{Hom}_{RK_{l,\mathcal{F}}^\dagger}(\eta_{l,\mathcal{F}}, -) : \mathcal{H}_R(K_{l,\mathcal{F}}^\dagger)_{e_{l,\mathcal{F}}} - \text{Mod} \rightleftarrows R - \text{Mod} : \eta_{l,\mathcal{F}} \otimes_R -.$$

Also, as in the case where the coefficient ring is \mathbb{C} , we will show in Section 3.6 how the theory of Weil representations produces extensions of Heisenberg representations to the bigger group

$$K_{l,\mathcal{F}} := G_{l,\mathcal{F}} K_{l,\mathcal{F}}^\dagger,$$

where $G_{l,\mathcal{F}}$ is the stabilizer of \mathcal{F} in G_l . We call such representations “Heisenberg–Weil representations”. Any Heisenberg–Weil representation $\kappa_{l,\mathcal{F}}$ provides an enhancement of the above pair of adjoint equivalences to the following pair of adjoint equivalences:

$$(3.3.3) \quad \text{Hom}_{RK_{l,\mathcal{F}}^\dagger}(\kappa_{l,\mathcal{F}}, -) : \mathcal{H}_R(K_{l,\mathcal{F}})_{e_{l,\mathcal{F}}} - \text{Mod} \rightleftarrows R[G_{l,\mathcal{F}}/G_{l,\mathcal{F},0+}] - \text{Mod} : \kappa_{l,\mathcal{F}} \otimes_R -.$$

Here, the category on the right hand side is justified by the isomorphism $G_{l,\mathcal{F}}/G_{l,\mathcal{F},0+} \xrightarrow{\sim} K_{l,\mathcal{F}}/K_{l,\mathcal{F}}^\dagger$ arising from the inclusion $G_{l,\mathcal{F}} \subseteq K_{l,\mathcal{F}}$.

So, a natural strategy to construct equivalences as in (3.2.7) is to try and arrange the “local equivalences” (3.3.3) in a sufficiently coherent way so that they respect face maps and actions. To this aim, we introduce the notion of a Heisenberg–Weil coefficient system below. This will be a coefficient system on a category that will serve as an intermediate between $[\mathcal{B}'_{\bullet/e}/G]$

and $[\mathcal{B}'_{l,\bullet/e}/G_l]$. At this point, it might be helpful to note that when $\overline{\mathcal{F}} \supseteq \mathcal{F}'$, we have the following inclusions: $K_{l,\mathcal{F}'}^+ \subseteq K_{l,\mathcal{F}}^+ \subseteq K_{l,\mathcal{F}} \subseteq K_{l,\mathcal{F}'}$ (the last one because e was chosen so that the stabilizers of e -facets coincide with their fixators). On the other hand, $K_{l,\mathcal{F}}^\dagger$ and $K_{l,\mathcal{F}'}^\dagger$ are not contained in one another, but we have $K_{l,\mathcal{F}}^\dagger = (K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger)G_{l,\mathcal{F},0^+}$. These facts allow us to make the following definition:

3.3.4 Definition.— We denote by $[\mathcal{B}'_{l,\bullet/e}/K_l]$ the following subcategory of $[\mathcal{B}'_{\bullet/e}/G]$:

- Its set of objects is the subset $\mathcal{B}'_{l,\bullet/e}$ of e -facets of \mathcal{B}' contained in \mathcal{B}'_l .
- For $\mathcal{F}, \mathcal{F}' \in \mathcal{B}'_{l,\bullet/e}$, morphisms are given by

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') := \{g \in K_{l,\mathcal{F}'} G_l, \overline{g\mathcal{F}} \supseteq \mathcal{F}'\} = \{g \in K_{l,\mathcal{F}}^\dagger G_l, \overline{g\mathcal{F}} \supseteq \mathcal{F}'\},$$

with composition given by multiplication in G .

As usual, a coefficient system \mathcal{W} on $[\mathcal{B}'_{l,\bullet/e}/K_l]$ is just a functor to $R - \mathrm{Mod}$. Note that this is equivalent to giving a G_l -equivariant coefficient system on $\mathcal{B}'_{l,\bullet/e}$ together with for every $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$ a smooth action of $K_{l,\mathcal{F}}^\dagger$ on $\mathcal{W}_{l,\mathcal{F}}$ such that, for each pair $\mathcal{F}, \mathcal{F}'$ such that $\overline{\mathcal{F}} \supseteq \mathcal{F}'$, the face map $\beta_{\mathcal{W},\mathcal{F},\mathcal{F}'}$ is $K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger$ -equivariant and, for each $g \in G_l$, the action map $g_{\mathcal{W},\mathcal{F}}$ induces a $K_{l,\mathcal{F}}^\dagger$ -equivariant isomorphism $g_{\mathcal{W},\mathcal{F}} : \mathcal{W}_{\mathcal{F}} \xrightarrow{\sim} g^* \mathcal{W}_{g\mathcal{F}}$ that coincides with the action of g on $\mathcal{W}_{\mathcal{F}}$ whenever $g \in G_{l,\mathcal{F}}$. Note that since $\beta_{\mathcal{W},\mathcal{F},\mathcal{F}'}$ is also $G_{l,\mathcal{F}} = (G_{l,\mathcal{F}} \cap G_{l,\mathcal{F}'})$ -equivariant, as part of the definition of a G_l -equivariant coefficient system on $\mathcal{B}'_{l,\bullet/e}$, it follows that $\beta_{\mathcal{W},\mathcal{F},\mathcal{F}'}$ is also $K_{l,\mathcal{F}}$ -equivariant.

3.3.5 Definition.— A coefficient system $\mathcal{W}_l : [\mathcal{B}'_{l,\bullet/e}/K_l] \rightarrow R - \mathrm{Mod}$ is called a *Heisenberg–Weil* coefficient system if :

1. For each $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$, the $RK_{l,\mathcal{F}}^\dagger$ -module $\mathcal{W}_{l,\mathcal{F}}$ is a Heisenberg representation for the triple $(K_{l,\mathcal{F}}^\dagger, K_{l,\mathcal{F}}^+, \check{\phi}_{l,\mathcal{F}}^+)$.
2. For each pair of facets $\mathcal{F}' \subseteq \overline{\mathcal{F}}$ in $\mathcal{B}'_{l,\bullet/e}$, the map $\beta_{\mathcal{W}_l,\mathcal{F},\mathcal{F}'}$ is a generator of the R -module $\mathrm{Hom}_{R(K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger)}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{W}_{l,\mathcal{F}'})$.

3.3.6 Proposition. — *Let \mathcal{W}_l be a Heisenberg–Weil coefficient system. Then the R -module $\mathrm{Hom}_{R(K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger)}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{W}_{l,\mathcal{F}'})$ is free of rank one, and $\beta_{\mathcal{W}_l,\mathcal{F},\mathcal{F}'}$ induces*

- i) a $K_{l,\mathcal{F}'}^\dagger$ -equivariant isomorphism $\mathrm{ind}_{K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger}^{K_{l,\mathcal{F}'}^\dagger} \mathcal{W}_{l,\mathcal{F}} \xrightarrow{\sim} \mathcal{W}_{l,\mathcal{F}'}$, and
- ii) a $K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger$ -equivariant isomorphism $\mathcal{W}_{l,\mathcal{F}} \xrightarrow{\sim} e'_{l,\mathcal{F}} \mathcal{W}_{l,\mathcal{F}'}$, where $e'_{l,\mathcal{F}}$ is the idempotent associated to the restriction of $\check{\phi}_{l,\mathcal{F}}$ to $K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger$.

Since $\beta_{\mathcal{W}_l,\mathcal{F},\mathcal{F}'}$ is also compatible with the action of G_l , the isomorphism in (ii) is actually $K_{l,\mathcal{F}}$ -equivariant and $e'_{l,\mathcal{F}} \mathcal{W}_{l,\mathcal{F}'} = e_{l,\mathcal{F}} \mathcal{W}_{l,\mathcal{F}'}$.

Proof. We show in Lemma 3.5.5 i) that the R -module $\mathrm{Hom}_{R(K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'})}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{W}_{l,\mathcal{F}'})$ is in general invertible, granted Property 1 of Definition 3.3.5. In the presence of a generator, it is thus free of rank 1. We show in Lemma 3.5.5 iii) that then a generator like $\beta_{\mathcal{W}_{l,\mathcal{F}},\mathcal{F}'}$ induces the isomorphisms in i) and ii). That $\beta_{\mathcal{W}_{l,\mathcal{F}},\mathcal{F}'}$ is $K_{l,\mathcal{F}}$ -equivariant was already observed above, and hence $e'_{l,\mathcal{F}}\mathcal{W}_{l,\mathcal{F}'} = e_{l,\mathcal{F}}\mathcal{W}_{l,\mathcal{F}'}$. \square

3.3.7 Theorem. — *Let R be a commutative $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra. Then there exists a Heisenberg–Weil coefficient system \mathcal{W}_l in $\mathrm{Coef}_R(\mathcal{B}'_{l,\bullet/e}/K_l)$.*

We postpone the proof of this theorem to Section 3.7, since it requires a long technical preparation done in Section 3.6. This theorem will be the main ingredient in the construction of the equivalence (3.2.7). We will also show the following uniqueness result in 3.7.12.

3.3.8 Proposition. — *If \mathcal{W}_l and \mathcal{W}'_l are two Heisenberg–Weil coefficient systems, then there is an invertible R -module L and a depth-0 character $\theta : G_l \rightarrow R^\times$ such that $\mathcal{W}'_l \simeq \mathcal{W}_l \otimes_R L_\theta$.*

3.4 Construction of the equivalences

In this section, we associate to any Heisenberg–Weil coefficient system a pair of equivalences as in (3.2.7) satisfying the extra properties of Theorem 3.1.1. More precisely, we prove the more precise version of Theorem, 3.1.1, which is Theorem 3.4.11 below, contingent on the proofs of the results stated in Section 3.3, which in turn are provided in Sections 3.5, 3.6 and 3.7 (which are independent of the present section, so there is no circular reasoning).

3.4.1 Overview. — To a Heisenberg–Weil coefficient system \mathcal{W}_l as in Definition 3.3.5, we will associate a pair of quasi-inverse equivalences of categories

$$\mathcal{I}_{\mathcal{W}_l} : \mathrm{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l) \xrightleftharpoons[\sim]{\sim} \mathrm{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G) : \mathcal{R}_{\mathcal{W}_l}$$

that respect cartesian objects. Since the construction of these functors, given in 3.4.2 and 3.4.3, might appear quite technical at first sight, we try to offer here a geometric intuition. This section is logically independent from the sequel and can be skipped.

Recall that \mathcal{W}_l is a coefficient system on the category $[\mathcal{B}'_{l,\bullet/e}/K_l]$ introduced above Definition 3.3.5. We have functors

$$[\mathcal{B}'_{l,\bullet/e}/\overline{G}_l] \xleftarrow{p} [\mathcal{B}'_{l,\bullet/e}/K_l] \xrightarrow{q} [\mathcal{B}'_{\bullet/e}/G].$$

Here q is the natural inclusion. On the other side, $[\mathcal{B}'_{l,\bullet/e}/\overline{G}_l]$ denotes the category with set of objects $\mathcal{B}'_{l,\bullet/e}$ and morphisms given by $\mathrm{Hom}(\mathcal{F}, \mathcal{F}') = \{g \in G_l, g\overline{\mathcal{F}} \supseteq \mathcal{F}'\}/G_{l,\mathcal{F},0+}$. The functor p is the identity on objects and is given on morphisms by the (well defined) maps

$$g = kg_l \in K_{l,\mathcal{F}'}^\dagger G_l (\text{such that } g\overline{\mathcal{F}} \supseteq \mathcal{F}') \mapsto \overline{g}_l \in G_l/G_{l,\mathcal{F},0+}.$$

Note that $\mathrm{Coef}_R([\mathcal{B}'_{l,\bullet/e}/\overline{G}_l])$ identifies to $\mathrm{Coef}_R^1([\mathcal{B}'_{l,\bullet/e}/G_l])$ via the pullback r^* along the projection functor $[\mathcal{B}'_{l,\bullet/e}/G_l] \xrightarrow{r} [\mathcal{B}'_{l,\bullet/e}/\overline{G}_l]$.

Associated with p and q are two pull-back functors (given by precomposition)

$$\mathrm{Coef}_R(\mathcal{B}'_{l,\bullet/e}/\overline{G}_l) \xrightarrow{p^*} \mathrm{Coef}_R(\mathcal{B}'_{l,\bullet/e}/K_l) \xleftarrow{q^*} \mathrm{Coef}_R(\mathcal{B}'_{\bullet/e}/G).$$

The functor p^* has a right adjoint p_* given by $p_*(\mathcal{W})_{\mathcal{F}} := \{w \in \mathcal{W}_{\mathcal{F}}, \forall \mathcal{F}' \subseteq \overline{\mathcal{F}}, \beta_{\mathcal{W},\mathcal{F},\mathcal{F}'}(w) \in (\mathcal{W}_{\mathcal{F}'})^{K_{l,\mathcal{F}'}}\}$ for $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$. The functor q^* has a left adjoint $q_!$, a description of which is done in 3.4.3. Now, on $\mathrm{Coef}_R(\mathcal{B}'_{l,\bullet/e}/K_l)$ we have tensor products, given on facets by $(\mathcal{W} \otimes_R \mathcal{W}')_{\mathcal{F}} = \mathcal{W}_{\mathcal{F}} \otimes_R \mathcal{W}'_{\mathcal{F}}$ with obvious face maps and action maps. We also have internal homs defined on facets by $\mathcal{H}om(\mathcal{W}, \mathcal{W}')_{\mathcal{F}} := \mathrm{Hom}(\mathcal{W}|_{\overline{\mathcal{F}}}, \mathcal{W}'|_{\overline{\mathcal{F}}})$, where the Hom on the right hand side denotes the R -module of morphisms between coefficient systems \mathcal{W} and \mathcal{W}' restricted to $\overline{\mathcal{F}}$. As in sheaf theory, these constructions satisfy the usual Hom -tensor adjunctions.

It now becomes natural to consider the adjoint pair of pull-push functors with kernel \mathcal{W}_l :

$$\mathcal{I}_{\mathcal{W}_l} : \mathcal{V}_l \mapsto q_!(\mathcal{W}_l \otimes_R p^* \mathcal{V}_l) \text{ and } \mathcal{R}_{\mathcal{W}_l} : \mathcal{V} \mapsto p_*(\mathcal{H}om_R(\mathcal{W}_l, q^* \mathcal{V})).$$

In the next three sections we provide more explicit definitions of these functors and a proof that they induce the desired equivalences between our categories of interest. However, we leave it to the reader to check that these explicit definitions indeed describe the above functors.

3.4.2 *A functor from $\mathrm{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)$ to $\mathrm{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l)$.* — This is the easier direction, since we start from a coefficient system defined on the target building. As above, we fix a Heisenberg–Weil coefficient system \mathcal{W}_l as in Definition 3.3.5, and let \mathcal{V} be a coefficient system in $\mathrm{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)$. We define a coefficient system \mathcal{V}_l on \mathcal{B}'_l as follows.

- $\forall \mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$, set $\mathcal{V}_{l,\mathcal{F}} := \mathrm{Hom}_{RK_{l,\mathcal{F}}^\dagger}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$.
- If $\overline{\mathcal{F}} \supseteq \mathcal{F}'$, define $\beta_{\mathcal{V}_l,\mathcal{F},\mathcal{F}'}$ as the unique map that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{V}_{l,\mathcal{F}} = \mathrm{Hom}_{RK_{l,\mathcal{F}}^\dagger}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{V}_{\mathcal{F}}) & \xrightarrow{\beta_{\mathcal{V}_l,\mathcal{F},\mathcal{F}'}} & \mathcal{V}_{l,\mathcal{F}'} = \mathrm{Hom}_{RK_{l,\mathcal{F}'}^\dagger}(\mathcal{W}_{l,\mathcal{F}'}, \mathcal{V}_{\mathcal{F}'}) \\ \downarrow u \mapsto \beta_{\mathcal{V}_l,\mathcal{F},\mathcal{F}'} \circ u & & \downarrow u \mapsto u \circ \beta_{\mathcal{W}_l,\mathcal{F},\mathcal{F}'} \\ \mathrm{Hom}_{R(K_{l,\mathcal{F}}^\dagger \cap K_{l,\mathcal{F}'}^\dagger)}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{V}_{\mathcal{F}'}) & \xlongequal{\quad} & \mathrm{Hom}_{R(K_{l,\mathcal{F}'}^\dagger \cap K_{l,\mathcal{F}}^\dagger)}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{V}_{\mathcal{F}'}) \end{array}$$

Existence and uniqueness follow from i) of Proposition 3.3.6 and Frobenius reciprocity.

- If $g \in G_l$, define $g_{\mathcal{V}_l,\mathcal{F}} : \mathcal{V}_{l,\mathcal{F}} \xrightarrow{\sim} \mathcal{V}_{l,g\mathcal{F}}$ as

$$\mathcal{V}_{l,\mathcal{F}} = \mathrm{Hom}_{RK_{l,\mathcal{F}}^\dagger}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{V}_{\mathcal{F}}) \xrightarrow{u \mapsto g_{\mathcal{V}_l,\mathcal{F}} \circ u \circ (g_{\mathcal{W}_l,\mathcal{F}})^{-1}} \mathrm{Hom}_{RK_{l,g\mathcal{F}}^\dagger}(\mathcal{W}_{l,g\mathcal{F}}, \mathcal{V}_{g\mathcal{F}}) = \mathcal{V}_{l,g\mathcal{F}}$$

Lemma. — *The above data define an object \mathcal{V}_l of $\mathrm{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l)$, that is Cartesian if \mathcal{V} is Cartesian.*

Proof. The transitivity of the action maps, i.e., the identities $(hg)_{\mathcal{V}_l, \mathcal{F}} = h_{\mathcal{V}_l, g\mathcal{F}} \circ g_{\mathcal{V}_l, \mathcal{F}}$, follows from that of the action maps for \mathcal{V} and \mathcal{W} .

Let us check the transitivity of the face maps. Given $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ such that $\overline{\mathcal{F}} \supseteq \overline{\mathcal{F}'} \supseteq \overline{\mathcal{F}''}$, we see that $\beta_{\mathcal{V}_l, \mathcal{F}', \mathcal{F}''} \circ \beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'}$ makes the diagram that defines $\beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}''}$ commute. Hence, by uniqueness, both maps are equal.

Let us now check compatibility of the face maps with the action. Given $\mathcal{F}, \mathcal{F}'$ such that $\overline{\mathcal{F}} \supseteq \overline{\mathcal{F}'}$ and $g \in G_l$, the map $g_{\mathcal{V}_l, \mathcal{F}'} \circ \beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'} \circ (g_{\mathcal{V}_l, \mathcal{F}})^{-1}$ makes the following diagram commute :

$$\begin{array}{ccc}
\mathcal{V}_{l, g\mathcal{F}} = \text{Hom}_{RK_{l, g\mathcal{F}}^\dagger}(\mathcal{W}_{l, g\mathcal{F}}, \mathcal{V}_{g\mathcal{F}}) & \xrightarrow{g_{\mathcal{V}_l, \mathcal{F}'} \circ \beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'} \circ (g_{\mathcal{V}_l, \mathcal{F}})^{-1}} & \mathcal{V}_{l, g\mathcal{F}'} = \text{Hom}_{RK_{l, g\mathcal{F}'}^\dagger}(\mathcal{W}_{l, g\mathcal{F}'}, \mathcal{V}_{g\mathcal{F}'}) \\
\downarrow w \mapsto (g_{\mathcal{V}, \mathcal{F}})^{-1} \circ u \circ g_{\mathcal{W}, \mathcal{F}} & & \downarrow w \mapsto (g_{\mathcal{V}, \mathcal{F}'})^{-1} \circ u \circ g_{\mathcal{W}, \mathcal{F}'} \\
\mathcal{V}_{l, \mathcal{F}} = \text{Hom}_{RK_{l, \mathcal{F}}^\dagger}(\mathcal{W}_{l, \mathcal{F}}, \mathcal{V}_{\mathcal{F}}) & \xrightarrow{\beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'}} & \mathcal{V}_{l, \mathcal{F}'} = \text{Hom}_{RK_{l, \mathcal{F}'}^\dagger}(\mathcal{W}_{l, \mathcal{F}'}, \mathcal{V}_{\mathcal{F}'}) \\
\downarrow w \mapsto \beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'} \circ u & & \downarrow w \mapsto u \circ \beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'} \\
\text{Hom}_{R(K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}^\dagger)}(\mathcal{W}_{l, \mathcal{F}}, \mathcal{V}_{\mathcal{F}'}) & \xlongequal{\quad\quad\quad} & \text{Hom}_{R(K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}^\dagger)}(\mathcal{W}_{l, \mathcal{F}}, \mathcal{V}_{\mathcal{F}'}) \\
\downarrow w \mapsto g_{\mathcal{V}, \mathcal{F}'} \circ u \circ (g_{\mathcal{W}, \mathcal{F}})^{-1} & & \downarrow w \mapsto g_{\mathcal{V}, \mathcal{F}'} \circ u \circ (g_{\mathcal{W}, \mathcal{F}})^{-1} \\
\text{Hom}_{R(K_{l, g\mathcal{F}}^\dagger \cap K_{l, g\mathcal{F}'}^\dagger)}(\mathcal{W}_{l, g\mathcal{F}}, \mathcal{V}_{g\mathcal{F}'}) & \xlongequal{\quad\quad\quad} & \text{Hom}_{R(K_{l, g\mathcal{F}}^\dagger \cap K_{l, g\mathcal{F}'}^\dagger)}(\mathcal{W}_{l, g\mathcal{F}}, \mathcal{V}_{g\mathcal{F}'})
\end{array}$$

Since \mathcal{V} is a coefficient system, we have $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'} \circ (g_{\mathcal{V}, \mathcal{F}})^{-1} = (g_{\mathcal{V}, \mathcal{F}'})^{-1} \circ \beta_{\mathcal{V}, g\mathcal{F}, g\mathcal{F}'}$, hence the composition of the left vertical maps is $u \mapsto \beta_{\mathcal{V}, g\mathcal{F}, g\mathcal{F}'} \circ u$. Similarly, \mathcal{W} being a G_l -equivariant coefficient system, we have $g_{\mathcal{W}, \mathcal{F}'} \circ \beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'} = \beta_{\mathcal{W}, g\mathcal{F}, g\mathcal{F}'} \circ g_{\mathcal{W}, \mathcal{F}}$, and the composition of the right vertical maps is $u \mapsto u \circ \beta_{\mathcal{W}, g\mathcal{F}, g\mathcal{F}'}$. It follows that the top map makes the defining diagram of $\beta_{\mathcal{V}_l, g\mathcal{F}, g\mathcal{F}'}$ commute, hence $\beta_{\mathcal{V}_l, g\mathcal{F}, g\mathcal{F}'} = g_{\mathcal{V}_l, \mathcal{F}'} \circ \beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'} \circ (g_{\mathcal{V}_l, \mathcal{F}})^{-1}$ as desired.

At this point, we have proved that \mathcal{V}_l is a G_l -equivariant coefficient system on $\mathcal{B}'_{l, \bullet/e}$. Since $G_{l, \mathcal{F}, 0+}$ visibly acts trivially on $\mathcal{V}_{l, \mathcal{F}}$, this coefficient system lies in $\text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l)$.

Let us now assume that \mathcal{V} is Cartesian. It remains to check that $\beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'}$ induces an isomorphism $\mathcal{V}_{l, \mathcal{F}} \xrightarrow{\sim} e_{G_{l, \mathcal{F}, 0+}} \mathcal{V}_{l, \mathcal{F}'}$ for all pairs of facets $\mathcal{F}' \subset \overline{\mathcal{F}} \subset \mathcal{B}'_{l, \bullet/e}$. The injectivity of $\beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'}$ follows from its defining diagram and the injectivity of $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'}$. The fact that the image of $\beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'}$ is invariant under the subgroup $G_{l, \mathcal{F}, 0+}$ of $G_{l, \mathcal{F}'}$ follows from its compatibility with the action maps and the fact that $G_{l, \mathcal{F}, 0+} \subseteq K_{l, \mathcal{F}}^\dagger$ acts trivially on $\mathcal{V}_{l, \mathcal{F}}$. Now, let $u \in \mathcal{V}_{l, \mathcal{F}'} = \text{Hom}_{RK_{l, \mathcal{F}'}^\dagger}(\mathcal{W}_{l, \mathcal{F}'}, \mathcal{V}_{\mathcal{F}'})$. The image $u(\beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'}(\mathcal{W}_{l, \mathcal{F}}))$ of the subspace $\beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'}(\mathcal{W}_{l, \mathcal{F}}) = e'_{l, \mathcal{F}} \mathcal{W}_{l, \mathcal{F}} \subseteq \mathcal{W}_{l, \mathcal{F}'}$ is contained in the $(\check{\phi}_{l, \mathcal{F}})|_{K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}^\dagger}$ -isotypic submodule of $\mathcal{V}_{\mathcal{F}'}$. If, additionally, u is $G_{l, \mathcal{F}, 0+}$ -invariant, then also $u(\beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'}(\mathcal{W}_{l, \mathcal{F}})) \subseteq (\mathcal{V}_{\mathcal{F}'})^{G_{l, \mathcal{F}, 0+}}$, because $G_{l, \mathcal{F}, 0+} \subset G_{l, \mathcal{F}} \cap G_{l, \mathcal{F}'}$ acts trivially on $\mathcal{W}_{l, \mathcal{F}}$ and hence also on $\beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'}(\mathcal{W}_{l, \mathcal{F}})$. Since $G_{l, \mathcal{F}, 0+} \cdot (K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}^\dagger) = K_{l, \mathcal{F}}^\dagger$ and $\check{\phi}_{l, \mathcal{F}}$ is trivial on $G_{l, \mathcal{F}, 0+}$, this means that $u(\beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'}(\mathcal{W}_{l, \mathcal{F}})) \subseteq e_{l, \mathcal{F}} \mathcal{V}_{l, \mathcal{F}'}$. But, $\mathcal{V} \in \text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)$, so we know that $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'}$ is an isomorphism $\mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} e_{l, \mathcal{F}} \mathcal{V}_{\mathcal{F}'}$. Applying the idempotent $e_{l, \mathcal{F}}$ on both sides, we see that $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'}$ induces an isomorphism $e_{l, \mathcal{F}} \mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} e_{l, \mathcal{F}} \mathcal{V}_{\mathcal{F}'}$. Composing $u \circ \beta_{\mathcal{W}, \mathcal{F}, \mathcal{F}'}$ with the inverse of that isomorphism yields a morphism $v : \mathcal{W}_{l, \mathcal{F}} \rightarrow e_{l, \mathcal{F}} \mathcal{V}_{\mathcal{F}'}$, which is both $K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}^\dagger$ -equivariant and $G_{l, \mathcal{F}, 0+}$ -equivariant. Since $G_{l, \mathcal{F}, 0+} \cdot (K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}^\dagger) = K_{l, \mathcal{F}}^\dagger$, we see that v belongs to $\mathcal{V}_{l, \mathcal{F}} = \text{Hom}_{RK_{l, \mathcal{F}}^\dagger}(\mathcal{W}_{l, \mathcal{F}}, \mathcal{V}_{\mathcal{F}})$. By construction, we have $\beta_{\mathcal{V}_l, \mathcal{F}, \mathcal{F}'}(v) = u$. \square

The assignment $\mathcal{V} \mapsto \mathcal{V}_l$ is clearly functorial in \mathcal{V} , providing a functor

$$\mathcal{R}_{\mathcal{W}_l} : \text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G) \longrightarrow \text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l),$$

that maps $\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}}$ into $\text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l)^{\text{cart}}$.

3.4.3 *A functor from $\text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l)$ to $\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)$.* — This direction carries additional complications since we start from a coefficient system only defined on a subset of the target building. So, let \mathcal{V}_l be an object of $\text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l)$. We define the following objects.

- For any $\mathcal{F} \in \mathcal{B}'_{\bullet/e}$, define

$$\mathcal{V}_{\mathcal{F}} := \left(\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l, \bullet/e}} \mathcal{V}_{\mathcal{F}, \mathcal{F}_l} \right)_{G_l} \quad \text{with } \mathcal{V}_{\mathcal{F}, \mathcal{F}_l} := \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) \otimes_{RK_{l, \mathcal{F}_l}} (\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l}).$$

Here, $G_{\mathcal{F}\mathcal{F}_l} := \{g \in G, g\mathcal{F}_l = \mathcal{F}\}$ is an open subset of G and \mathcal{C}_c^∞ denotes smooth, compactly supported R -valued functions⁸. In particular, $\mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l})$ has a right RK_{l, \mathcal{F}_l} -module structure arising from multiplication in G , i.e., $g \in G$ acts on a function by right-translation by g^{-1} , while $\mathcal{W}_{l, \mathcal{F}_l}$ is a left RK_{l, \mathcal{F}_l} -module by definition, and so is $\mathcal{V}_{l, \mathcal{F}_l}$ through the projection map $K_{l, \mathcal{F}_l} \rightarrow G_{l, \mathcal{F}_l}$. Finally, $h \in G_l$ acts as the sum of all maps $\mathcal{V}_{\mathcal{F}, \mathcal{F}_l} \rightarrow \mathcal{V}_{\mathcal{F}, h\mathcal{F}_l}$, $f \otimes w \otimes v \mapsto fh^{-1} \otimes h_{\mathcal{W}_{l, \mathcal{F}_l}}(w) \otimes h_{\mathcal{V}_{l, \mathcal{F}_l}}(v)$.

- For $g \in G$, let $g_{\mathcal{V}, \mathcal{F}} : \mathcal{V}_{\mathcal{F}} \rightarrow \mathcal{V}_{g\mathcal{F}}$ be induced by the sum over $\mathcal{F}_l \in \mathcal{B}'_{l, \bullet/e}$ of all maps $\mathcal{V}_{\mathcal{F}, \mathcal{F}_l} \rightarrow \mathcal{V}_{g\mathcal{F}, \mathcal{F}_l}$, $f \otimes w \otimes v \mapsto gf \otimes w \otimes v$, where $gf(h) = f(g^{-1}h)$ for $h \in G_{(g\mathcal{F})\mathcal{F}_l}$.
- For $\mathcal{F}, \mathcal{F}'$ such that $\overline{\mathcal{F}} \supseteq \mathcal{F}'$, we let $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'} : \mathcal{V}_{\mathcal{F}} \rightarrow \mathcal{V}_{\mathcal{F}'}$ be the map induced by the sum over $\mathcal{F}_l, \mathcal{F}'_l \in \mathcal{B}'_{l, \bullet/e}$ with $\overline{\mathcal{F}}_l \supseteq \mathcal{F}'_l$ of all maps

$$p_{\mathcal{F}\mathcal{F}_l, \mathcal{F}'\mathcal{F}'_l} \otimes \beta_{\mathcal{W}_{l, \mathcal{F}_l, \mathcal{F}'_l}} \otimes \beta_{\mathcal{V}_{l, \mathcal{F}_l, \mathcal{F}'_l}} : \mathcal{V}_{\mathcal{F}, \mathcal{F}_l} \rightarrow \mathcal{V}_{\mathcal{F}', \mathcal{F}'_l}$$

where $p_{\mathcal{F}\mathcal{F}_l, \mathcal{F}'\mathcal{F}'_l} : \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) \rightarrow \mathcal{C}_c^\infty(G_{\mathcal{F}'\mathcal{F}'_l})$ denotes restriction of functions to $G_{\mathcal{F}\mathcal{F}_l} \cap G_{\mathcal{F}'\mathcal{F}'_l}$ followed by extension by zero to $G_{\mathcal{F}'\mathcal{F}'_l}$. These maps descend to G_{l, \mathcal{F}_l} -coinvariants since $G_{l, \mathcal{F}_l} \subseteq K_{l, \mathcal{F}_l} \subseteq K_{l, \mathcal{F}'_l}$ (by our choice of e), and then the sum descends to G_l -coinvariants.

Lemma. — *The above data define an object \mathcal{V} in $\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)$, that is Cartesian if \mathcal{V}_l is cartesian.*

Proof. Both the transitivity of the action maps and their compatibility with face maps are straightforward. To prove transitivity for face maps, let $\mathcal{F}, \mathcal{F}'$ and \mathcal{F}'' such that $\overline{\mathcal{F}} \supseteq \overline{\mathcal{F}'} \supseteq \overline{\mathcal{F}''}$. Using transitivity of face maps for the coefficient systems \mathcal{W}_l and \mathcal{V}_l , we are left to prove that for any two facets $\mathcal{F}_l, \mathcal{F}''_l \in \mathcal{B}'_{l, \bullet/e}$ such that $\overline{\mathcal{F}}_l \supseteq \mathcal{F}''_l$, we have

$$p_{\mathcal{F}\mathcal{F}_l, \mathcal{F}''\mathcal{F}''_l} = \sum_{\mathcal{F}'_l, \overline{\mathcal{F}}_l \supseteq \mathcal{F}'_l \supseteq \mathcal{F}''_l} p_{\mathcal{F}'\mathcal{F}'_l, \mathcal{F}''\mathcal{F}''_l} \circ p_{\mathcal{F}\mathcal{F}_l, \mathcal{F}'\mathcal{F}'_l},$$

⁸We set $\mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) = \{0\}$ when $G_{\mathcal{F}\mathcal{F}_l}$ is empty.

or equivalently, that

$$G_{\mathcal{F}\mathcal{F}_l} \cap G_{\mathcal{F}'\mathcal{F}'_l} = \bigsqcup_{\mathcal{F}'_l, \overline{\mathcal{F}'_l} \supseteq \overline{\mathcal{F}'_l} \supseteq \mathcal{F}'_l} G_{\mathcal{F}\mathcal{F}_l} \cap G_{\mathcal{F}'\mathcal{F}'_l} \cap G_{\mathcal{F}'\mathcal{F}'_l}.$$

This follows from the observation that for any $g \in G_{\mathcal{F}\mathcal{F}_l} \cap G_{\mathcal{F}'\mathcal{F}'_l}$, the facet $\mathcal{F}'_l(g) := g^{-1}\mathcal{F}'_l$ satisfies $\overline{\mathcal{F}'_l} \supseteq \overline{\mathcal{F}'_l(g)} \supseteq \mathcal{F}'_l$ and, in particular, belongs to $\mathcal{B}'_{\bullet/e}$.

At this point, we know that \mathcal{V} is a smooth G -equivariant coefficient system on $\mathcal{B}'_{\bullet/e}$. To show that it belongs to $\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)$, it suffices to prove that $e_{I, \mathcal{F}}\mathcal{V}_{\mathcal{F}, \mathcal{F}_l} = \mathcal{V}_{\mathcal{F}, \mathcal{F}_l}$ for all facets \mathcal{F}_l . We may assume that \mathcal{F}_l is G -conjugate to \mathcal{F} since otherwise both sides vanish. Observe that $\mathcal{V}_{\mathcal{F}, \mathcal{F}_l} \xleftarrow{\sim} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l})e_{l, \mathcal{F}_l} \otimes_{RK_{l, \mathcal{F}_l}}(\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l})$ since e_{l, \mathcal{F}_l} acts as identity on $\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l}$. Similarly $e_{I, \mathcal{F}}\mathcal{V}_{\mathcal{F}, \mathcal{F}_l} \xleftarrow{\sim} e_{I, \mathcal{F}}\mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l})e_{l, \mathcal{F}_l} \otimes_{RK_{l, \mathcal{F}_l}}(\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l})$. But $e_{I, \mathcal{F}}\mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l})e_{l, \mathcal{F}_l} = \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l})e_{I, \mathcal{F}_l}e_{l, \mathcal{F}_l} = \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l})e_{l, \mathcal{F}_l}$ since $e_{I, \mathcal{F}_l}e_{l, \mathcal{F}_l} = e_{l, \mathcal{F}_l}$ by Lemma 2.6.1.

Let us now assume that \mathcal{V}_l is Cartesian. It then remains to show that $\beta_{\mathcal{V}, \mathcal{F}, \mathcal{F}'}$ induces an isomorphism $\mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} e_{I, \mathcal{F}}\mathcal{V}_{\mathcal{F}'}$ whenever $\overline{\mathcal{F}} \supseteq \mathcal{F}'$. To this aim, it is useful to note that $\mathcal{V}_{\mathcal{F}}$ can also be written as

$$\mathcal{V}_{\mathcal{F}} = \left(\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{\bullet/e}} \mathcal{V}_{\mathcal{F}, \mathcal{F}_l}^\dagger \right)_{G_l} \quad \text{with } \mathcal{V}_{\mathcal{F}, \mathcal{F}_l}^\dagger := \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) \otimes_{RK_{l, \mathcal{F}_l}^\dagger}(\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l}).$$

Indeed, the G_l -coinvariants on the sum take care of the difference between $K_{l, \mathcal{F}_l}^\dagger$ and K_{l, \mathcal{F}_l} coinvariants on each summand. Since $g^{-1}\mathcal{F}' \in \mathcal{B}'_{\bullet/e}$ for any $g \in G_{\mathcal{F}\mathcal{F}_l}$, we have a further decomposition $\mathcal{V}_{\mathcal{F}, \mathcal{F}_l}^\dagger = \bigoplus_{\mathcal{F}'_l \in \mathcal{B}'_{\bullet/e}, \overline{\mathcal{F}'_l} \supseteq \mathcal{F}_l} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l} \cap G_{\mathcal{F}'\mathcal{F}'_l}) \otimes_{RK_{l, \mathcal{F}_l}^\dagger}(\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l})$, which yields the following expression :

$$(3.4.4) \quad \mathcal{V}_{\mathcal{F}} = \left(\bigoplus_{\mathcal{F}_l, \mathcal{F}'_l \in \mathcal{B}'_{\bullet/e}, \overline{\mathcal{F}'_l} \supseteq \mathcal{F}_l} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l} \cap G_{\mathcal{F}'\mathcal{F}'_l}) \otimes_{RK_{l, \mathcal{F}_l}^\dagger}(\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l}) \right)_{G_l}.$$

On the other hand, for $g \in G_{\mathcal{F}'\mathcal{F}'_l}$, the facet $g^{-1}\mathcal{F}$ need not be contained in $\mathcal{B}'_{\bullet/e}$, but we still have a decomposition

$$\mathcal{V}_{\mathcal{F}'_l, \mathcal{F}'_l}^\dagger = \bigoplus_{\mathcal{G} \in \mathcal{B}'_{\bullet/e}, \overline{\mathcal{G}} \supseteq \mathcal{F}'_l} \mathcal{C}_c^\infty(G_{\mathcal{F}'\mathcal{F}'_l} \cap G_{\mathcal{F}\mathcal{G}}) \otimes_{RK_{l, \mathcal{F}'_l}^\dagger}(\mathcal{W}_{l, \mathcal{F}'_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}),$$

hence also

$$\begin{aligned} e_{I, \mathcal{F}}\mathcal{V}_{\mathcal{F}'_l, \mathcal{F}'_l}^\dagger &= \bigoplus_{\mathcal{G} \in \mathcal{B}'_{\bullet/e}, \overline{\mathcal{G}} \supseteq \mathcal{F}'_l} e_{I, \mathcal{F}}\mathcal{C}_c^\infty(G_{\mathcal{F}'\mathcal{F}'_l} \cap G_{\mathcal{F}\mathcal{G}}) \otimes_{RK_{l, \mathcal{F}'_l}^\dagger}(\mathcal{W}_{l, \mathcal{F}'_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}) \\ &= \bigoplus_{\mathcal{G} \in \mathcal{B}'_{\bullet/e}, \overline{\mathcal{G}} \supseteq \mathcal{F}'_l} \mathcal{C}_c^\infty(G_{\mathcal{F}'\mathcal{F}'_l} \cap G_{\mathcal{F}\mathcal{G}})e_{I, \mathcal{G}} \otimes_{RK_{l, \mathcal{F}'_l}^\dagger}(\mathcal{W}_{l, \mathcal{F}'_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}). \end{aligned}$$

The summand associated to \mathcal{G} above is non-zero only if $G_{\mathcal{F}'\mathcal{F}'_l} \cap G_{\mathcal{F}\mathcal{G}} \neq \emptyset$ and $e_{I, \mathcal{G}}e_{l, \mathcal{F}'_l} \neq 0$. By Proposition 2.5.2 ii), the latter condition implies the existence of a $l' \in I$ such that $\mathcal{G} \in \mathcal{B}'_{l'}$ and

$l' \in K_{l, \mathcal{F}'_l}^\dagger$. Since $K_{l, \mathcal{F}'_l}^\dagger \subseteq G_{\mathcal{F}_l, 0+}$ fixes \mathcal{G} , this implies that $\mathcal{G} \in \mathcal{B}'_l$, so we can rewrite the last sum as

$$e_{I, \mathcal{F}} \mathcal{V}_{\mathcal{F}'_l, \mathcal{F}'_l}^\dagger = \bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l, \bullet/e}, \bar{\mathcal{F}}_l \supseteq \mathcal{F}'_l} \mathcal{C}_c^\infty(G_{\mathcal{F}'_l} \cap G_{\mathcal{F}_l}) e_{I, \mathcal{F}_l} \otimes_{RK_{l, \mathcal{F}'_l}^\dagger} (\mathcal{W}_{l, \mathcal{F}'_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}).$$

Using i) of Proposition 3.3.6, we may rewrite it further as

$$e_{I, \mathcal{F}} \mathcal{V}_{\mathcal{F}'_l, \mathcal{F}'_l}^\dagger \xleftarrow[\oplus(\text{id} \otimes \beta_{\mathcal{W}_{l, \mathcal{F}_l, \mathcal{F}'_l} \otimes \text{id}})]{\simeq} \bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l, \bullet/e}, \bar{\mathcal{F}}_l \supseteq \mathcal{F}'_l} \mathcal{C}_c^\infty(G_{\mathcal{F}'_l} \cap G_{\mathcal{F}_l}) e_{I, \mathcal{F}_l} \otimes_{R(K_{l, \mathcal{F}'_l}^\dagger \cap K_{l, \mathcal{F}_l}^+)} (\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l})$$

Recall the idempotent $e'_{l, \mathcal{F}}$ of Proposition 3.3.6 ii), which is supported on $K_{l, \mathcal{F}'_l}^\dagger \cap K_{l, \mathcal{F}_l}^+ \subseteq K_{l, \mathcal{F}'_l}^\dagger \cap K_{l, \mathcal{F}_l}^+$ and acts trivially on $\mathcal{W}_{l, \mathcal{F}_l} \otimes \mathcal{V}_{l, \mathcal{F}'_l}$. Since $K_{l, \mathcal{F}'_l}^\dagger \cap K_{l, \mathcal{F}_l}^+$ contains the group K'_{l, \mathcal{F}_l} introduced in Remark 2.5.11, Remark 2.6.2 tells us that $e_{I, \mathcal{F}_l} e'_{l, \mathcal{F}_l} = e_{l, \mathcal{F}_l}$, hence each e_{I, \mathcal{F}_l} in the last sum can be replaced by e_{l, \mathcal{F}_l} . We thus get the following expression of $e_{I, \mathcal{F}} \mathcal{V}_{\mathcal{F}'_l}$:

$$e_{I, \mathcal{F}} \mathcal{V}_{\mathcal{F}'_l} \xleftarrow[\oplus(\text{id} \otimes \beta_{\mathcal{W}_{l, \mathcal{F}_l, \mathcal{F}'_l} \otimes \text{id}})]{\simeq} \left(\bigoplus_{\substack{\mathcal{F}_l, \mathcal{F}'_l \in \mathcal{B}'_{l, \bullet/e}, \\ \bar{\mathcal{F}}_l \supseteq \mathcal{F}'_l}} \mathcal{C}_c^\infty(G_{\mathcal{F}'_l} \cap G_{\mathcal{F}_l}) e_{l, \mathcal{F}_l} \otimes_{R(K_{l, \mathcal{F}'_l}^\dagger \cap K_{l, \mathcal{F}_l}^+)} (\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}) \right)_{G_l}.$$

Now, since $K_{l, \mathcal{F}_l}^\dagger = (K_{l, \mathcal{F}'_l}^\dagger \cap K_{l, \mathcal{F}_l}^+) G_{l, \mathcal{F}_l, 0+}$, we also have

$$e_{I, \mathcal{F}} \mathcal{V}_{\mathcal{F}'_l} \xleftarrow[\oplus(\text{id} \otimes \beta_{\mathcal{W}_{l, \mathcal{F}_l, \mathcal{F}'_l} \otimes \text{id}})]{\simeq} \left(\bigoplus_{\mathcal{F}_l, \mathcal{F}'_l \in \mathcal{B}'_{l, \bullet/e}, \bar{\mathcal{F}}_l \supseteq \mathcal{F}'_l} \mathcal{C}_c^\infty(G_{\mathcal{F}'_l} \cap G_{\mathcal{F}_l}) e_{l, \mathcal{F}_l} \otimes_{RK_{l, \mathcal{F}_l}^\dagger} (\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}) \right)_{G_l}.$$

Moreover, since e_{l, \mathcal{F}_l} acts trivially on $\mathcal{W}_{l, \mathcal{F}_l}$ and through $e_{G_{l, \mathcal{F}_l, 0+}}$ on $\mathcal{V}_{l, \mathcal{F}'_l}$, we have

$$e_{l, \mathcal{F}_l} (\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}) = \mathcal{W}_{l, \mathcal{F}_l} \otimes_R e_{G_{l, \mathcal{F}_l, 0+}} \mathcal{V}_{l, \mathcal{F}'_l} \xleftarrow[\text{id} \otimes \beta_{\mathcal{V}_{l, \mathcal{F}_l, \mathcal{F}'_l}}]{\simeq} \mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}_l}.$$

Combining this with (3.4.4) we obtain

$$(3.4.5) \quad e_{I, \mathcal{F}} \mathcal{V}_{\mathcal{F}'_l} \xleftarrow{\simeq} \left(\bigoplus_{\mathcal{F}_l, \mathcal{F}'_l \in \mathcal{B}'_{l, \bullet/e}, \bar{\mathcal{F}}_l \supseteq \mathcal{F}'_l} \mathcal{C}_c^\infty(G_{\mathcal{F}'_l} \cap G_{\mathcal{F}_l}) \otimes_{RK_{l, \mathcal{F}_l}^\dagger} (\mathcal{W}_{l, \mathcal{F}_l} \otimes_R \mathcal{V}_{l, \mathcal{F}'_l}) \right)_{G_l} = \mathcal{V}_{\mathcal{F}},$$

where the isomorphism agrees by construction with the map $\beta_{\mathcal{V}_{\mathcal{F}}, \mathcal{F}'_l}$. This finishes the proof of the lemma. \square

Again, the assignment $\mathcal{V}_l \mapsto \mathcal{V}$ is easily seen to be functorial in \mathcal{V}_l , providing a functor

$$\mathcal{I}_{\mathcal{W}_l} : \text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l) \longrightarrow \text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)$$

that maps $\text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l)^{\text{cart}}$ into $\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}}$.

3.4.6 Theorem. — *The functors $\mathcal{R}_{\mathcal{W}_l}$ and $\mathcal{I}_{\mathcal{W}_l}$ are quasi-inverse equivalences of categories*

$$\mathcal{I}_{\mathcal{W}_l} : \text{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l) \xrightleftharpoons[\sim]{\sim} \text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G) : \mathcal{R}_{\mathcal{W}_l}$$

that respect cartesian objects on both sides.

Proof. Let us simplify our notation $\mathcal{I} := \mathcal{I}_{\mathcal{W}_l}$ and $\mathcal{R} := \mathcal{R}_{\mathcal{W}_l}$ in this proof. We first construct an isomorphism $\text{Id}_{\text{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l)} \xrightarrow{\sim} \mathcal{R} \circ \mathcal{I}$. So let us start with an object $\mathcal{V}_l \in \text{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l)$. For any $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$, we have

$$\mathcal{R} \circ \mathcal{I}(\mathcal{V}_l)_{\mathcal{F}} = \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, \left(\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) \otimes_{RK_{l,\mathcal{F}_l}} (\mathcal{W}_{l,\mathcal{F}_l} \otimes_R \mathcal{V}_{l,\mathcal{F}_l}) \right)_{G_l} \right).$$

We thus can define an R -linear map

$$(3.4.6a) \quad \mathcal{V}_{l,\mathcal{F}} \longrightarrow \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}}) \otimes_{RK_{l,\mathcal{F}}} (\mathcal{W}_{l,\mathcal{F}} \otimes_R \mathcal{V}_{l,\mathcal{F}}) \right) \longrightarrow \mathcal{R} \circ \mathcal{I}(\mathcal{V}_l)_{\mathcal{F}},$$

where the first map sends $v \in \mathcal{V}_l$ to $(w \mapsto e_{l,\mathcal{F}} \otimes (w \otimes v))$. Let us prove that this map is an isomorphism. We have the following sequence of isomorphisms explained below:

$$\begin{aligned} \mathcal{R} \circ \mathcal{I}(\mathcal{V}_l)_{\mathcal{F}} &= \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, \left(\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) \otimes_{RK_{l,\mathcal{F}_l}} (\mathcal{W}_{l,\mathcal{F}_l} \otimes_R \mathcal{V}_{l,\mathcal{F}_l}) \right)_{G_l} \right) \\ &\xleftarrow{\sim} \left(\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}} \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) \otimes_{RK_{l,\mathcal{F}_l}} (\mathcal{W}_{l,\mathcal{F}_l} \otimes_R \mathcal{V}_{l,\mathcal{F}_l}) \right) \right)_{G_l} \\ &\xleftarrow{\sim} \left(\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}} \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, e_{l,\mathcal{F}} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) e_{l,\mathcal{F}_l} \otimes_{RK_{l,\mathcal{F}_l}} (\mathcal{W}_{l,\mathcal{F}_l} \otimes_R \mathcal{V}_{l,\mathcal{F}_l}) \right) \right)_{G_l} \\ &= \left(\bigoplus_{\mathcal{F}_l \in G_l \cdot \mathcal{F}} \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, e_{l,\mathcal{F}} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) e_{l,\mathcal{F}_l} \otimes_{RK_{l,\mathcal{F}_l}} (\mathcal{W}_{l,\mathcal{F}_l} \otimes_R \mathcal{V}_{l,\mathcal{F}_l}) \right) \right)_{G_l} \\ &\xleftarrow{\sim} \left(\text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, e_{l,\mathcal{F}} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}}) e_{l,\mathcal{F}} \otimes_{RK_{l,\mathcal{F}}} (\mathcal{W}_{l,\mathcal{F}} \otimes_R \mathcal{V}_{l,\mathcal{F}}) \right) \right)_{G_l, \mathcal{F}} \\ &\xrightarrow{\sim} \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} \left(\mathcal{W}_{l,\mathcal{F}}, \left(\mathcal{C}_c^\infty(K_{l,\mathcal{F}}) \otimes_{RK_{l,\mathcal{F}}} (\mathcal{W}_{l,\mathcal{F}} \otimes_R \mathcal{V}_{l,\mathcal{F}}) \right)_{G_l, \mathcal{F}} \right) \\ &= \text{Hom}_{RK_{l,\mathcal{F}}^\dagger} (\mathcal{W}_{l,\mathcal{F}}, \mathcal{W}_{l,\mathcal{F}} \otimes_R \mathcal{V}_{l,\mathcal{F}}). \end{aligned}$$

The first natural map (between the second and first line in the above sequence of isomorphism) is an isomorphism since $\text{Hom}_{RK_{l,\mathcal{F}}^\dagger}(\mathcal{W}_{l,\mathcal{F}}, -)$ commutes with colimits because $\mathcal{W}_{l,\mathcal{F}}$ is a finitely generated projective $RK_{l,\mathcal{F}}^\dagger$ -module. The second map is induced by the inclusions $e_{l,\mathcal{F}} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) e_{l,\mathcal{F}_l} \subseteq \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l})$. It is an isomorphism because the central idempotent

$e_{\iota, \mathcal{F}} \in \mathcal{H}_R(K_{\iota, \mathcal{F}}^\dagger)$ acts trivially on $\mathcal{W}_{\iota, \mathcal{F}}$ and, similarly, the central idempotent $e_{\iota, \mathcal{F}_\iota} \in \mathcal{H}_R(K_{\iota, \mathcal{F}_\iota})$ acts trivially on $(\mathcal{W}_{\iota, \mathcal{F}_\iota} \otimes \mathcal{V}_{\iota, \mathcal{F}_\iota})$. The second equality says that a facet \mathcal{F}_ι provides a non-zero contribution to the big sum only if it is a G_ι translate of \mathcal{F} . To see this, observe that, for $g \in G_{\mathcal{F}\mathcal{F}_\iota}$, $e_{\iota, \mathcal{F}} g e_{\iota, \mathcal{F}_\iota} = e_{\iota, \mathcal{F}} e_{g_\iota, \mathcal{F}} g$ is non-zero in RG only if $g_\iota \in K_{\iota, \mathcal{F}_\iota}^\dagger$, due to Lemma 2.6.1. In this case, g can be written $g = k g_\iota$ with $k \in K_{\iota, \mathcal{F}}^\dagger$ and $g_\iota \in G_\iota$, and it follows that $\mathcal{F}_\iota = g^{-1} \mathcal{F} = (g_\iota)^{-1} \mathcal{F} \in G_\iota \cdot \mathcal{F}$. The third map is well defined and an isomorphism since G_ι permutes transitively the summands and $G_{\iota, \mathcal{F}}$ is the stabilizer of the summand associated to $\mathcal{F}_\iota = \mathcal{F}$. To see that the fourth map is an isomorphism, we first use again the projectivity of $W_{\mathcal{F}_\iota}$ as an $RK_{\iota, \mathcal{F}}$ -module to exchange the Hom and the coinvariants (recall that $G_{\iota, \mathcal{F}}$ only acts on the target of the Hom). Then, as above, we observe that, for $g \in G_{\mathcal{F}}$, the product $e_{\iota, \mathcal{F}} g e_{\iota, \mathcal{F}}$ is non-zero if and only if $g \in K_{\iota, \mathcal{F}}^\dagger G_{\iota, \mathcal{F}} = K_{\iota, \mathcal{F}}$. This allows to rewrite $e_{\iota, \mathcal{F}} \mathcal{C}_c^\infty(G_{\mathcal{F}}) e_{\iota, \mathcal{F}} = \mathcal{C}_c^\infty(K_{\iota, \mathcal{F}}) e_{\iota, \mathcal{F}}$. Then the idempotent $e_{\iota, \mathcal{F}}$ can be moved to the right hand side of the $\otimes_{RK_{\iota, \mathcal{F}}}$, where it acts trivially.

The precomposition of this isomorphism with the map $\mathcal{V}_{\iota, \mathcal{F}} \rightarrow \mathcal{R} \circ \mathcal{I}(\mathcal{V}_\iota)_\mathcal{F}$ in (3.4.6a) is just $v \mapsto (w \mapsto w \otimes v)$ and it is an isomorphism of R -modules by the pair of adjoint equivalences (3.3.2) because $\mathcal{W}_{\iota, \mathcal{F}}$ is a Heisenberg representation (and it is even an isomorphism of $RG_{\iota, \mathcal{F}}$ -modules by (3.3.3)). It is easy to see that this isomorphism is compatible with the action maps, and a bit more tedious, but still straightforward, to check it is compatible with face maps. Also it is clearly functorial in \mathcal{V}_ι , so that we have just constructed an isomorphism $\text{Id}_{\text{Coef}_R^1(\mathcal{B}'_{\bullet, \bullet/e}/G_\iota)} \xrightarrow{\sim} \mathcal{R} \circ \mathcal{I}$.

Let us now construct an isomorphism $\mathcal{I} \circ \mathcal{R} \xrightarrow{\sim} \text{Id}_{\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet, \bullet/e}/G)}$. Let \mathcal{V} be an object in $\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet, \bullet/e}/G)$ and \mathcal{F} a facet in $\mathcal{B}'_{\bullet, \bullet/e}$. By construction, we have

$$\mathcal{I} \circ \mathcal{R}(\mathcal{V})_\mathcal{F} = \left(\bigoplus_{\mathcal{F}_\iota \in \mathcal{B}'_{\bullet, \bullet/e}} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_\iota}) \otimes_{RK_{\iota, \mathcal{F}_\iota}} \left(\mathcal{W}_{\iota, \mathcal{F}_\iota} \otimes_R \text{Hom}_{RK_{\iota, \mathcal{F}_\iota}^\dagger}(\mathcal{W}_{\iota, \mathcal{F}_\iota}, \mathcal{V}_{\mathcal{F}_\iota}) \right) \right)_{G_\iota}.$$

For fixed \mathcal{F}_ι , consider the map

$$\begin{aligned} \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_\iota}) \otimes_R \left(\mathcal{W}_{\iota, \mathcal{F}_\iota} \otimes_R \text{Hom}_{RK_{\iota, \mathcal{F}_\iota}^\dagger}(\mathcal{W}_{\iota, \mathcal{F}_\iota}, \mathcal{V}_{\mathcal{F}_\iota}) \right) &\rightarrow \mathcal{V}_\mathcal{F} \\ h \otimes (w \otimes \theta) &\mapsto h_{\mathcal{V}, \mathcal{F}_\iota}(\theta(w)) \end{aligned}.$$

By definition of the actions, it descends to a map

$$\mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_\iota}) \otimes_{RK_{\iota, \mathcal{F}_\iota}} \left(\mathcal{W}_{\iota, \mathcal{F}_\iota} \otimes_R \text{Hom}_{RK_{\iota, \mathcal{F}_\iota}^\dagger}(\mathcal{W}_{\iota, \mathcal{F}_\iota}, \mathcal{V}_{\mathcal{F}_\iota}) \right) \longrightarrow \mathcal{V}_\mathcal{F}$$

and the sum of all these maps descends to the G_ι -coinvariants, providing a map

$$(3.4.6g) \quad \mathcal{I} \circ \mathcal{R}(\mathcal{V})_\mathcal{F} \longrightarrow \mathcal{V}_\mathcal{F}.$$

Let us prove that this map is an isomorphism. To this aim, observe first that each evaluation map $\mathcal{W}_{\iota, \mathcal{F}_\iota} \otimes_R \text{Hom}_{RK_{\iota, \mathcal{F}_\iota}^\dagger}(\mathcal{W}_{\iota, \mathcal{F}_\iota}, \mathcal{V}_{\mathcal{F}_\iota}) \rightarrow \mathcal{V}_{\mathcal{F}_\iota}$ induces an isomorphism of R -modules

$$(3.4.6h) \quad \mathcal{W}_{\iota, \mathcal{F}_\iota} \otimes_R \text{Hom}_{RK_{\iota, \mathcal{F}_\iota}^\dagger}(\mathcal{W}_{\iota, \mathcal{F}_\iota}, \mathcal{V}_{\mathcal{F}_\iota}) \xrightarrow{\sim} e_{\iota, \mathcal{F}_\iota} \mathcal{V}_{\mathcal{F}_\iota}$$

because $\mathcal{W}_{l,\mathcal{F}_l}$ is a Heisenberg representation with central character $\check{\phi}_{l,\mathcal{F}_l}$. This is even an isomorphism of $RK_{l,\mathcal{F}}$ -modules by (3.3.3).

On the other hand, we know from Lemma 2.6.1 that $K_{l,\mathcal{F}_l} = K_{l,\mathcal{F}_l}^\dagger G_{l,\mathcal{F}_l}$ is the centralizer of e_{l,\mathcal{F}_l} in $G_{\mathcal{F}_l}$, and that the various idempotents e_{hl,\mathcal{F}_l} for $h \in G_{\mathcal{F}_l}/K_{l,\mathcal{F}_l}$ are pairwise orthogonal. Therefore, the action map $\mathcal{C}_c^\infty(G_{\mathcal{F}_l}) \otimes_{RK_{l,\mathcal{F}_l}} e_{l,\mathcal{F}_l} \mathcal{V}_{\mathcal{F}_l} \longrightarrow \mathcal{V}_{\mathcal{F}_l}$ induces an isomorphism

$$\mathcal{C}_c^\infty(G_{\mathcal{F}_l}) \otimes_{RK_{l,\mathcal{F}_l}} e_{l,\mathcal{F}_l} \mathcal{V}_{\mathcal{F}_l} \xrightarrow{\sim} \bigoplus_{h \in G_{\mathcal{F}_l}/K_{l,\mathcal{F}_l}} e_{hl,\mathcal{F}_l} \mathcal{V}_{\mathcal{F}_l} = \sum_{h \in G_{\mathcal{F}_l}/K_{l,\mathcal{F}_l}} e_{hl,\mathcal{F}_l} \mathcal{V}_{\mathcal{F}_l} \subseteq \mathcal{V}_{\mathcal{F}_l}.$$

This in turn induces an isomorphism

$$(3.4.6i) \quad \mathcal{C}_c^\infty(G_{\mathcal{F}\mathcal{F}_l}) \otimes_{RK_{l,\mathcal{F}_l}} e_{l,\mathcal{F}_l} \mathcal{V}_{\mathcal{F}_l} \xrightarrow{\sim} \bigoplus_{l' \in (G_{\mathcal{F}\mathcal{F}_l} \cdot l) / \sim_{\mathcal{F}}} e_{l',\mathcal{F}} \mathcal{V}_{\mathcal{F}} = e_{I_{\mathcal{F}\mathcal{F}_l},\mathcal{F}} \mathcal{V}_{\mathcal{F}},$$

where the equivalence $\sim_{\mathcal{F}}$ on the subset $I_{\mathcal{F}\mathcal{F}_l} := G_{\mathcal{F}\mathcal{F}_l} \cdot l$ of $I_{\mathcal{F}}$ is that of 2.6.1, and where we have written $e_{I_{\mathcal{F}\mathcal{F}_l},\mathcal{F}} := \sum_{l' \in (I_{\mathcal{F}\mathcal{F}_l}) / \sim_{\mathcal{F}}} e_{l',\mathcal{F}}$, which is an idempotent in $\mathcal{H}_R(G_{\mathcal{F}})$. Combining (3.4.6h) and (3.4.6i) yields the following isomorphism

$$\mathcal{I} \circ \mathcal{R}(\mathcal{V})_{\mathcal{F}} \xrightarrow{\sim} \left(\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}} e_{I_{\mathcal{F}\mathcal{F}_l},\mathcal{F}} \mathcal{V}_{\mathcal{F}} \right)_{G_l}.$$

Since G_l acts only on the index set, we can rewrite the right hand side as $\bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}/G_l} e_{I_{\mathcal{F}\mathcal{F}_l},\mathcal{F}} \mathcal{V}_{\mathcal{F}}$, and the map (3.4.6g) factors as

$$\mathcal{I} \circ \mathcal{R}(\mathcal{V})_{\mathcal{F}} \xrightarrow{\sim} \bigoplus_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}/G_l} e_{I_{\mathcal{F}\mathcal{F}_l},\mathcal{F}} \mathcal{V}_{\mathcal{F}} \xrightarrow{\Sigma} \mathcal{V}_{\mathcal{F}}.$$

So, in order to show that this map is an isomorphism, it suffices to show that the idempotents $e_{I_{\mathcal{F}\mathcal{F}_l},\mathcal{F}}$ are pairwise orthogonal in $\mathcal{H}_R(G_{\mathcal{F}})$ and that their sum is $e_{I,\mathcal{F}}$. Equivalently, we need to show that $I_{\mathcal{F}} / \sim_{\mathcal{F}} = \bigsqcup_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}/G_l} I_{\mathcal{F}\mathcal{F}_l} / \sim_{\mathcal{F}}$. Obviously, we have $I_{\mathcal{F}} = \bigsqcup_{\mathcal{F}_l \in \mathcal{B}'_{l,\bullet/e}/G_l} I_{\mathcal{F}\mathcal{F}_l}$, so it is a matter of showing that this partition splits the equivalence relation $\sim_{\mathcal{F}}$. In other word, given $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{B}'_{l,\bullet/e}$ and $g_i \in G_{\mathcal{F}\mathcal{F}_i}$ such that $e_{g_1 l, \mathcal{F}} = e_{g_2 l, \mathcal{F}}$, we need to show that \mathcal{F}_1 and \mathcal{F}_2 are G_l -conjugate. But, as in Lemma 2.6.1, there is k in $K_{g_1 l, \mathcal{F}}^\dagger$ such that $g_2 l = k g_1 l$, whence an element $g_l \in G_l$ such that $g_2 g_l = k g_1$. Since k fixes \mathcal{F} , it follows that $\mathcal{F}_1 = g_1^{-1} \mathcal{F} = g_1^{-1} k^{-1} \mathcal{F} = g_l^{-1} g_2^{-1} \mathcal{F} = g_l^{-1} \mathcal{F}_2$. Hence, \mathcal{F}_1 and \mathcal{F}_2 are G_l -conjugate, and we have proved that the natural map (3.4.6g) is an isomorphism.

As in the other direction, the above identifications are natural enough to be compatible with face maps and action maps on both sides, and we have thus constructed an isomorphism $\mathcal{I} \circ \mathcal{R} \xrightarrow{\sim} \text{Id}_{\text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)}$, as desired. \square

3.4.10 We now reap the rewards of the above constructions by setting

$$\begin{aligned} \mathcal{I}_{\mathcal{W}_l} &: \text{Rep}_R^1(G_l) \xrightarrow{(\pi_l^*)^1} \text{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l)^{\text{cart}} \xrightarrow{\mathcal{I}_{\mathcal{W}_l}} \text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}} \xrightarrow{\pi_1} \text{Rep}_R^{\phi,I}(G) \\ \mathcal{R}_{\mathcal{W}_l} &: \text{Rep}_R^{\phi,I}(G) \xrightarrow{(\pi^*)^{\phi,I}} \text{Coef}_R^{\phi,I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}} \xrightarrow{\mathcal{R}_{\mathcal{W}_l}} \text{Coef}_R^1(\mathcal{B}'_{l,\bullet/e}/G_l)^{\text{cart}} \xrightarrow{\pi_{l,1}} \text{Rep}_R^1(G_l) \end{aligned}$$

3.4.11 Theorem. — Let \mathcal{W}_l be a Heisenberg–Weil coefficient system as in Definition 3.3.5. The two above functors are quasi-inverse equivalences of categories :

$$\mathcal{I}_{\mathcal{W}_l} : \text{Rep}_R^1(G_l) \xrightleftharpoons{\sim} \text{Rep}_R^{\phi, I}(G) : \mathcal{R}_{\mathcal{W}_l}.$$

Moreover, for any e -vertex $\mathcal{F}_0 \in \mathcal{B}'_{l, \bullet/e}$ and any representation ρ of G_{l, \mathcal{F}_0} that is trivial on $G_{l, \mathcal{F}_0, 0+}$, there is an isomorphism of RG -modules

$$\mathcal{I}_{\mathcal{W}_l}(\text{ind}_{G_{l, \mathcal{F}_0}}^{G_l}(\rho)) \simeq \text{ind}_{K_{l, \mathcal{F}_0}}^G(\mathcal{W}_{l, \mathcal{F}_0} \otimes_R \rho)$$

through which $\mathcal{I}_{\mathcal{W}_l}$ induces an isomorphism of R -algebras

$$\text{End}_{RG_l}(\text{ind}_{G_{l, \mathcal{F}_0}}^{G_l}(\rho)) \xrightarrow{\sim} \text{End}_{RG}(\text{ind}_{K_{l, \mathcal{F}_0}}^G(\mathcal{W}_{l, \mathcal{F}_0} \otimes_R \rho)).$$

Proof. The first statement is now a consequence of Theorem 3.4.6, (3.2.4) and (3.2.6). To prove the second statement, we first note that the following diagrams are 1-commutative (i.e., commutative up to isomorphism of functors).

$$\begin{array}{ccc} \text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G) & \xrightarrow{\pi_!} & \text{Rep}_R^{\phi, I}(G) \\ \mathcal{I}_{\mathcal{W}_l} \uparrow \downarrow \mathcal{R}_{\mathcal{W}_l} & & \mathcal{I}_{\mathcal{W}_l} \uparrow \downarrow \mathcal{R}_{\mathcal{W}_l} \\ \text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l) & \xrightarrow{\pi_{l,!}} & \text{Rep}_R^1(G_l) \end{array}$$

Indeed, this follows by adjunction from the 1-commutativity of the following diagram, which holds by construction.

$$\begin{array}{ccc} \text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G) & \xleftarrow{(\pi^*)^{\phi, I}} & \text{Rep}_R^{\phi, I}(G) \\ \mathcal{I}_{\mathcal{W}_l} \uparrow \downarrow \mathcal{R}_{\mathcal{W}_l} & & \mathcal{I}_{\mathcal{W}_l} \uparrow \downarrow \mathcal{R}_{\mathcal{W}_l} \\ \text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l) & \xleftarrow{(\pi_l^*)^1} & \text{Rep}_R^1(G_l) \end{array}$$

Now, denote by $\mathcal{V}_{l, \mathcal{F}_0, \rho}$ the coefficient system on \mathcal{B}'_l defined by

$$\begin{aligned} \mathcal{V}_{l, \mathcal{F}_0, \rho}(\mathcal{F}) &:= \mathcal{C}^\infty(G_{l, \mathcal{F}\mathcal{F}_0}) \otimes_{RG_{l, \mathcal{F}_0}} \rho, \quad \forall \mathcal{F} \in \mathcal{B}'_{l, \bullet/e} \\ g_{\mathcal{F}} : \mathcal{V}_{l, \mathcal{F}_0, \rho}(\mathcal{F}) &\xrightarrow{g \otimes \text{Id}} \mathcal{V}_{l, \mathcal{F}_0, \rho}(g\mathcal{F}), \quad \forall \mathcal{F} \in \mathcal{B}'_{l, \bullet/e}, \forall g \in G \\ \beta_{\mathcal{F}, \mathcal{F}'} &= 0, \quad \forall \mathcal{F}, \mathcal{F}' \text{ s.t. } \overline{\mathcal{F}'} \subsetneq \overline{\mathcal{F}} \end{aligned}$$

Observe that $\mathcal{V}_{l, \mathcal{F}_0, \rho}(\mathcal{F}) = \{0\}$ unless \mathcal{F} is G_l -conjugate to \mathcal{F}_0 , hence it follows that

$$\pi_{l,!}(\mathcal{V}_{l, \mathcal{F}_0, \rho}) = \text{ind}_{G_{l, \mathcal{F}_0}}^{G_l}(\rho).$$

By definition, $\mathcal{V}_{l, \mathcal{F}_0, \rho}$ belongs to $\text{Coef}_R^1(\mathcal{B}'_{l, \bullet/e}/G_l)$ and $\mathcal{I}_{\mathcal{W}_l}(\mathcal{V}_{l, \mathcal{F}_0, \rho})$ is again a coefficient system with trivial face maps, that vanishes outside the G -orbit of \mathcal{F}_0 and is given by

$$\mathcal{I}_{\mathcal{W}_l}(\mathcal{V}_{l, \mathcal{F}_0, \rho})(\mathcal{F}) = \mathcal{C}^\infty(G_{\mathcal{F}\mathcal{F}_0}) \otimes_{RK_{l, \mathcal{F}_0}}(\mathcal{W}_{l, \mathcal{F}_0} \otimes_R \rho)$$

for all facets $\mathcal{F} \in \mathcal{B}'_{\bullet/e}$. It follows that

$$\pi_!(\mathcal{I}_{\mathcal{W}_l}(\mathcal{V}_{l,\mathcal{F}_0,\rho})) = \text{ind}_{K_{l,\mathcal{F}_0}}^G(\mathcal{W}_{l,\mathcal{F}_0} \otimes_R \rho).$$

By the first 1-commutative diagram above, we conclude that

$$\mathcal{I}_{\mathcal{W}_l}(\text{ind}_{G_{l,\mathcal{F}_0}}^{G_l}(\rho)) = \mathcal{I}_{\mathcal{W}_l}(\pi_{l,!}(\mathcal{V}_{l,\mathcal{F}_0,\rho})) \simeq \pi_!(\mathcal{I}_{\mathcal{W}_l}(\mathcal{V}_{l,\mathcal{F}_0,\rho})) = \text{ind}_{K_{l,\mathcal{F}_0}}^G(\mathcal{W}_{l,\mathcal{F}_0} \otimes_R \rho).$$

Finally, the last statement follows from the second one and the fact that $\mathcal{I}_{\mathcal{W}_l}$ is an equivalence of categories. \square

3.4.12 Dependence on the choices. — Our construction depends on two choices: the integer e and the Heisenberg–Weil coefficient system \mathcal{W}_l . Let us first fix e , and rename $\mathcal{I}_{\mathcal{W}_l}^e : \text{Rep}_R^1(G_l) \rightleftharpoons \text{Rep}_R^{\phi,I}(G) : \mathcal{R}_{\mathcal{W}_l}^e$ the pair of inverse equivalences of Theorem 3.4.11, to emphasize dependence on e . The effect of changing \mathcal{W}_l is clear from Proposition 3.3.8. Given another Heisenberg–Weil coefficient system \mathcal{W}'_l , and with L, θ as in that statement, we have

$$(3.4.13) \quad \mathcal{I}_{\mathcal{W}'_l}^e = \mathcal{I}_{\mathcal{W}_l}^e \circ (L_\theta \otimes_R -) \text{ and } \mathcal{R}_{\mathcal{W}'_l}^e = (L_\theta)^{\otimes -1} \otimes_R - \circ \mathcal{R}_{\mathcal{W}_l}^e.$$

Now, let us change e to some multiple e' . There is a functor $[\mathcal{B}'_{\bullet/e'}/G] \rightarrow [\mathcal{B}'_{\bullet/e}/G]$ that takes an e' -facet \mathcal{G} to the unique e -facet $\mathcal{F}_e(\mathcal{G})$ that contains \mathcal{G} . Composing with this functor induces a linear exact functor $\text{Coef}_R^{\phi,I}([\mathcal{B}'_{\bullet/e'}/G]) \xrightarrow{\mathcal{C}_{e,e'}} \text{Coef}_R^{\phi,I}([\mathcal{B}'_{\bullet/e}/G])$, which satisfies $\mathcal{C}_{e,e'} \circ (\pi^*)_e^{\phi,I} = (\pi^*)_{e'}^{\phi,I}$ (here we use the notation of (3.2.4) with an extra index emphasizing dependence on e). In particular, $\mathcal{C}_{e,e'}$ is an equivalence of categories. Similarly, we have a functor $[\mathcal{B}'_{l,\bullet/e'}/G_l] \rightarrow [\mathcal{B}'_{l,\bullet/e}/G_l]$ that induces an equivalence of categories $\text{Coef}_R^1([\mathcal{B}'_{l,\bullet/e'}/G_l]) \xrightarrow{\mathcal{C}_{l,e,e'}} \text{Coef}_R^1([\mathcal{B}'_{l,\bullet/e}/G_l])$ such that $\mathcal{C}_{l,e,e'} \circ (\pi_l^*)_e^1 = (\pi_l^*)_{e'}^1$. Finally, composing with $[\mathcal{B}'_{l,\bullet/e'}/K_l] \rightarrow [\mathcal{B}'_{l,\bullet/e}/K_l]$, any Heisenberg–Weil coefficient system \mathcal{W}_l for the e -facet subdivision induces a Heisenberg–Weil coefficient system $\mathcal{C}_{l,e,e'}(\mathcal{W}_l) =: \mathcal{W}'_l$ for the e' -facet subdivision.

Proposition. — *With the above notation, the equivalences of categories $\mathcal{R}_{\mathcal{W}_l}^e, \mathcal{I}_{\mathcal{W}_l}^e$, resp. $\mathcal{R}_{\mathcal{W}'_l}^{e'}, \mathcal{I}_{\mathcal{W}'_l}^{e'}$ of Theorem 3.4.11 respectively associated to (e, \mathcal{W}_l) and (e', \mathcal{W}'_l) satisfy*

$$\mathcal{R}_{\mathcal{W}_l}^e \simeq \mathcal{R}_{\mathcal{W}'_l}^{e'} \text{ and } \mathcal{I}_{\mathcal{W}_l}^e \simeq \mathcal{I}_{\mathcal{W}'_l}^{e'}.$$

Proof. It follows from their construction that the functors of Theorem 3.4.6 satisfy the commutation relations $\mathcal{C}_{l,e,e'} \circ \mathcal{R}_{\mathcal{W}_l}^e = \mathcal{R}_{\mathcal{W}'_l}^{e'} \circ \mathcal{C}_{e,e'}$ and $\mathcal{C}_{e,e'} \circ \mathcal{I}_{\mathcal{W}_l}^e = \mathcal{I}_{\mathcal{W}'_l}^{e'} \circ \mathcal{C}_{l,e,e'}$, from which the proposition follows. \square

3.4.14 Corollary. — *Let \mathcal{W}_l be a Heisenberg–Weil coefficient system and $\mathcal{I}_{\mathcal{W}_l}$ the equivalence of categories of Theorem 3.4.11. Then for any $x \in \mathcal{B}'_l$ and any R -representation ρ of $G_{l,x}$ that is trivial on $G_{l,x,0^+}$, there is an isomorphism of RG -modules*

$$\mathcal{I}_{\mathcal{W}_l}(\text{ind}_{G_{l,x}}^{G_l}(\rho)) \simeq \text{ind}_{K_{l,x}}^G(\mathcal{W}_{l,x} \otimes_R \rho).$$

Here $\mathcal{W}_{l,x}$ denotes the $RK_{l,x}$ -module $\mathcal{W}_{l,\mathcal{F}}$ where \mathcal{F} is the e -facet that contains x .

Proof. The case where \mathcal{F} is an e -vertex of $\mathcal{B}'_{\iota, \bullet/e}$ is done in Theorem 3.4.11. In general, let y be the barycenter of \mathcal{F} . Then y becomes a vertex in $\mathcal{B}'_{\iota, \bullet/e'}$ for some multiple e' of e . Thanks to the last proposition, we deduce from Theorem 3.4.11 that $\mathcal{I}_{\mathcal{W}_\iota}(\text{ind}_{G_{\iota, y}}^{G_\iota}(\rho)) \simeq \text{ind}_{K_{\iota, y}}^G(\mathcal{W}_{\iota, y} \otimes_R \rho)$, and we conclude since $K_{\iota, x} = K_{\iota, \mathcal{F}} = K_{\iota, y}$, $\mathcal{W}_{\iota, x} = \mathcal{W}_{\iota, \mathcal{F}} = \mathcal{W}_{\iota, y}$ and accordingly for $G_{\iota, x}$ and $G_{\iota, x, 0+}$. \square

3.4.15 Remark. — We will see in Corollary 3.7.11 below that given a point $x \in \mathcal{B}'_\iota$ and any twisted $RK_{\iota, x}$ -Heisenberg–Weil representation $\kappa_{\iota, x}$ (as defined in Definition 3.6.8 below) whose restriction to $K_{\iota, x}^+$ is $\check{\phi}_{\iota, x}^+$ -isotypic, there exists a Heisenberg–Weil coefficient system such that $\mathcal{W}_{\iota, x} \simeq \kappa_{\iota, x}$.

3.4.16 Proof of Theorem 3.1.1 (admitting the results of Section 3.3). — That theorem, and actually a more precise version of it, now follows from Theorem 3.4.11 and Corollary 3.4.14. Of course, it still remains to prove the results claimed in Section 3.3 which we have assumed in the above proof. This is the subject of the next subsections.

3.5 Heisenberg representations

In this subsection, we adapt the theory of Heisenberg representations to coefficients in any commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R . As in the previous subsections, we have fixed an embedding $\iota \in I$. According to 2.5.13, for any $x \in \mathcal{B}_\iota$, the triple $(K_{\iota, x}^\dagger, K_{\iota, x}^+, \check{\phi}_{\iota, x}^+)$ is a “Heisenberg triple” in the following sense.

3.5.1 Definition.— A triple $(K^\dagger, K^+, \check{\phi})$ consisting of a pro- p -group K^\dagger , an open normal subgroup $K^+ \subset K^\dagger$ and a continuous homomorphism $\check{\phi} : K^+ \rightarrow \mu_{p^\infty}$ centralized by K^\dagger , is called a *Heisenberg triple* if K^\dagger/K^+ is an \mathbb{F}_p -vector space and the map $(k, k') \mapsto \check{\phi}(kk'k^{-1}(k')^{-1})$ defines a $(\mu_p$ -valued) perfect pairing on K^\dagger/K^+ .

Note that the pairing is necessarily symplectic since $p \neq 2$. For any commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R , we still denote by $\check{\phi}$ the character $K^\dagger \rightarrow \mu_{p^\infty} \subset R^\times$, and we denote by $e \in \mathcal{H}_R(K^\dagger)$ the corresponding idempotent, which is central and supported on K^+ . The category of smooth RK^\dagger -modules whose restriction to K^+ is $\check{\phi}$ -isotypic identifies with the category $e\mathcal{H}_R(K^\dagger) - \text{Mod}$ of modules over the finite R -algebra $e\mathcal{H}_R(K^\dagger)$.

3.5.2 Lemma. — Let $(K^\dagger, K^+, \check{\phi})$ be a Heisenberg triple. For a commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R , there is a smooth RK^\dagger -module η such that

- i) η is finitely generated and projective over R .
- ii) $\eta|_{K^+}$ is $\check{\phi}$ -isotypic.
- iii) $\text{End}_{RK^\dagger}(\eta) = R$.

Any such η has R -rank $\sqrt{[K^\dagger : K^+]}$ and is a projective generator of $e\mathcal{H}_R(K^\dagger) - \text{Mod}$. In particular, we have adjoint equivalences of categories

$$\text{Hom}_{RK^\dagger}(\eta, -) : e\mathcal{H}_R(K^\dagger) - \text{Mod} \rightleftarrows R - \text{Mod} : \eta \otimes_R -$$

Moreover, if η' is another RK^\dagger -module satisfying (i), (ii) and (iii), then $\text{Hom}_{RK^\dagger}(\eta, \eta')$ is an invertible R -module (i.e. finitely generated projective of rank 1).

Proof. We first construct a representation η that satisfies (i), (ii) and (iii). Set $V := K^\dagger/K^+$, pick a Lagrangian subspace $W \subseteq V$ and denote by $K_W^\dagger \subseteq K^\dagger$ the preimage of W in K^\dagger . We thus have $[K^\dagger : K_W^\dagger] = [K_W^\dagger : K^+] = \sqrt{[K^\dagger : K^+]}$, and we denote this integer by a . There are exactly a distinct characters $\nu : K_W^\dagger \rightarrow \mu_{p^\infty} \subseteq R^\times$ that extend $\check{\phi}$ and they are transitively permuted under the conjugation action of K^\dagger . Now pick such a character ν , denote by R_ν the associated RK_W^\dagger -module with underlying R -module R , and put

$$\eta := \text{ind}_{K_W^\dagger}^{K^\dagger} R_\nu.$$

This certainly satisfies (i) and (ii). Moreover, we have $(\text{ind}_{K_W^\dagger}^{K^\dagger} R_\nu)|_{K_W^\dagger} = \bigoplus_{\nu'} R_{\nu'}$ by the Mackey formula, so (iii) follows from Frobenius reciprocity. For the same reason, the R -module $\text{Hom}_{RK^\dagger}(\eta, \text{ind}_{K_W^\dagger}^{K^\dagger} R_{\nu'})$ is free of rank 1 for any character $\nu' : K_W^\dagger \rightarrow \mu_{p^\infty} \subseteq R^\times$ that extends $\check{\phi}$. Pick a generator f of this R -module. Its image contains the K_W^\dagger - ν -eigenspace of $\text{ind}_{K_W^\dagger}^{K^\dagger} R_{\nu'}$, which generates the latter RK^\dagger -module, hence f is surjective. Being a morphism of free R -modules of the same rank, it is also injective, hence an isomorphism. Now observe that each $R_{\nu'}$ is a projective RK_W^\dagger -module since K_W^\dagger is a pro- p -group and $p \in R^\times$. Therefore $\bigoplus_{\nu'} R_{\nu'}$ is a projective generator of $(e\mathcal{H}_R(K_W^\dagger)) - \text{Mod}$, and Frobenius reciprocity implies that η is a projective generator of $(e\mathcal{H}_R(K^\dagger)) - \text{Mod}$. This formally implies that the adjoint pair of functors

$$\text{Hom}_{RK^\dagger}(\eta, -) : (e\mathcal{H}_R(K^\dagger)) - \text{Mod} \rightleftarrows R - \text{Mod} : \eta \otimes_R -$$

are equivalences of categories. Finally, let η' be another smooth RK^\dagger -module satisfying (i), (ii) and (iii). Since it is R -projective, it is RK^\dagger -projective. Therefore, the R -module $P := \text{Hom}_{RK^\dagger}(\eta, \eta')$ is finitely generated projective and satisfies $\text{End}_R P = R$. So P has rank 1 and $\eta' \simeq \eta \otimes_R P$ is also a projective generator of $e\mathcal{H}_R(K^\dagger) - \text{Mod}$. \square

3.5.3 Definition.— A representation η as in the previous lemma will be called a *Heisenberg representation* for the Heisenberg triple $(K^\dagger, K^+, \check{\phi})$.

3.5.4 Note that, when R is a field or when $R = \overline{\mathbb{Z}}[\frac{1}{p}]$, any invertible R -module is free, so η is *uniquely determined* by properties (i), (ii) and (iii) up to isomorphism. In general, η is only unique up to isomorphism and twisting by an invertible R -module. In any case, we always have a canonical evaluation isomorphism

$$\eta \otimes_R \text{Hom}_{RK^\dagger}(\eta, \eta') \xrightarrow{\sim} \eta'$$

for any two choices of Heisenberg representations η and η' .

As already noted before, for any $x \in \mathcal{B}_l$, the triple $(K_{l,x}^\dagger, K_{l,x}^+, \check{\phi}_{l,x}^+)$ is a Heisenberg triple, by 2.5.13. We will generally denote by $\eta_{l,x}$ a Heisenberg representation for this triple. Note that this notion only depends on the e-facet $\mathcal{F}_e(x)$ of \mathcal{B}_l that contains x . This allows us to use the notation $\eta_{l,\mathcal{F}}$ instead of $\eta_{l,x}$ for the facet $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$ that contains x , whenever we need it.

3.5.5 Lemma. — Let $x, x' \in \mathcal{B}_l$ and choose Heisenberg representations $\eta_{l,x}$ and $\eta_{l,x'}$.

i) The R -module $\mathbb{H}_{x,x'} := \mathrm{Hom}_{R(K_{l,x}^\dagger \cap K_{l,x'}^\dagger)}(\eta_{l,x}, \eta_{l,x'})$ is invertible.

ii) The evaluation map $\eta_{l,x} \otimes_R \mathbb{H}_{x,x'} \longrightarrow \eta_{l,x'}$ factors as:

$$\eta_{l,x} \otimes_R \mathbb{H}_{x,x'} \twoheadrightarrow (e_{l,x',x} \eta_{l,x}) \otimes_R \mathbb{H}_{x,x'} \xrightarrow{\sim} e_{l,x,x'} \eta_{l,x'} \hookrightarrow \eta_{l,x'},$$

where $e_{l,x,x'}$ is the idempotent associated to the restriction of $\check{\phi}_{l,x}^+$ to $K_{l,x}^+ \cap K_{l,x'}^+$, and the first map is the projection obtained by applying this idempotent.

iii) If $\overline{\mathcal{F}_e(x)} \supseteq \mathcal{F}_e(x')$, then the evaluation maps induce:

(a) an $RK_{l,x'}^\dagger$ -isomorphism $\mathrm{ind}_{K_{l,x}^\dagger \cap K_{l,x'}^\dagger}^{K_{l,x'}^\dagger} \eta_{l,x} \otimes_R \mathbb{H}_{x,x'} \xrightarrow{\sim} \eta_{l,x'}$

(b) an $R(K_{l,x}^\dagger \cap K_{l,x'}^\dagger)$ -isomorphism $\eta_{l,x} \otimes_R \mathbb{H}_{x,x'} \xrightarrow{\sim} e_{l,x,x'} \eta_{l,x'}$.

Proof. i) By the previous lemma, we may assume $x \neq x'$. Let $(\mathbf{P}, \bar{\mathbf{P}})$ be the pair of F -rational opposite parabolic subgroups of \mathbf{G} constructed in 2.4.10. As usual, we denote by \mathbf{M} their common Levi component and by \mathbf{U} and $\bar{\mathbf{U}}$ the respective unipotent radicals. Then both $K_{l,x}^\dagger$ and $K_{l,x'}^\dagger$ have the Iwahori decomposition with respect to this pair, and the following equalities and inclusions hold:

$$(3.5.5a) \quad M \cap K_{l,x}^\dagger = M \cap K_{l,x'}^\dagger, \quad \bar{U} \cap K_{l,x}^+ \supseteq \bar{U} \cap K_{l,x'}^+, \quad U \cap K_{l,x}^\dagger \subseteq U \cap K_{l,x'}^+.$$

By Lemma 2.4.9, $\bar{U} \cap K_{l,x}^+$ acts trivially on $\eta_{l,x}$ and $U \cap K_{l,x'}^+$ acts trivially on $\eta_{l,x'}$. Hence any $K_{l,x}^\dagger \cap K_{l,x'}^\dagger$ -equivariant map $\eta_{l,x} \longrightarrow \eta_{l,x'}$ factors through $(\eta_{l,x})_{U \cap K_{l,x}^\dagger}$ and lands in $(\eta_{l,x'})^{\bar{U} \cap K_{l,x'}^\dagger}$. In view of the Iwahori decomposition

$$K_{l,x}^\dagger \cap K_{l,x'}^\dagger = (U \cap K_{l,x}^\dagger)(M \cap K_{l,x}^\dagger)(\bar{U} \cap K_{l,x'}^\dagger),$$

this means that

$$(3.5.5b) \quad \mathrm{Hom}_{R(K_{l,x}^\dagger \cap K_{l,x'}^\dagger)}(\eta_{l,x}, \eta_{l,x'}) = \mathrm{Hom}_{R(M \cap K_{l,x}^\dagger)}\left((\eta_{l,x})_{U \cap K_{l,x}^\dagger}, (\eta_{l,x'})^{\bar{U} \cap K_{l,x'}^\dagger}\right).$$

By Proposition 2.5.13 ii), the restriction to $(M \cap K_{l,x}^\dagger)/(M \cap K_{l,x}^+)$ of the bilinear form θ associated to $\check{\phi}_{l,x}^+$ is non-degenerate, so that $(M \cap K_{l,x}^\dagger, M \cap K_{l,x}^+, (\check{\phi}_{l,x}^+)|_{M \cap K_{l,x}^+})$ is a Heisenberg triple. We claim that both $(\eta_{l,x})_{U \cap K_{l,x}^\dagger}$ and $(\eta_{l,x'})^{\bar{U} \cap K_{l,x'}^\dagger}$ are such Heisenberg representations for this triple. According to the last statement of Lemma 3.5.2, this concludes our proof.

To prove the claim, we may argue with any particular choice of $\eta_{l,x}$, and in particular with a model $\eta_{l,x} = \mathrm{ind}_{K_W^\dagger}^{K_{l,x}^\dagger} R_\nu$ as in the proof of Lemma 3.5.2, for a suitable Lagrangian $W \subseteq K_{l,x}^\dagger/K_{l,x}^+$ and character ν . Namely, choose first W of the form $W_{\bar{U}} \oplus W_M$ where $W_{\bar{U}} = (\bar{U} \cap K_{l,x}^\dagger)/(\bar{U} \cap K_{l,x}^+)$ and W_M is a Lagrangian in $(M \cap K_{l,x}^\dagger)/(M \cap K_{l,x}^+)$, and pick any ν that is trivial on W_U . Then using Mackey's formula we see that $(\mathrm{ind}_{K_W^\dagger}^{K_{l,x}^\dagger} R_\nu)_{U \cap K_{l,x}^\dagger} = (\mathrm{ind}_{P \cap K_W^\dagger}^{P \cap K_{l,x}^\dagger} R_\nu)_{U \cap K_{l,x}^\dagger} \xrightarrow{\sim}$

$\text{ind}_{M \cap K_{\ell,x}^\dagger}^{M \cap K_{\ell,x}^\dagger} R_\nu$, which is indeed a Heisenberg representation for $(M \cap K_{\ell,x}^\dagger, \check{\phi}_{\ell,x}^+)$. We argue similarly for $(\eta_{\ell,x'})^{\bar{U} \cap K_{\ell,x'}^\dagger}$, by computing with a Lagrangian of the form $W_U \oplus W_M$ in $K_{\ell,x'}^\dagger / K_{\ell,x'}^+$.

ii) By the above discussion, the evaluation map $\eta_{\ell,x} \otimes_R \mathbb{H}_{x,x'} \longrightarrow \eta_{\ell,x'}$ factorizes as

$$(3.5.5c) \quad \eta_{\ell,x} \otimes_R \mathbb{H}_{x,x'} \twoheadrightarrow (\eta_{\ell,x})_{U \cap K_{\ell,x}^\dagger} \otimes_R \mathbb{H}_{x,x'} \xrightarrow{\sim} (\eta_{\ell,x'})^{\bar{U} \cap K_{\ell,x'}^\dagger} \hookrightarrow \eta_{\ell,x'},$$

where the map in the middle is an isomorphism of $R(M \cap K_{\ell,x}^\dagger)$ -modules and is also an evaluation map through the identification (3.5.5b).

On the other hand, by (3.5.5a) we have $K_{\ell,x}^+ \cap K_{\ell,x'}^\dagger = (K_{\ell,x}^+ \cap K_{\ell,x'}^+)(\bar{U} \cap K_{\ell,x'}^\dagger)$. Since $\check{\phi}_{\ell,x}^+$ is trivial on $\bar{U} \cap K_{\ell,x}^+$ and coincides with $\check{\phi}_{\ell,x'}^+$ on $K_{\ell,x}^+ \cap K_{\ell,x'}^+$, we have in $\mathcal{H}_R(K_{\ell,x'}^\dagger)$ the equality $e_{\ell,x,x'} e_{\ell,x'} = e_{\bar{U} \cap K_{\ell,x}^\dagger} e_{\ell,x'}$, and therefore $(\eta_{\ell,x'})^{\bar{U} \cap K_{\ell,x'}^\dagger} = e_{\ell,x,x'} \eta_{\ell,x'}$. Exchanging the roles of x and x' , we also have $e_{\ell,x',x} e_{\ell,x} = e_{U \cap K_{\ell,x}^\dagger} e_{\ell,x}$ in $\mathcal{H}_R(K_{\ell,x}^\dagger)$. Therefore, the projection $\eta_{\ell,x} \twoheadrightarrow (\eta_{\ell,x})_{U \cap K_{\ell,x}^\dagger}$ factors as $\eta_{\ell,x} \twoheadrightarrow e_{\ell,x',x} \eta_{\ell,x} \xrightarrow{\sim} (\eta_{\ell,x})_{U \cap K_{\ell,x}^\dagger}$, where the first map is given by the action of $e_{\ell,x',x}$. This completes the proof of ii).

iii) Under the assumption that $\mathcal{F}_e(x') \subseteq \overline{\mathcal{F}_e(x)}$, the segment $[x, x']$ has a non-empty open intersection with the facet $\mathcal{F}_e(x)$. By definition of the e -facet decomposition, it follows that for each $i = 0, \dots, d$, we have $U \cap G_{\ell,x,(r_{i-1}/2)+}^i = U \cap G_{\ell,x,(r_{i-1}/2)}^i$ and $\bar{U} \cap G_{\ell,x,(r_{i-1}/2)+}^i = \bar{U} \cap G_{\ell,x,(r_{i-1}/2)}^i$, from which we deduce that $U \cap K_{\ell,x}^\dagger = U \cap K_{\ell,x}^+$ and $\bar{U} \cap K_{\ell,x}^\dagger = \bar{U} \cap K_{\ell,x}^+$. In particular, we have $K_{\ell,x}^\dagger = (M \cap K_{\ell,x}^\dagger) K_{\ell,x}^+$ and $(\eta_{\ell,x})^{U \cap K_{\ell,x}^\dagger} = \eta_{\ell,x} = (\eta_{\ell,x})_{U \cap K_{\ell,x}^\dagger}$, so that iii)(b) follows from ii).

Now, pick a Lagrangian subspace W_M in $(M \cap K_{\ell,x}^\dagger) / (M \cap K_{\ell,x}^+) = K_{\ell,x}^\dagger / K_{\ell,x}^+$, denote by K_{ℓ,x,W_M}^\dagger its preimage in $K_{\ell,x}^\dagger$, and pick a character ν of K_{ℓ,x,W_M}^\dagger that extends $\check{\phi}_{\ell,x}^+$. Then we know that $\text{ind}_{K_{\ell,x,W_M}^\dagger}^{K_{\ell,x}^\dagger} \nu$ is a Heisenberg representation for $\check{\phi}_{\ell,x}^+$, and we observe that

$$(\text{ind}_{K_{\ell,x,W_M}^\dagger}^{K_{\ell,x}^\dagger} \nu)|_{K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger} = \text{ind}_{K_{\ell,x,W_M}^\dagger \cap K_{\ell,x'}^\dagger}^{K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger} \nu$$

because $K_{\ell,x}^\dagger = K_{\ell,x}^+(K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger)$ and $K_{\ell,x,W_M}^\dagger = K_{\ell,x}^+(K_{\ell,x,W_M}^\dagger \cap K_{\ell,x'}^\dagger)$. Now, the point is that the group $K_{\ell,x,W_M}^\dagger \cap K_{\ell,x'}^\dagger$ is the preimage of the Lagrangian subspace $W_M \oplus (\bar{U} \cap K_{\ell,x'}^\dagger) / (\bar{U} \cap K_{\ell,x'}^+)$ of $K_{\ell,x'}^\dagger / K_{\ell,x'}^+$, while the restriction of ν to $K_{\ell,x}^+$ coincides with $\check{\phi}_{\ell,x'}^+$ (by Lemma 2.4.9). It follows that

$$\text{ind}_{K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger}^{K_{\ell,x'}^\dagger} (\text{ind}_{K_{\ell,x,W_M}^\dagger}^{K_{\ell,x}^\dagger} \nu) = \text{ind}_{K_{\ell,x,W_M}^\dagger \cap K_{\ell,x'}^\dagger}^{K_{\ell,x'}^\dagger} \nu$$

is a Heisenberg representation for $\check{\phi}_{\ell,x'}^+$. This proves iii)(a) when $\eta_{\ell,x} = \text{ind}_{K_{\ell,x,W_M}^\dagger}^{K_{\ell,x}^\dagger} \nu$ and $\eta_{\ell,x'} =$

$\text{ind}_{K_{\ell,x,W_M}^\dagger \cap K_{\ell,x'}^\dagger}^{K_{\ell,x'}^\dagger} \nu$, and the general case follows from the last statement of Lemma 3.5.2. \square

3.5.6 Lemma. — *Suppose given three points $x, x', x'' \in \mathcal{B}_\ell$ and Heisenberg representations $\eta_{\ell,x}, \eta_{\ell,x'},$ and $\eta_{\ell,x''}$ of $K_{\ell,x}^\dagger, K_{\ell,x'}^\dagger,$ and $K_{\ell,x''}^\dagger$, respectively. Assume that*

$$(K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger \cap K_{\ell,x''}^\dagger)(K_{\ell,x}^+ \cap K_{\ell,x''}^+) = K_{\ell,x}^\dagger \cap K_{\ell,x''}^\dagger.$$

Then the inclusion $\mathrm{Hom}_{R(K_{\ell,x}^\dagger \cap K_{\ell,x''}^\dagger)}(\eta_{\ell,x}, \eta_{\ell,x''}) \subseteq \mathrm{Hom}_{R(K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger \cap K_{\ell,x''}^\dagger)}(\eta_{\ell,x}, \eta_{\ell,x''})$ is an equality, and the resulting composition map

$$\mathrm{Hom}_{R(K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger)}(\eta_{\ell,x}, \eta_{\ell,x'}) \otimes_R \mathrm{Hom}_{R(K_{\ell,x'}^\dagger \cap K_{\ell,x''}^\dagger)}(\eta_{\ell,x'}, \eta_{\ell,x''}) \longrightarrow \mathrm{Hom}_{R(K_{\ell,x}^\dagger \cap K_{\ell,x''}^\dagger)}(\eta_{\ell,x}, \eta_{\ell,x''})$$

is an isomorphism.

Proof. By definition, $(\eta_{\ell,x})|_{K_{\ell,x}^+ \cap K_{\ell,x''}^+}$ is $\check{\phi}_{\ell,x}^+$ -isotypic, while $(\eta_{\ell,x''})|_{K_{\ell,x}^+ \cap K_{\ell,x''}^+}$ is $\check{\phi}_{\ell,x''}^+$ -isotypic. By lemma 2.5.12, both characters $\check{\phi}_{\ell,x}^+$ and $\check{\phi}_{\ell,x''}^+$ agree on $K_{\ell,x}^+ \cap K_{\ell,x''}^+$. Therefore, any R -linear map $\eta_{\ell,x} \longrightarrow \eta_{\ell,x''}$ is actually $K_{\ell,x}^+ \cap K_{\ell,x''}^+$ -equivariant, and the claimed equality of the lemma follows immediately from its hypothesis.

In particular, the composition map of the lemma is well defined. Let us first prove that it is non-zero. To this aim, according to the factorization in point ii) of Lemma 3.5.5, it is enough to prove that the map $e_{\ell,x,x'} \eta_{\ell,x'} \xrightarrow{e_{\ell,x'',x'} * -} e_{\ell,x'',x'} \eta_{\ell,x'}$ is non-zero. Since $\eta_{\ell,x'}$ is a projective generator of $\mathcal{H}_R(K_{\ell,x'}^\dagger) e_{\ell,x'} - \mathrm{Mod}$ with endomorphism ring R , the evaluation map

$$\eta_{\ell,x'} \otimes_R H \xrightarrow{\sim} \mathcal{H}_R(K_{\ell,x'}^\dagger) e_{\ell,x'}, \text{ with } H := \mathrm{Hom}_{RK_{\ell,x'}^\dagger}(\eta_{\ell,x'}, \mathcal{H}_R(K_{\ell,x'}^\dagger) e_{\ell,x'})$$

is an isomorphism of $RK_{\ell,x'}^\dagger$ -modules. Therefore, it is enough to show that the product $e_{\ell,x'',x'} e_{\ell,x,x'}$ is non-zero in $\mathcal{H}_R(K_{\ell,x'}^\dagger) e_{\ell,x'}$. As in (2.5.4), this is equivalent to $\check{\phi}_{\ell,x}^+$ and $\check{\phi}_{\ell,x''}^+$ agreeing on $(K_{\ell,x}^+ \cap K_{\ell,x'}^+) \cap (K_{\ell,x'}^+ \cap K_{\ell,x''}^+)$. But this follows from Lemma 2.5.12.

We have just proved that the composition map of the lemma is non-zero, for any commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R . Since it is compatible with change of scalars, this means it remains non-zero after any such change of scalars, in particular after reducing modulo any maximal ideal. Now, this composition map is an R -linear map between two invertible R -modules. Since it is non-zero modulo any maximal ideal, it is an isomorphism. \square

3.5.7 Corollary. — *Let C be a chamber in \mathcal{B}'_ℓ (i.e., facet of maximal dimension for the usual Bruhat–Tits polysimplicial structure on \mathcal{B}'_ℓ), and let x, x', x'' be three points in the closure \bar{C} of C , such that x'' is in C and $\mathcal{F}_e(x') \subseteq \mathcal{F}_e(x)$. Then the composition map*

$$\mathrm{Hom}_{R(K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger)}(\eta_{\ell,x}, \eta_{\ell,x'}) \otimes_R \mathrm{Hom}_{R(K_{\ell,x'}^\dagger \cap K_{\ell,x''}^\dagger)}(\eta_{\ell,x'}, \eta_{\ell,x''}) \longrightarrow \mathrm{Hom}_{R(K_{\ell,x}^\dagger \cap K_{\ell,x''}^\dagger)}(\eta_{\ell,x}, \eta_{\ell,x''})$$

is well defined and is an isomorphism.

Proof. Since $\mathcal{F}_e(x') \subseteq \overline{\mathcal{F}_e(x)}$, we have for all $i > 0$, the inclusion $G_{\ell,x,r_{i-1}/2}^i \subseteq G_{\ell,x',r_{i-1}/2}^i$, from which we get the equality $K_{\ell,x}^\dagger = (K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger) G_{\ell,x,0+}$. In particular, an element $g \in K_{\ell,x}^\dagger \cap K_{\ell,x''}^\dagger$ can be written $g = kg_\ell$ with $k \in K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger$ and $g_\ell \in G_{\ell,x,0+}$. On the other hand, since $x \in \bar{C}$ and $x'' \in C$, we have $G_{\ell,x,0+} \subseteq G_{\ell,C,0+} = G_{\ell,x'',0+}$, hence also $G_{\ell,x,0+} \subseteq K_{\ell,x}^+ \cap K_{\ell,x''}^+ \subseteq K_{\ell,x''}^+$. It follows that $g_\ell \in K_{\ell,x''}^+$, hence also $k \in K_{\ell,x''}^+$. So we have proved $K_{\ell,x}^\dagger \cap K_{\ell,x''}^\dagger = (K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger \cap K_{\ell,x''}^\dagger) G_{\ell,x,0+}$, which implies $K_{\ell,x}^\dagger \cap K_{\ell,x''}^\dagger = (K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger \cap K_{\ell,x''}^\dagger) (K_{\ell,x}^+ \cap K_{\ell,x''}^+)$. It remains to apply the previous lemma. \square

3.6 A family of Heisenberg–Weil representations

The aim of this section is to construct a family $(\kappa_{\iota,x})_{x \in \mathcal{B}'_\iota}$ of smooth $RK_{\iota,x}$ -modules such that :

- i) for all $x \in \mathcal{B}'_\iota$, the restriction $(\kappa_{\iota,x})|_{K_{\iota,x}^\dagger}$ is a Heisenberg representation for $\check{\phi}_{\iota,x}^+$,
- ii) for any two $x, x' \in \mathcal{B}'_\iota$, the inclusion $\text{Hom}_{R(K_{\iota,x} \cap K_{\iota,x'})}(\kappa_{\iota,x}, \kappa_{\iota,x'}) \subseteq \text{Hom}_{R(K_{\iota,x}^\dagger \cap K_{\iota,x'}^\dagger)}(\kappa_{\iota,x}, \kappa_{\iota,x'})$ is an equality.

Working over the base ring $\overline{\mathbb{Z}}[\frac{1}{p}]$, it would not be difficult to get a non-constructive proof of existence of $RK_{\iota,x}$ -modules satisfying i). However, the lack of uniqueness of such extensions makes it difficult to choose them such that ii) is also satisfied. Following [Yu01, §11] and [Gér77], the theory of Heisenberg–Weil representations produces explicit extensions that are even already defined over the base ring $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$, although they still do not satisfy ii). To ensure property ii), we will twist these extensions by the quadratic characters introduced in [FKS23] and [AFMO24b].

3.6.1 Generalized Heisenberg–Weil representation over $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebras – an explicit model. — Let C be a cyclic group of order p^N for some positive integer N and denote by μ_p its subgroup of order p . Let V be an \mathbb{F}_p -vector space with a non-degenerate symplectic form θ valued in $\mu_p \subseteq C$. We define the group $C \boxtimes V$ to have underlying set $C \times V$ and group law given by

$$(a_1, v_1) \cdot (a_2, v_2) = (a_1 \cdot a_2 \cdot \sqrt{\theta(v_1, v_2)}, v_1 + v_2).$$

Note that the special case of $N = 1$ yields the usual Heisenberg \mathbb{F}_p -group $\mu_p \boxtimes V$.

If $\nu : C \rightarrow \mu_{p^\infty}$ is an injective homomorphism, the triple $(C \boxtimes V, C, \nu)$ is a Heisenberg triple in the sense of Definition 3.5.1. Pick a Lagrangian subspace $W \subseteq V$ and, for any commutative $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{p}]$ -algebra R , set

$$\eta_{\nu,W,R} := \text{Ind}_{C \times W}^{C \boxtimes V} (\nu \times 1),$$

where ν is seen as a character $C \rightarrow \mu_{p^\infty} \subset R^\times$. This is a Heisenberg representation in the sense of Definition 3.5.3. We drop the subscript R and simply write $\eta_{\nu,W}$ instead of $\eta_{\nu,W,R}$ if R is clear from the context. We let $\text{Sp}(V)$ act on $C \boxtimes V$ by acting on the second coordinate, i.e., for $g \in \text{Sp}(V)$ and $(a, v) \in C \boxtimes V$, we have $g(a, v) = (a, gv)$. Note that this action preserves the character ν of the center $C = C \times \{0\} \subseteq C \boxtimes V$ and it preserves the subgroup $\mu_p \boxtimes V$.

If $R = \mathbb{C}$, the restriction of $\eta_{\nu,W,\mathbb{C}}$ to $\mu_p \boxtimes V$ is a Heisenberg representation in the classical sense, with central character $\nu|_{\mu_p}$. According to [Gér77, Theorem 2.4], there is a representation $\omega_{\mathbb{C}}$ of $\text{Sp}(V)$ on the vector space $V_{\eta_{\nu,W,\mathbb{C}}}$ underlying $\eta_{\nu,W,\mathbb{C}}$, such that the product map

$$\omega_{\mathbb{C}} \times \eta_{\nu,W,\mathbb{C}} : \text{Sp}(V) \times (\mu_p \boxtimes V) \rightarrow \text{Aut}_{\mathbb{C}}(V_{\eta_{\nu,W,\mathbb{C}}})$$

is a homomorphism. Since $\text{End}_{\mathbb{C}(\mu_p \boxtimes V)}(\eta_{\nu,W,\mathbb{C}}) = \mathbb{C}$, this property makes $\omega_{\mathbb{C}}$ unique up to a twist by a character, and therefore unique, since $\text{Sp}(V) = [\text{Sp}(V), \text{Sp}(V)]$, unless $p = 3$ and $\dim_{\mathbb{F}_p} V = 2$, in which case, we follow the choice made in [Gér77, Theorem 2.4].

Lemma. — *The submodule $V_{\eta_{\nu,W,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}}$ of $V_{\eta_{\nu,W,\mathbb{C}}}$ is stable under $\omega_{\mathbb{C}}(\text{Sp}(V))$.*

Proof. Pick a Lagrangian subspace $W^- \subset V$ that satisfies $V = W \oplus W^-$. Restriction of functions from $C\boxtimes V$ to $\{1\} \times W^-$ provides an isomorphism from the \mathbb{C} -vector space $V_{\eta_{\nu,W},\mathbb{C}}$ to the space $\mathbb{C}[W^-]$ of \mathbb{C} -valued functions on W^- , which also sends the $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -submodule $V_{\eta_{\nu,W},\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}$ onto the submodule $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}][W^-]$ of $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -valued functions on W^- . The proof of [Gér77, Theorem 2.4] contains an explicit description of the representation $\omega_{\mathbb{C}}$ transported on $\mathbb{C}[W^-]$ via the above isomorphism. The group $\mathrm{Sp}(V)$ is generated by the parabolic subgroup P that stabilizes $W \subset V$ and an element $s \in \mathrm{Sp}(V)$ that exchanges W and W^- . Gérardin ([Gér77, (2.7) and (2.8)]) shows that the action of P is given by

$$\begin{aligned} (\omega_{\mathbb{C}}(p)(F))(y) &= \mathrm{sgn}(\det_{\mathbb{F}_p}(p|_W))F(p^{-1}(y)) && \text{if } p \in P \text{ stabilizes } W^- \\ (\omega_{\mathbb{C}}(p)(F))(y) &= \chi'(p)F(y) && \text{if } p \in P \text{ fixes } W \text{ pointwise,} \end{aligned}$$

for $F \in \mathbb{C}[W^-]$ and some explicit quadratic character χ' . Moreover, by [Gér77, (2.18)], we have for $F \in \mathbb{C}[W^-]$ and $y \in W^-$

$$(3.6.1a) \quad (\omega_{\mathbb{C}}(s)(F))(y) \in \mu_4 \cdot p^{-\dim W/2} \sum_{z \in W^-} F(z) \cdot \nu \circ \theta(z, (s|_W)^{-1}y) dz.$$

Since $\mathrm{Sp}(V)$ is generated by P and s , these formulas show that the submodule $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}][W^-] \subset \mathbb{C}[W^-]$ is preserved by the action of the full $\mathrm{Sp}(V)$. Going back through the above isomorphism, we deduce that $V_{\eta_{\nu,W},\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}$ is stable under $\omega_{\mathbb{C}}(\mathrm{Sp}(V))$. \square

Let us denote by $\omega_{\nu,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]} : \mathrm{Sp}(V) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]} V_{\eta_{\nu,W},\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}$ the representation provided by the above lemma. By definition of $\omega_{\mathbb{C}}$, we have

$$\eta_{\nu,W,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}(gk) = \omega_{\nu,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}(g)\eta_{\nu,W,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}(k)\omega_{\nu,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}(g)^{-1}$$

for all $g \in \mathrm{Sp}(V)$ and $k \in \mu_p \tilde{\boxtimes} V$. Since $C = C \times \{0\}$ is centralized by $\mathrm{Sp}(V)$ and acts via a character on $V_{\eta_{\nu,W},\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}$, the same equality holds for $g \in \mathrm{Sp}(V)$ and $k \in C \tilde{\boxtimes} V$. We thus get a representation $\omega_{\nu,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]} \times \eta_{\nu,W,\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}$ of the full group $\mathrm{Sp}(V) \times (C \tilde{\boxtimes} V)$ on $V_{\eta_{\nu,W},\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]}$.

Definition.— Let R be a commutative $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra. We denote by $\omega_{\nu,W,R} \times \eta_{\nu,W,R}$ (or simply $\omega_{\nu,W} \times \eta_{\nu,W}$ if the context is clear), the $R(\mathrm{Sp}(V) \times (C \tilde{\boxtimes} V))$ -module obtained by base change from the above $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}](\mathrm{Sp}(V) \times (C \tilde{\boxtimes} V))$ -module. We call it the (η, W) -Heisenberg–Weil representation and we call its restriction $\omega_{\nu,W}$ to $\mathrm{Sp}(V)$ the (η, W) -Weil representation.

Note that the restriction of the (η, W) -Heisenberg–Weil representation $\omega_{\nu,W} \times \eta_{\nu,W}$ to $C \tilde{\boxtimes} V$ is the Heisenberg representation $\eta_{\nu,W}$ we started with.

From now on R denotes a commutative $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra.

Remark.— If $p \equiv 1 \pmod{4}$, then by [Gér77, (2.17) and (2.18)] the μ_4 in (3.6.1a) can be replaced by $\mu_2 = \{\pm 1\}$ and we could work with the ring $\mathbb{Z}[\mu_{p^\infty}, \frac{1}{\sqrt{p}}]$ instead of $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ throughout.

3.6.2 Lemma. — Let V^+ be a subspace of W , and write $V^+ \oplus V^0$ for the orthogonal complement of V^+ in V . Then the restriction of θ to V^0 is non-degenerate and we may form $C\tilde{\boxtimes}V^0$. Let P be the subgroup of $\mathrm{Sp}(V)$ that stabilizes V^+ . Then the restriction of the (η, W) -Heisenberg–Weil representation to $P \ltimes (C\tilde{\boxtimes}V)$ satisfies:

$$(3.6.2a) \quad (\omega_{\nu, W} \times \eta_{\nu, W})|_{P \ltimes (C\tilde{\boxtimes}V)} \simeq \mathrm{Ind}_{P \ltimes ((C\tilde{\boxtimes}V^0) \times V^+)}^{P \ltimes (C\tilde{\boxtimes}V)} (\omega_{\nu, W \cap V^0} \times \eta_{\nu, W \cap V^0}) \otimes (\chi^{V^+} \times 1),$$

where $\omega_{\nu, W \cap V^0} \times \eta_{\nu, W \cap V^0}$ is the $(\eta, W \cap V^0)$ -Heisenberg–Weil representation of $\mathrm{Sp}(V^0) \ltimes (C\tilde{\boxtimes}V^0)$ on which $P \ltimes ((C\tilde{\boxtimes}V^0) \times V^+)$ acts via the projection to $\mathrm{Sp}(V^0) \ltimes ((C\tilde{\boxtimes}V^0) \times \{0\})$, and χ^{V^+} denotes the character $P \rightarrow \{\pm 1\} \subset R^\times$ given by $p \mapsto \mathrm{sgn}(\det_{\mathbb{F}_p}(p|_{V^+}))$.

Proof. Note that $W = V^+ \oplus (W \cap V^0)$ and that $W \cap V^0$ is a Lagrangian of V^0 . Hence we have by definition and transitivity of induction

$$\eta_{\nu, W} = \mathrm{Ind}_{C \times W}^{C\tilde{\boxtimes}V} (\nu \times 1) \simeq \mathrm{Ind}_{(C\tilde{\boxtimes}V^0) \times V^+}^{C\tilde{\boxtimes}V} \left(\mathrm{Ind}_{C \times (W \cap V^0)}^{C\tilde{\boxtimes}V^0} (\nu \times 1) \times 1 \right) = \mathrm{Ind}_{(C\tilde{\boxtimes}V^0) \times V^+}^{C\tilde{\boxtimes}V} (\eta_{\nu, W \cap V^0})$$

This means the two sides of (3.6.2a) are isomorphic as representations of $C\tilde{\boxtimes}V$, and we use this isomorphism to identify the underlying R -modules on both side. Since both sides are representations of $P \ltimes (C\tilde{\boxtimes}V)$, the action of $p \in P$ on each side provides an isomorphism between $\eta_{\nu, W}$ and $\eta_{\nu, pW}$. Since $\mathrm{End}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W}) = R$, we therefore conclude that the action of P on both sides of (3.6.2a) agrees up to twist by a character valued in the roots of unity contained in R^\times . It is enough to determine this character for $R = \mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$, where we can base change to \mathbb{C} and apply [Gér77, Theorem 2.4.(b)]⁹ to see that the desired character is χ^{V^+} as claimed. \square

3.6.3 Definition / Lemma.— Let η be a Heisenberg representation for the Heisenberg triple $(C\tilde{\boxtimes}V, C, \nu)$. Then the evaluation isomorphism $\eta_{\nu, W} \otimes_R \mathrm{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W}, \eta) \xrightarrow{\sim} \eta$ turns the representation $(\omega_{\nu, W} \times \eta_{\nu, W}) \otimes 1$ on the R -module $V_{\eta_{\nu, W}} \otimes_R \mathrm{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W}, \eta)$ into a representation $\omega \times \eta$ of $\mathrm{Sp}(V) \ltimes (C\tilde{\boxtimes}V)$ on V_η that is independent of the choice of Lagrangian $W \subset V$. We call the representation $\omega \times \eta$ the *Heisenberg–Weil representation associated to η* and ω the *Weil representations associated to the Heisenberg representation η* .

Proof. Let W and W' be two Lagrangian subspaces of V . Then there exists $g \in \mathrm{Sp}(V)$ such that $gW = W'$. The action of g on functions on $C\tilde{\boxtimes}V$ provides an R -linear isomorphism $I_g : V_{\eta_{\nu, W}, \nu} \xrightarrow{\sim} V_{\eta_{\nu, W'}, \nu}$ that intertwines $\eta_{\nu, W}$ with $(\eta_{\nu, W'}, \nu)^g := \eta_{\nu, W', \nu} \circ \mathrm{act}_g$. Therefore, composing with $\omega_{\nu, W'}(g)^{-1}$, we get an isomorphism $\eta_{\nu, W} \xrightarrow{\sim} \eta_{\nu, W'}$, which is a generator of the invertible R -module $\mathrm{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W}, \eta_{\nu, W'})$. From the uniqueness of the Weil representation or from the explicit formulas in the proof of Lemma 3.6.1, we see that this isomorphism has to be $\mathrm{Sp}(V)$ -equivariant, giving an isomorphism $\omega_{\nu, W} \times \eta_{\nu, W} \xrightarrow{\sim} \omega_{\nu, W'} \times \eta_{\nu, W'}$. It follows that the evaluation isomorphism $\eta_{\nu, W} \otimes_R \mathrm{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W}, \eta_{\nu, W'}) \xrightarrow{\sim} \eta_{\nu, W'}$ sends $(\omega_{\nu, W} \times \eta_{\nu, W}) \otimes 1$ to $\omega_{\nu, W'} \times \eta_{\nu, W'}$, and hence the transport of $(\omega_{\nu, W} \times \eta_{\nu, W}) \otimes 1$ via

$$\eta_{\nu, W} \otimes_R \mathrm{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W}, \eta) \xleftarrow{\sim} \eta_{\nu, W} \otimes_R \mathrm{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W}, \eta_{\nu, W'}) \otimes_R \mathrm{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W'}, \eta) \xrightarrow{\sim} \eta$$

⁹The statement of [Gér77, Theorem 2.4.(b)] omits the factor $(\chi^{V^+} \times 1)$, which is a typo. See also [Fin22, Lemma 3.2].

agrees with the transport of $(\omega_{\nu, W'} \times \eta_{\nu, W'}) \otimes 1$ via $\eta_{\nu, W'} \otimes_R \text{Hom}_{R(C\tilde{\boxtimes}V)}(\eta_{\nu, W'}, \eta) \xrightarrow{\sim} \eta$. \square

3.6.4 Lemma. — Let η and η' be two Heisenberg representations for $(C\tilde{\boxtimes}V, C, \nu)$ with associated Weil representations ω and ω' . Then the inclusion

$$\text{Hom}_{R(\text{Sp}(V) \times (C\tilde{\boxtimes}V))}(\omega \times \eta, \omega' \times \eta') \subseteq \text{Hom}_{R(C\tilde{\boxtimes}V)}(\eta, \eta')$$

is an equality.

Proof. By Definition/Lemma 3.6.3, evaluation induces an isomorphism

$$(\omega \times \eta) \otimes_R \text{Hom}_{R(C\tilde{\boxtimes}V)}(\eta, \eta') \xrightarrow{\sim} \omega' \times \eta'.$$

Since $\text{Hom}_{R(C\tilde{\boxtimes}V)}(\eta, \eta')$ is a flat R -module, this induces in turn an isomorphism

$$\text{Hom}_{R(\text{Sp}(V) \times (C\tilde{\boxtimes}V))}(\omega \times \eta, \omega' \times \eta') \xrightarrow{\sim} \text{End}_{R(\text{Sp}(V) \times (C\tilde{\boxtimes}V))}(\omega \times \eta) \otimes_R \text{Hom}_{R(C\tilde{\boxtimes}V)}(\eta, \eta').$$

This reduces the lemma to the case where $\eta = \eta'$, where it follows from

$$R = \text{Hom}_{R(C\tilde{\boxtimes}V)}(\eta, \eta) \supseteq \text{Hom}_{R(\text{Sp}(V) \times (C\tilde{\boxtimes}V))}(\omega \times \eta, \omega \times \eta) \supseteq R.$$

\square

3.6.5 From p -adic groups to the abstract Heisenberg–Weil setting. — We take up our usual setup associated to a point x in \mathcal{B}_l . We denote by C_x the image $\check{\phi}_{l,x}^+(K_{l,x}^+)$ of the character $\check{\phi}_{l,x}^+$ in μ_{p^∞} . Recall from 2.5.13 that the map $(a, b) \mapsto [a, b] := \check{\phi}_{l,x}^+(aba^{-1}b^{-1})$ yields a non-degenerate symplectic form $V_x \times V_x \rightarrow C_x$ on $V_x := K_{l,x}^+/K_{l,x}^+$ that factors through the subgroup $\mu_p \subseteq C_x$ of order p . By construction, the conjugation action of $G_{l,x}$ on $K_{l,x}^+$ induces an action on V_x that preserves this non-degenerate symplectic form. We denote by $\sigma : G_{l,x} \rightarrow \text{Sp}(V_x)$ the corresponding morphism.

Lemma. — There exists a homomorphism $f : K_{l,x}^+ \rightarrow C_x\tilde{\boxtimes}V_x$ such that

i) The following diagram is commutative

$$\begin{array}{ccccc} K_{l,x}^+ & \hookrightarrow & K_{l,x}^+ & \twoheadrightarrow & V_x \\ \check{\phi}_{l,x}^+ \downarrow & & \downarrow f & & \parallel \\ C_x & \hookrightarrow & C_x\tilde{\boxtimes}V_x & \twoheadrightarrow & V_x \end{array}$$

ii) The product map $\sigma \times f : G_{l,x} \times K_{l,x}^+ \rightarrow \text{Sp}(V_x) \times (C_x\tilde{\boxtimes}V_x)$ is a homomorphism.

Moreover, such a f is unique up to post-composition by conjugation by an element in $C_x\tilde{\boxtimes}V_x$. More precisely, if f' is another homomorphism satisfying i) and ii), then there is $w \in (V_x)^{\sigma(G_{l,x})}$ such that $\forall k \in K_{l,x}^+, f'(k) = (c(k).[w, v(k)], v(k))$, where $f(k) = (c(k), v(k))$.

Proof. Recall the groups $J_{\iota,x}^i$ and $J_{\iota,x}^{i+}$ introduced in 2.5.8 and the equality $K_{\iota,x}^\dagger = G_{\iota,x,0+} \prod_{i=1}^d J_{\iota,x}^i$. Set

$$J^{\text{der}} := \prod_{i=1}^d (J_{\iota,x}^i \cap \mathbf{G}_{\iota,\text{der}}^i(F)) \quad \text{and} \quad J_+^{\text{der}} := \prod_{i=1}^d (J_{\iota,x}^{i+} \cap \mathbf{G}_{\iota,\text{der}}^i(F))$$

By [Fin21, Corollary 7.2] we have $K_{\iota,x}^\dagger = G_{\iota,x,0+} J^{\text{der}}$. The desired morphism f has to coincide with $\check{\phi}_{\iota,x}^+$ on $G_{\iota,x,0+}$ since $G_{\iota,x,0+} \subset K_{\iota,x}^\dagger$. We will follow Yu to define f on J^{der} . Note that $J^{\text{der}}/J_+^{\text{der}} \xrightarrow{\sim} K_{\iota,x}^\dagger/K_{\iota,x}^+ = V_x$ and $\check{\phi}_{\iota,x}^+(J^{\text{der}}) = \mu_p \subset C_x$. Therefore, setting $N := \ker(\check{\phi}_{\iota,x}^+|_{J^{\text{der}}})$, the group J/N is a Heisenberg p -group in the sense of [Yu01, §10]. We can now apply the proof of [Yu01, Proposition 11.4] to our setting by replacing Yu's J by our J^{der} , his J_+ by our J_+^{der} , his N by ours, his G' by our G_ι , his y by our x , his $(G', G)(F)_{y,(r+,s+)}$ by our $\prod_{i=1}^d (G_\iota^{i-1} \cap (G_\iota^i)^{\text{der}}, (G_\iota^i)^{\text{der}})_{x,r_{i-1}+,r_{i-1}/2+}$, and his $G_\alpha(F)_{y,s}$ by our $U_\alpha \cap G(F)_{x,r_i/2}$, where i is the lowest index such that the root α occurs in \mathfrak{g}_ι^i . His argument goes through and provides us with a group homomorphism f^{Yu} that fits in a commutative diagram

$$\begin{array}{ccccc} J_+^{\text{der}} & \hookrightarrow & J^{\text{der}} & \twoheadrightarrow & V_x \\ \check{\phi}_{\iota,x}^+ \downarrow & & \downarrow f^{\text{Yu}} & & \parallel \\ \mu_p & \hookrightarrow & \mu_p \tilde{\boxtimes} V_x & \twoheadrightarrow & V_x \end{array}$$

and such that $\sigma \times f^{\text{Yu}} : G_{\iota,x} \times J^{\text{der}} \rightarrow \text{Sp}(V_x) \times (\mu_p \tilde{\boxtimes} V_x)$ is a homomorphism. Since f^{Yu} coincides with $\check{\phi}_{\iota,x}^+$ on $G_{\iota,x,0+} \cap J^{\text{der}} \subset J_+^{\text{der}}$, we can define a morphism $f : K_{\iota,x}^\dagger \rightarrow C_x \tilde{\boxtimes} V_x$ by setting $f(k) := \check{\phi}_{\iota,x}^+(g) f^{\text{Yu}}(j)$ for any $k = gj$ with $g \in G_{\iota,x,0+}$ and $j \in J^{\text{der}}$. The commutativity of the diagram in i) of the lemma follows from that of the diagram above. Property ii) follows from the same property for f^{Yu} and the fact that $G_{\iota,x}$ centralizes $\check{\phi}_{\iota,x}^+$.

Now, let f' be another morphism satisfying i) and ii). There is an automorphism α of the extension $C_x \hookrightarrow C_x \tilde{\boxtimes} V_x \twoheadrightarrow V_x$ such that $f' = \alpha \circ f$. Any such automorphism has the form $(c, v) \mapsto (c\Lambda(v), v)$ for some morphism $\Lambda : V_x \rightarrow C_x$. Since V_x is an \mathbb{F}_p -vector space, Λ factors through μ_p and is a linear form. Hence there exists $w \in V_x$ such that $\Lambda(v) = [w, v]$ for all $v \in V_x$. It follows that, writing $f(k) = (c(k), v(k))$ and $f'(k) = (c'(k), v'(k))$ for $k \in K_{\iota,x}^\dagger$, we have $v'(k) = v(k)$ and $c'(k) = c(k)[w, v(k)]$. Now, property ii) for f , resp. f' , means that $f(gkg^{-1}) = (c(k), \sigma(g).v(k))$, resp. $f'(gkg^{-1}) = (c'(k), \sigma(g).v'(k))$, from which we deduce that $c'(k) = c(k)[w, \sigma(g).v(k)]$, for all $g \in G_{\iota,x}$ and $k \in K_{\iota,x}^\dagger$. It follows that $[\sigma(g)^{-1}w, v(k)] = [w, \sigma(g).v(k)] = [w, v(k)]$ for all $k \in K_{\iota,x}^\dagger$. Since $k \mapsto v(k)$ is surjective onto V_x , we deduce that $w \in (V_x)^{\sigma(G_x)}$. \square

3.6.6 *A family of non-twisted Heisenberg–Weil representations.* — Let $\eta_{\iota,x}$ be a Heisenberg representation for the triple $(K_{\iota,x}^\dagger, K_{\iota,x}^+, \check{\phi}_{\iota,x}^+)$. We will extend this representation to a representation of $K_{\iota,x}$ using the theory of Heisenberg–Weil representations.

Pick a morphism $f : K_{\iota,x}^\dagger \rightarrow C_x \tilde{\boxtimes} V_x$ as in Lemma 3.6.5. Note that for any Heisenberg representation η of $C_x \tilde{\boxtimes} V_x$, the pullback representation $\eta \circ f$ is a Heisenberg representation of $K_{\iota,x}^\dagger$. It then follows from the classification of Heisenberg representations that all Heisenberg representations of $K_{\iota,x}^\dagger$ arise by pullback from f . In particular, there is a unique Heisenberg representation η_f of $C_x \tilde{\boxtimes} V_x$ on the module $V_{\eta_{\iota,x}}$ such that $\eta_{\iota,x} = \eta_f \circ f$. Denote by ω_f the Weil representation $\text{Sp}(V_x) \rightarrow \text{Aut}_R(V_{\eta_{\iota,x}})$ associated to η_f by Definition/Lemma 3.6.3.

Lemma. — The representation $\omega_{\iota,x} := \omega_f \circ \sigma : G_{\iota,x} \longrightarrow \text{Aut}_R(V_{\eta_{\iota,x}})$ is independent of f .

Proof. Let f' be another morphism as in Lemma 3.6.5 and let w be an element of V_x as in the last sentence of that lemma. Then $\eta_{f'} = \eta_f \circ \text{Inn}_{(1,w)}$ where $\text{Inn}_{(1,w)}$ denotes inner conjugation by $(1, w)$ in $C_x \widetilde{\boxtimes} V_x$. It follows that the R -linear automorphism $\eta_f(1, w)$ of $V_{\eta_{\iota,x}}$ induces an isomorphism of representations $\eta_f \xrightarrow{\sim} \eta_{f'}$. By Lemma 3.6.4, it also induces an isomorphism $\omega_f \xrightarrow{\sim} \omega_{f'}$. Hence for any $u \in \text{Sp}(V_x)$, we have

$$\omega_{f'}(u) = \eta_f(1, w) \circ \omega_f(u) \circ \eta_f(1, w)^{-1} = \omega_f(u) \circ \eta_f(1, u^{-1} \cdot w) \circ \eta_f(1, w)^{-1}.$$

Since w is fixed under $\sigma(G_{\iota,x})$, we get $\omega_{f'} \circ \sigma = \omega_f \circ \sigma$, as desired. \square

We call $\omega_{\iota,x}$ the Weil representation associated to $\eta_{\iota,x}$. By construction, the product map $\omega_{\iota,x} \times \eta_{\iota,x} : G_{\iota,x} \times K_{\iota,x}^\dagger \longrightarrow \text{Aut}_R(V_{\eta_{\iota,x}})$ is a group homomorphism. However, this representation does not descend along the multiplication map $G_{\iota,x} \times K_{\iota,x}^\dagger \longrightarrow K_{\iota,x}$. To be able to descend along this map, we need to twist $\omega_{\iota,x}$ by a character of $G_{\iota,x}$ that extends $\check{\phi}_{\iota,x}^+$. We will use the restriction of the character $\check{\varphi}_0 : G_\iota \longrightarrow \mu_{p^\infty} \subset R^\times$ chosen in 2.4.5. This choice is independent of x and $\eta_{\iota,x}$.

Definition.— We define the representation $\kappa_{\iota,x}^{\text{nt}}$ of $K_{\iota,x}$ to be the unique R -representation such that the composition of the morphism

$$\begin{aligned} G_{\iota,x} \times K_{\iota,x}^\dagger &\twoheadrightarrow K_{\iota,x} \\ (k_1, k_2) &\mapsto k_1 k_2 \end{aligned}$$

with $\kappa_{\iota,x}^{\text{nt}}$ yields the representation $(\check{\varphi}_0|_{G_{\iota,x}} \omega_{\iota,x}) \times \eta_{\iota,x}$. We call $\kappa_{\iota,x}^{\text{nt}}$ the *non-twisted $RK_{\iota,x}$ -Heisenberg–Weil representation associated with $\eta_{\iota,x}$* .

Corollary. — Let $\eta_{\iota,x}$ and $\eta'_{\iota,x}$ be two Heisenberg representations of $K_{\iota,x}^\dagger$ to which we associate the representations $\kappa_{\iota,x}^{\text{nt}}$ and $\kappa'_{\iota,x}^{\text{nt}}$ of $K_{\iota,x}$ via Definition 3.6.6. Then

$$\text{Hom}_{RK_{\iota,x}}(\kappa_{\iota,x}^{\text{nt}}, \kappa'_{\iota,x}^{\text{nt}}) = \text{Hom}_{RK_{\iota,x}^\dagger}(\eta_{\iota,x}, \eta'_{\iota,x})$$

Proof. This follows from Lemma 3.6.4 since $K_{\iota,x}^\dagger$ surjects onto $C_x \widetilde{\boxtimes} V_x$. \square

3.6.7 The quadratic twist. — We write $\varepsilon_x^{\vec{G}_\iota}$, or also simply ε_x , for the quadratic character of $K_{\iota,x}$ defined in [AFMO24b, Notation 3.6.4] arising from [FKS23] attached to the sequence \vec{G}_ι and the generic characters ψ_i , i.e.,

$$\varepsilon_x = \prod_{i=1}^d \varepsilon_x^{G_\iota^i / G_\iota^{i-1}},$$

where $\varepsilon_x^{G_\iota^i / G_\iota^{i-1}}$ is trivial on $K_{\iota,x}^\dagger$, and on $K_{\iota,x} / K_{\iota,x}^\dagger \simeq G_{\iota,x} / G_{\iota,x,0+}$ it is given by the restriction of the quadratic character $\varepsilon_x^{G_\iota^i / G_\iota^{i-1}}$ on $G_{\iota,x}^i$ defined in [FKS23, Lemma 4.1.2] for $r = r_{i-1}$. The only property about the quadratic character ε_x that we use is that it satisfies the following.

Lemma. ([FKS23]) — Let $x, x' \in \mathcal{B}'_\iota$. Then

$$(3.6.7a) \quad (\varepsilon_x) |_{K_{\iota,x} \cap K_{\iota,x'}} \cdot \delta_{x'}^x = (\varepsilon_{x'}) |_{K_{\iota,x} \cap K_{\iota,x'}} \cdot \delta_x^{x'},$$

where $\delta_{x'}^x(g) = \text{sgn}(\det_{\mathbb{F}_p}(u_g))$ with u_g denoting the action induced via conjugation by $g \in K_{\iota,x} \cap K_{\iota,x'}$ on the \mathbb{F}_p -vector space

$$((K_{\iota,x}^\dagger \cap K_{\iota,x'}^\dagger)K_{\iota,x}^+)/K_{\iota,x}^+.$$

Proof. This follows from applying [FKS23, Lemma 4.1.2] to each $\varepsilon_x^{G_i^i/G_i^{i-1}}$ and taking the product. \square

3.6.8 Definition.— Given a Heisenberg representation $\eta_{\iota,x}$ of $K_{\iota,x}^\dagger$, we define the *twisted $RK_{\iota,x}$ -Heisenberg–Weil representation associated with $\eta_{\iota,x}$* as

$$\kappa_{\iota,x} := \varepsilon_x \cdot \kappa_{\iota,x}^{\text{nt}}.$$

More generally, an $RK_{\iota,x}$ -module is called a *twisted $RK_{\iota,x}$ -Heisenberg–Weil representation* if it is the twisted $RK_{\iota,x}$ -Heisenberg–Weil representation associated to some Heisenberg representation of $K_{\iota,x}^\dagger$.

3.6.9 Remark. — If R is an algebraically closed field, the representation $\kappa_{\iota,x}$ agrees with the following representation constructed in [AFMO24b]: if $r_{d-1} = r_d$, resp. $r_{d-1} < r_d$, set

$$\begin{aligned} \text{HW}_x &= ((\mathbf{G}_\iota = \mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \dots \subsetneq \mathbf{G}^d), (r_0, \dots, r_{d-1}), x, \mathbf{G}^0(F)_x, (\psi_0, \dots, \psi_{d-1})), \text{ resp.} \\ \text{HW}_x &= ((\mathbf{G}_\iota = \mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \dots \subsetneq \mathbf{G}^d \subseteq \mathbf{G}^{d+1} := \mathbf{G}^d), (r_0, \dots, r_d), x, \mathbf{G}^0(F)_x, (\psi_0, \dots, \psi_d)) \end{aligned}$$

for $x \in \mathcal{B}_\iota$. Then HW_x is a Heisenberg–Weil datum as defined in [AFMO24b, Definition 3.6.1] by Lemma 2.5.7. Note that by our construction we have fixed embeddings $\mathcal{B}_\iota = \mathcal{B}(\mathbf{G}^0, F) \subseteq \mathcal{B}(\mathbf{G}^1, F) \subseteq \dots \subseteq \mathcal{B}(\mathbf{G}, F)$ so we do not record them as part of the Heisenberg–Weil datum. Then it follows from [Gér77, Corollary 2.5] and [AFMO24b, Lemma 3.6.8] that (κ_x, V_{κ_x}) is isomorphic to the representation of $K_{\iota,x}$ obtained from HW_x via the twisted Heisenberg–Weil construction as in [AFMO24b, Notation 3.6.3].

3.6.10 Corollary. — Let $\eta_{\iota,x}$ and $\eta'_{\iota,x}$ be two Heisenberg representations of $K_{\iota,x}^\dagger$ with associated twisted $RK_{\iota,x}$ -Heisenberg–Weil representations $\kappa_{\iota,x}$ and $\kappa'_{\iota,x}$. Then

$$\text{Hom}_{RK_{\iota,x}}(\kappa_{\iota,x}, \kappa'_{\iota,x}) = \text{Hom}_{RK_{\iota,x}^\dagger}(\eta_{\iota,x}, \eta'_{\iota,x})$$

Proof. This follows from Corollary 3.6.6 and Definition 3.6.8. \square

3.6.11 Proposition. — Let $x, x' \in \mathcal{B}'_\iota$ and let $\eta_{\iota,x}$ and $\eta_{\iota,x'}$ be two Heisenberg representations with associated twisted $RK_{\iota,x}$ -Heisenberg–Weil representations $\kappa_{\iota,x}$ and $\kappa_{\iota,x'}$. Then

$$\text{Hom}_{R(K_{\iota,x} \cap K_{\iota,x'})}(\kappa_{\iota,x}, \kappa_{\iota,x'}) = \text{Hom}_{R(K_{\iota,x}^\dagger \cap K_{\iota,x'}^\dagger)}(\eta_{\iota,x}, \eta_{\iota,x'}).$$

Proof. If $x = x'$, the claimed equality holds since both sides are equal to R . So let us assume $x \neq x'$, and let $(\mathbf{P}, \bar{\mathbf{P}})$ be the pair of opposite parabolic subgroups of \mathbf{G} with unipotent radicals \mathbf{U} and $\bar{\mathbf{U}}$, respectively, that are attached to the segment (x, x') as in the proof of Lemma 2.5.12 and Lemma 3.5.5, i.e., we have

$$M \cap K_{\ell,x} = M \cap K_{\ell,x'}, \quad \bar{\mathbf{U}} \cap K_{\ell,x}^+ \supseteq \bar{\mathbf{U}} \cap K_{\ell,x'}^+, \quad U \cap K_{\ell,x} \subseteq U \cap K_{\ell,x'}^+,$$

$$K_{\ell,x} \cap K_{\ell,x'} = (U \cap K_{\ell,x})(M \cap K_{\ell,x})(\bar{\mathbf{U}} \cap K_{\ell,x'})$$

and

$$(3.6.11a) \quad K_{\ell,x}^\dagger \cap K_{\ell,x'}^\dagger = (U \cap K_{\ell,x}^\dagger)(M \cap K_{\ell,x}^\dagger)(\bar{\mathbf{U}} \cap K_{\ell,x'}^\dagger)$$

Hence the conjugation action of $K_{\ell,x} \cap K_{\ell,x'}$ preserves the subspace

$$V_U := (U \cap K_{\ell,x}^\dagger)/(U \cap K_{\ell,x}^+) = ((K_{\ell,x}^\dagger \cap K_{\ell,x'}^+)K_{\ell,x}^+)/K_{\ell,x}^+$$

of $K_{\ell,x}^\dagger/K_{\ell,x}^+ = V_x$. Therefore the image of $K_{\ell,x} \cap K_{\ell,x'}$ in $\mathrm{Sp}(V_x)$ is contained in the parabolic subgroup P^{Sp} that is the stabilizer of V_U in $\mathrm{Sp}(V_x)$. We write $V_M := (M \cap K_{\ell,x}^\dagger)/(M \cap K_{\ell,x}^+)$. Then by Lemma 3.6.2 the restriction of the Heisenberg–Weil representation $\omega \rtimes \eta_{\ell,x}$ associated to $\eta_{\ell,x}$ to $P^{\mathrm{Sp}} \times (C_x \tilde{\boxtimes} V_x)$ satisfies:

$$(\omega \rtimes \eta_{\ell,x})|_{P^{\mathrm{Sp}} \times (C_x \tilde{\boxtimes} V_x)} \simeq \mathrm{Ind}_{P^{\mathrm{Sp}} \times ((C_x \tilde{\boxtimes} V_M) \times V_U)}^{P^{\mathrm{Sp}} \times (C_x \tilde{\boxtimes} V_x)} (\omega^M \rtimes \eta_{\ell,x}^M) \otimes (\chi^U \rtimes 1),$$

where $\omega^M \rtimes \eta_{\ell,x}^M$ denotes a Heisenberg–Weil representation of $\mathrm{Sp}(V_M) \times (C_x \tilde{\boxtimes} V_M)$ with central character $\check{\phi}_{\ell,x}^+$ on which $P^{\mathrm{Sp}} \times (V_U \times (C_x \tilde{\boxtimes} V_M))$ acts via the projection to $\mathrm{Sp}(V_M) \times (\{0\} \times (C_x \tilde{\boxtimes} V_M))$, and χ^U denotes the character $P^{\mathrm{Sp}} \rightarrow \{\pm 1\}$ given by $p \mapsto \mathrm{sgn}(\det_{\mathbb{F}_p}(p|_{V_U}))$. Note that the composition of $K_{\ell,x} \cap K_{\ell,x'} \rightarrow P^{\mathrm{Sp}}$ with χ^U is the character $\delta_{x'}^x$ defined in 3.6.7. Thus we obtain

$$(\kappa_{\ell,x})|_{K_{\ell,x} \cap K_{\ell,x'}} \simeq \left(\mathrm{Ind}_{(U \cap K_{\ell,x})(M \cap K_{\ell,x})(\bar{\mathbf{U}} \cap K_{\ell,x}^+)}^{(K_{\ell,x} \cap K_{\ell,x'})K_{\ell,x}^\dagger} \varepsilon_x^{\bar{\mathbf{G}}_\ell} \varepsilon_x^{\bar{\mathbf{M}}_\ell} \kappa_{\ell,x}^M \otimes \delta_{x'}^x \right) |_{K_{\ell,x} \cap K_{\ell,x'}},$$

where $\kappa_{\ell,x}^M$ denotes the representation of $M \cap K_{\ell,x}$ constructed analogously to $\kappa_{\ell,x}$ in Definition 3.6.8 replacing $\bar{\mathbf{G}}_\ell$ by $\bar{\mathbf{M}}_\ell$ and $\eta_{\ell,x}$ by $\eta_{\ell,x}^M$ in the construction, and $\varepsilon_x^{\bar{\mathbf{M}}_\ell} \kappa_{\ell,x}^M$ is viewed as a representation of $(U \cap K_{\ell,x})(M \cap K_{\ell,x})(\bar{\mathbf{U}} \cap K_{\ell,x}^+)$ by letting $(U \cap K_{\ell,x})$ and $(\bar{\mathbf{U}} \cap K_{\ell,x}^+)$ act trivially. From this, we deduce that

$$(3.6.11b) \quad (\kappa_{\ell,x}|_{K_{\ell,x} \cap K_{\ell,x'}})_{U \cap K_{\ell,x}^\dagger} \simeq \left(\varepsilon_x^{\bar{\mathbf{G}}_\ell} \varepsilon_x^{\bar{\mathbf{M}}_\ell} \kappa_{\ell,x}^M \otimes \delta_{x'}^x \right) |_{K_{\ell,x} \cap K_{\ell,x'}},$$

and that the surjection $V_{\kappa_{\ell,x}} \twoheadrightarrow (V_{\kappa_{\ell,x}})_{U \cap K_{\ell,x}^\dagger}$ is $K_{\ell,x} \cap K_{\ell,x'}$ -equivariant. Similarly, we obtain

$$(3.6.11c) \quad (\kappa_{\ell,x'}|_{K_{\ell,x} \cap K_{\ell,x'}})^{\bar{\mathbf{U}} \cap K_{\ell,x}^\dagger} \simeq \left(\varepsilon_{x'}^{\bar{\mathbf{G}}_\ell} \varepsilon_{x'}^{\bar{\mathbf{M}}_\ell} \kappa_{\ell,x'}^M \otimes \delta_x^{x'} \right) |_{K_{\ell,x} \cap K_{\ell,x'}},$$

and in particular that the subspace $V_{\kappa_{l,x'}}^{\bar{U} \cap K_{l,x'}^\dagger}$ of $V_{\kappa_{l,x'}}$ is $K_{l,x} \cap K_{l,x'}$ -stable. We can thus consider the following commutative diagram of inclusions

$$\begin{array}{ccc} \mathrm{Hom}_{R(K_{l,x} \cap K_{l,x'})} \left((\kappa_{l,x})_{U \cap K_{l,x}^\dagger}, (\kappa_{l,x'})_{\bar{U} \cap K_{l,x'}^\dagger} \right) & \subset & \mathrm{Hom}_{R(K_{l,x} \cap K_{l,x'})} (\kappa_{l,x}, \kappa_{l,x'}) \\ \cap & & \cap \\ \mathrm{Hom}_{R(K_{l,x}^\dagger \cap K_{l,x'}^\dagger)} \left((\eta_{l,x})_{U \cap K_{l,x}^\dagger}, (\eta_{l,x'})_{\bar{U} \cap K_{l,x'}^\dagger} \right) & \subset & \mathrm{Hom}_{R(K_{l,x}^\dagger \cap K_{l,x'}^\dagger)} (\eta_{l,x}, \eta_{l,x'}) \end{array}$$

where both horizontal inclusions are obtained by precomposition with projection and composition with inclusion. By (3.5.5b), the bottom inclusion is an equality. On the other hand, since the image of x and x' in the reduced building of M agree, we have $\varepsilon_x^{\bar{M}_l} = \varepsilon_{x'}^{\bar{M}_l}$ and

$$\mathrm{Hom}_{R(M \cap K_{l,x})} (\kappa_{l,x}^M, \kappa_{l,x'}^M) = \mathrm{Hom}_{R(M \cap K_{l,x}^\dagger)} (\eta_{l,x}^M, \eta_{l,x'}^M)$$

by Corollary 3.6.10. Combining this equality with (3.6.11b), (3.6.11c) and (3.6.7a), we obtain that the left vertical inclusion in the above diagram is an equality. It follows that the two remaining inclusions are equalities too. \square

3.7 Construction of a Heisenberg–Weil coefficient system

In this section we finally prove Theorem 3.3.7, so we let R be a commutative $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra.

We will first prove the result under the following assumption, which we assume from now on until the end of 3.7.5.

3.7.1 Assumption. — *We have $Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F), \circ} = Z(\hat{\mathbf{G}})^{W_F, \circ}$. Equivalently, the maximal split central torus of \mathbf{G}_l coincides with that of \mathbf{G} , i.e., $Z(\mathbf{G}_l)/Z(\mathbf{G})$ is anisotropic.*

The role of this assumption is to ensure that the subset \mathcal{B}'_l of \mathcal{B}' is the *reduced* building of G_l . In particular, it is equipped with the canonical Bruhat–Tits polysimplicial structure. Note that each Bruhat–Tits polysimplex is the union of the e -facets it contains. However, beware that our e -facets were defined with respect to G , and they may not be e -facets with respect to G_l .

Notation.— For every chamber \mathcal{C} (i.e., polysimplex of maximal dimension) of \mathcal{B}'_l , we denote by $\mathcal{F}_{\mathcal{C}}$ the unique e -facet that contains the barycenter of \mathcal{C} . By our choice of e , this is actually an e -vertex, whose stabilizer $G_{l, \mathcal{F}_{\mathcal{C}}}$ coincides with the stabilizer $G_{l, \mathcal{C}}$ of \mathcal{C} .

3.7.2 Basic choices. — We choose and fix the following data :

- i) a chamber \mathcal{C}_0 of \mathcal{B}'_l , whose closure we denote by $\bar{\mathcal{C}}_0$.
- ii) a set \mathcal{S} of representatives of G_{l, \mathcal{C}_0} -orbits of e -facets of \mathcal{B}' lying in $\bar{\mathcal{C}}_0$. Note that this set must contain $\mathcal{F}_{\mathcal{C}_0}$ and is also a set of representatives of G_l -orbits of e -facets lying in \mathcal{B}'_l .
- iii) for every $\mathcal{F}_0 \in \mathcal{S}$, a twisted RK_{l, \mathcal{F}_0} -Heisenberg–Weil representation $(\kappa_{l, \mathcal{F}_0}, \mathcal{W}_{l, \mathcal{F}_0}^0)$ such that the invertible R -module $\mathrm{Hom}_{R(K_{l, \mathcal{F}_0} \cap K_{l, \mathcal{F}_{\mathcal{C}_0}})} (\kappa_{l, \mathcal{F}_0}, \kappa_{l, \mathcal{F}_{\mathcal{C}_0}}) =: H_{\mathcal{F}_0, \mathcal{F}_{\mathcal{C}_0}}$ is free. This

can be achieved as follows: Start with an arbitrary choice $(\kappa'_{l,\mathcal{F}_0})_{\mathcal{F}_0 \in \mathcal{S}}$ and set $\kappa_{l,\mathcal{F}_0} := \kappa'_{l,\mathcal{F}_0} \otimes_R (H_{\mathcal{F}_0, \mathcal{F}_{C_0}})^*$ where $(H_{\mathcal{F}_0, \mathcal{F}_{C_0}})^* := \text{Hom}_R(H_{\mathcal{F}_0, \mathcal{F}_{C_0}}, R)$ denotes the inverse R -module.

iv) a generator $\alpha_{\mathcal{F}_0, \mathcal{F}_{C_0}}$ of the R -module $\text{Hom}_{R(K_{l,\mathcal{F}_0} \cap K_{l,\mathcal{F}_{C_0}})}(\kappa_{l,\mathcal{F}_0}, \kappa_{l,\mathcal{F}_{C_0}})$.

3.7.3 Construction of a Heisenberg–Weil coefficient system under Assumption 3.7.1, step 1: modules and actions. — For any $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$, we let $\mathcal{F}_0 \in \mathcal{S}$ be the unique element in the G_l -orbit of \mathcal{F} . As before, we denote by $G_{l,\mathcal{F}\mathcal{F}_0} := \{g \in G_l, g\mathcal{F}_0 = \mathcal{F}\}$ the transporter of \mathcal{F}_0 to \mathcal{F} , an open subset of G_l , and we set

$$K_{l,\mathcal{F}\mathcal{F}_0} := G_{l,\mathcal{F}\mathcal{F}_0} K_{l,\mathcal{F}_0} = K_{l,\mathcal{F}} G_{l,\mathcal{F}\mathcal{F}_0},$$

which is an open subset of G . Note that, since $G_{l,\mathcal{F}}$ is contained in $K_{l,\mathcal{F}}$, the set $K_{l,\mathcal{F}\mathcal{F}_0}$ is a right K_{l,\mathcal{F}_0} -coset, and a left $K_{l,\mathcal{F}}$ -coset. Now we set

$$\mathcal{W}_{l,\mathcal{F}} := \mathcal{C}_c^\infty(K_{l,\mathcal{F}\mathcal{F}_0}) \otimes_{RK_{l,\mathcal{F}_0}} \mathcal{W}_{l,\mathcal{F}_0}^0.$$

For any facet $\mathcal{F}' \in \mathcal{B}'_{l,\bullet/e}$ in the G_l -orbit of \mathcal{F} , the set $K_{l,\mathcal{F}'\mathcal{F}}$ is the hom set $\text{Hom}(\mathcal{F}, \mathcal{F}')$ in $[\mathcal{B}'_{l,\bullet/e}/K_l]$. Therefore, the maps

$$\begin{aligned} \mathcal{W}_{l,\mathcal{F},\mathcal{F}'} : K_{l,\mathcal{F}'\mathcal{F}} &\rightarrow \text{Hom}_R(\mathcal{W}_{l,\mathcal{F}}, \mathcal{W}_{l,\mathcal{F}'}) \\ g &\mapsto \lambda_g \otimes \text{Id} \end{aligned},$$

where $\lambda_g : \mathcal{C}_c^\infty(K_{l,\mathcal{F}\mathcal{F}_0}) \rightarrow \mathcal{C}_c^\infty(K_{l,\mathcal{F}'\mathcal{F}_0})$ is induced by left translation by g^{-1} , define a coefficient system on the full subcategory of $[\mathcal{B}'_{l,\bullet/e}/K_l]$ given by the orbit of \mathcal{F}_0 . In particular, $\mathcal{W}_{l,\mathcal{F}}$ defines a representation of $K_{l,\mathcal{F}}$ on $\mathcal{W}_{l,\mathcal{F}}$ and, by construction, its restriction to $K_{l,\mathcal{F}}^\dagger$ is a Heisenberg representation for $\check{\phi}_{l,\mathcal{F}}^+$. To maintain consistency of our notation with that of G_l -equivariant coefficient systems, we also write

$$g_{\mathcal{W},\mathcal{F}} := \mathcal{W}_{l,\mathcal{F},g\mathcal{F}}(g), \quad \text{for any } g \in G_l$$

3.7.4 Construction of a Heisenberg–Weil coefficient system under Assumption 3.7.1, step 2: face maps. — With \mathcal{F} and \mathcal{F}_0 as in the previous paragraph, pick also a chamber \mathcal{C} such that $\mathcal{F} \subseteq \bar{\mathcal{C}}$. We can then find $g \in G_l$ such that $\mathcal{C} = g\mathcal{C}_0$ and $\mathcal{F} = g\mathcal{F}_0$. Such a g is unique modulo right multiplication by $G_{l,\mathcal{F}_0} \cap G_{l,\mathcal{C}_0}$, and we have $K_{l,\mathcal{F}\mathcal{F}_0} \cap K_{l,\mathcal{F}\mathcal{C}_0} = g(K_{l,\mathcal{F}_0} \cap K_{l,\mathcal{F}_{C_0}}) = (K_{l,\mathcal{F}} \cap K_{l,\mathcal{F}_{C_0}})g$. We thus can define a generator $\alpha_{\mathcal{F},\mathcal{C}}$ of $\text{Hom}_{R(K_{l,\mathcal{F}} \cap K_{l,\mathcal{F}_{C_0}})}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{W}_{l,\mathcal{F}_{C_0}})$ by requiring the commutativity of the following diagram of $R(K_{l,\mathcal{F}} \cap K_{l,\mathcal{F}_{C_0}})$ -modules (where can denotes canonical inclusions):

$$\begin{array}{ccc} \mathcal{W}_{l,\mathcal{F}} = \mathcal{C}_c^\infty(K_{l,\mathcal{F}\mathcal{F}_0}) \otimes_{RK_{l,\mathcal{F}_0}} \mathcal{W}_{l,\mathcal{F}_0}^0 & \overset{\alpha_{\mathcal{F},\mathcal{C}}}{\dashrightarrow} & \mathcal{W}_{l,\mathcal{F}_{C_0}} = \mathcal{C}_c^\infty(K_{l,\mathcal{F}_{C_0}\mathcal{F}_{C_0}}) \otimes_{RK_{l,\mathcal{F}_{C_0}}} \mathcal{W}_{l,\mathcal{F}_{C_0}}^0 \\ \uparrow \simeq \text{can} \otimes \text{Id} & & \nearrow \text{can} \otimes \alpha_{\mathcal{F}_0, \mathcal{C}_0} \\ \mathcal{C}_c^\infty(K_{l,\mathcal{F}\mathcal{F}_0} \cap K_{l,\mathcal{F}_{C_0}\mathcal{F}_{C_0}}) \otimes_{R(K_{l,\mathcal{F}_0} \cap K_{l,\mathcal{F}_{C_0}})} \mathcal{W}_{l,\mathcal{F}_0}^0 & & \end{array}$$

By construction, these maps satisfy the following compatibility with the action maps :

$$\forall g \in G_\iota, g_{\mathcal{W}, \mathcal{F}_C} \circ \alpha_{\mathcal{F}, \mathcal{C}} = \alpha_{g_{\mathcal{F}}, g_{\mathcal{C}}} \circ g_{\mathcal{W}, \mathcal{F}}.$$

Now, let $\mathcal{F}' \subseteq \overline{\mathcal{F}}$ be another e -facet in the closure of \mathcal{F} . It is also contained in $\overline{\mathcal{C}}$, so we have a map $\alpha_{\mathcal{F}', \mathcal{C}}$ as above. By Corollary 3.5.7, there is a unique $R(K_{\iota, \mathcal{F}}^\dagger \cap K_{\iota, \mathcal{F}'}^\dagger)$ -linear map

$$\beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}} : \mathcal{W}_{\iota, \mathcal{F}} \longrightarrow \mathcal{W}_{\iota, \mathcal{F}'}$$

such that $\alpha_{\mathcal{F}, \mathcal{C}} = \alpha_{\mathcal{F}', \mathcal{C}} \circ \beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}}$. It is a generator of the R -module $\text{Hom}_{R(K_{\iota, \mathcal{F}}^\dagger \cap K_{\iota, \mathcal{F}'}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{W}_{\iota, \mathcal{F}'})$ and, by Proposition 3.6.11, it actually belongs to $\text{Hom}_{R(K_{\iota, \mathcal{F}} \cap K_{\iota, \mathcal{F}'})}(\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{W}_{\iota, \mathcal{F}'})$. By uniqueness, we have $\beta_{\mathcal{F}', \mathcal{F}'', \mathcal{C}} \circ \beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}} = \beta_{\mathcal{F}, \mathcal{F}'', \mathcal{C}}$ and, by construction again, we have the following compatibility with the action maps :

$$\forall g \in G_\iota, g_{\mathcal{W}, \mathcal{F}'} \circ \beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}} = \beta_{g_{\mathcal{F}}, g_{\mathcal{F}'}, g_{\mathcal{C}}} \circ g_{\mathcal{W}, \mathcal{F}}.$$

Lemma. — $\beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}}$ is independent of \mathcal{C} . We will denote it simply by $\beta_{\mathcal{F}, \mathcal{F}'}$

Proof. Recall that $G_{\iota, \mathcal{F}}$ acts transitively on the set of chambers that contain \mathcal{F} . Moreover, by our choice of e , we have $G_{\iota, \mathcal{F}} \subseteq G_{\iota, \mathcal{F}'}$ and $K_{\iota, \mathcal{F}} \subseteq K_{\iota, \mathcal{F}'}$, hence also $G_{\iota, \mathcal{F}} \subseteq K_{\iota, \mathcal{F}} \cap K_{\iota, \mathcal{F}'}$. Now, when $g \in G_{\iota, \mathcal{F}}$, the last displayed property reads $g_{\mathcal{W}, \mathcal{F}'} \circ \beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}} = \beta_{\mathcal{F}, \mathcal{F}', g_{\mathcal{C}}} \circ g_{\mathcal{W}, \mathcal{F}}$. On the other hand, the $K_{\iota, \mathcal{F}} \cap K_{\iota, \mathcal{F}'}$ -equivariance of $\beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}}$ means that $g_{\mathcal{W}, \mathcal{F}'} \circ \beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}} = \beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}} \circ g_{\mathcal{W}, \mathcal{F}}$. It follows that $\beta_{\mathcal{F}, \mathcal{F}', \mathcal{C}} = \beta_{\mathcal{F}, \mathcal{F}', g_{\mathcal{C}}}$. \square

3.7.5 Construction of a Heisenberg–Weil coefficient system under Assumption 3.7.1, step 3: morphisms. — Let now \mathcal{F} and \mathcal{F}' be arbitrary and recall that $\text{Hom}(\mathcal{F}, \mathcal{F}') = \{g \in K_{\iota, \mathcal{F}'} G_\iota, \overline{g\mathcal{F}} \supseteq \mathcal{F}'\}$.

$$\text{For } g \in \text{Hom}(\mathcal{F}, \mathcal{F}'), \text{ set } \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}(g) := \mathcal{W}_{\iota, g^{-1}\mathcal{F}', \mathcal{F}'}(g) \circ \beta_{\mathcal{F}, g^{-1}\mathcal{F}'}$$

This is consistent with our previous definition when $g\mathcal{F} = \mathcal{F}'$ since $\beta_{\mathcal{F}, \mathcal{F}} = \text{Id}$.

Lemma. — We have $\mathcal{W}_{\iota, \mathcal{F}', \mathcal{F}''}(h) \circ \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}(g) = \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}''}(hg)$ for all $g \in \text{Hom}(\mathcal{F}, \mathcal{F}')$ and $h \in \text{Hom}(\mathcal{F}', \mathcal{F}'')$.

Proof. Choose decompositions $g = kg_\iota$ and $h = lh_\iota$ with $k \in K_{\iota, \mathcal{F}'}$, $l \in K_{\iota, \mathcal{F}''}$ and $g_\iota, h_\iota \in G_\iota$. We have

$$\begin{aligned} \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}(g) &= \mathcal{W}_{\iota, \mathcal{F}', \mathcal{F}'}(k) \circ (g_\iota)_{\mathcal{W}, g_\iota^{-1}\mathcal{F}'} \circ \beta_{\mathcal{F}, g_\iota^{-1}\mathcal{F}'} \\ \mathcal{W}_{\iota, \mathcal{F}', \mathcal{F}''}(h) &= \mathcal{W}_{\iota, \mathcal{F}'', \mathcal{F}''}(l) \circ (h_\iota)_{\mathcal{W}, h_\iota^{-1}\mathcal{F}''} \circ \beta_{\mathcal{F}', h_\iota^{-1}\mathcal{F}''} \end{aligned}$$

Since $\beta_{\mathcal{F}', h_\iota^{-1}\mathcal{F}''}$ is $K_{\iota, \mathcal{F}'} \cap K_{\iota, h_\iota^{-1}\mathcal{F}''}$ -equivariant and $K_{\iota, \mathcal{F}'} \subset K_{\iota, h_\iota^{-1}\mathcal{F}''}$, we have

$$\beta_{\mathcal{F}', h_\iota^{-1}\mathcal{F}''} \circ \mathcal{W}_{\iota, \mathcal{F}', \mathcal{F}'}(k) = \mathcal{W}_{\iota, h_\iota^{-1}\mathcal{F}'', h_\iota^{-1}\mathcal{F}''}(k) \circ \beta_{\mathcal{F}', h_\iota^{-1}\mathcal{F}''}.$$

On the other hand, we have already seen that

$$\beta_{\mathcal{F}', h_\iota^{-1}\mathcal{F}''} \circ (g_\iota)_{\mathcal{W}, g_\iota^{-1}\mathcal{F}'} = (g_\iota)_{\mathcal{W}, (h_\iota g_\iota)^{-1}\mathcal{F}''} \circ \beta_{g_\iota^{-1}\mathcal{F}', (h_\iota g_\iota)^{-1}\mathcal{F}''}.$$

From the properties of \mathcal{W}_l on G_l -orbits, we have

$$\mathcal{W}_{l, \mathcal{F}'', \mathcal{F}''}(l) \circ (h_l)_{\mathcal{W}, h_l^{-1} \mathcal{F}''} \circ \mathcal{W}_{l, h_l^{-1} \mathcal{F}'', h_l^{-1} \mathcal{F}''}(k) \circ (g_l)_{\mathcal{W}, (h_l g_l)^{-1} \mathcal{F}''} = \mathcal{W}_{l, (hg)^{-1} \mathcal{F}'', \mathcal{F}''}(hg).$$

Finally, the transitivity of face maps implies

$$\beta_{g_l^{-1} \mathcal{F}', (h_l g_l)^{-1} \mathcal{F}''} \beta_{\mathcal{F}, g_l^{-1} \mathcal{F}'} = \beta_{\mathcal{F}, (h_l g_l)^{-1} \mathcal{F}''} = \beta_{\mathcal{F}, (hg)^{-1} \mathcal{F}''}.$$

The four last displayed equalities imply the lemma. \square

Proposition. — *Let R be a commutative $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra, and assume Assumption 3.7.1. Then there exists a Heisenberg–Weil coefficient system \mathcal{W}_l in $\text{Coef}_R(\mathcal{B}'_{l, \bullet/e}/K_l)$.*

Proof. By Lemma 3.7.5 the maps $\mathcal{F} \mapsto \mathcal{W}_{l, \mathcal{F}}$ and $g \in \text{Hom}(\mathcal{F}, \mathcal{F}') \mapsto \mathcal{W}_{l, \mathcal{F}, \mathcal{F}'}(g)$ define a functor $[\mathcal{B}'_{l, \bullet/e}/K_l] \rightarrow R\text{-Mod}$. We have already seen that the action of $K_{l, \mathcal{F}}^\dagger$ on $\mathcal{W}_{l, \mathcal{F}}$ through $\mathcal{W}_{l, \mathcal{F}, \mathcal{F}}$ is a Heisenberg representation, and each face map $\beta_{\mathcal{F}, \mathcal{F}'} = \mathcal{W}_{l, \mathcal{F}, \mathcal{F}'}(1)$ is a generator of $\text{Hom}_{R(K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}^\dagger)}(\mathcal{W}_{l, \mathcal{F}}, \mathcal{W}_{l, \mathcal{F}'})$ by construction. \square

We now stop imposing Assumption 3.7.1.

3.7.6 Minimal factorization through a rational Levi subgroup. — Consider the centralizer $\mathcal{M} := C_{L\mathbf{G}}(Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F), \circ})$ in $L\mathbf{G}$ of the maximal W_F -invariant central torus in $C_{\hat{\mathbf{G}}}(\phi)$. This is a Levi subgroup of $L\mathbf{G}$ in the sense of Borel [Bor79, §3.4], so, after conjugating ϕ , we may assume \mathcal{M} is of the form $L\mathbf{M}$ for some (non-twisted) F -rational Levi subgroup \mathbf{M} of \mathbf{G} . By construction, both ϕ and φ factor through \mathcal{M} , providing us with a wild inertia parameter $\phi_M \in \Phi(P_F, \mathbf{M})$ (since any relevant extension φ' for G factors through \mathcal{M} and thus provides a relevant extension for M). Note that, by definition, $\hat{\mathbf{M}} = \mathcal{M}^\circ$ is the centralizer in $\hat{\mathbf{G}}$ of $(\hat{\mathbf{S}}_\phi)^{W_F, \circ}$. On the other side, pick a $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in I and consider the centralizer $C_{\mathbf{G}}(\iota(\mathbf{S}_\phi^{\text{split}}))$ of the image of the maximal split subtorus of \mathbf{S}_ϕ . This is a rational Levi subgroup of \mathbf{G} in the G -conjugacy class of \mathbf{M} . So, after conjugating ι , we may assume it factors through \mathbf{M} , so that $\mathbf{M} = C_{\mathbf{G}}(\iota(\mathbf{S}_\phi^{\text{split}}))$. By construction of \mathcal{M} , we have $C_{\hat{\mathbf{G}}}(\phi) \subseteq \hat{\mathbf{M}}$, hence $\mathbf{S}_\phi = \mathbf{S}_{\phi_M}$, and ι determines an embedding ι_M in I_{ϕ_M} , for which we have $\mathbf{M}_{\iota_M} = \mathbf{G}_\iota$.

By construction, Assumption 3.7.1 is satisfied for $(\hat{\mathbf{M}}, \phi_M, I_M)$. So, Proposition 3.7.5 applies and provides us with a Heisenberg–Weil coefficient system related to these data. We need to compare the buildings at stake.

Let us take up the notations \mathcal{B}_M and \mathcal{B}_{ι_M} of 2.8.8. These are subsets of \mathcal{B} , and we denote by \mathcal{B}'_M and \mathcal{B}'_{ι_M} their images in \mathcal{B}' . In our setup here, we actually have $\mathcal{B}'_\iota = \mathcal{B}'_{\iota_M}$. These subsets are stable under the e -facet decomposition of \mathcal{B}' , and we reserve the notation $\mathcal{B}'_{M, \bullet/e}$, resp. $\mathcal{B}'_{\iota_M, \bullet/e}$, for the corresponding posets. On the other hand, the reduced Bruhat–Tits building $\mathcal{B}'(\mathbf{M}, F)$ of \mathbf{M} also comes with its own e -facet decomposition, and its subset $\mathcal{B}'(\mathbf{M}, F)_{\iota_M}$ is stable under this decomposition. We denote by $\mathcal{B}'(\mathbf{M}, F)_{\bullet/e}$, resp., $\mathcal{B}'(\mathbf{M}, F)_{\iota_M, \bullet/e}$, the associated posets. There is a canonical projection $\mathcal{B}'_M \rightarrow \mathcal{B}'(\mathbf{M}, F)$ that takes any e -facet of \mathcal{B}'_M into an e -facet of $\mathcal{B}'(\mathbf{M}, F)$. This projection thus induces a functor

$$\pi : [\mathcal{B}'_{\iota, \bullet/e}/(M \cap G_\iota)] = [\mathcal{B}'_{\iota_M, \bullet/e}/M_{\iota_M}] \rightarrow [\mathcal{B}'(\mathbf{M}, F)_{\iota_M, \bullet/e}/M_{\iota_M}].$$

Denote by $K_{\iota_M, \pi(\mathcal{F})}$, $K_{\iota_M, \pi(\mathcal{F})}^\dagger$, etc., the objects analogous to $K_{l, \mathcal{F}}$, $K_{l, \mathcal{F}}^\dagger$, etc., constructed for $(\mathbf{M}, \phi_M, \iota_M \in I_M)$ and the facet $\pi(\mathcal{F})$ in place of $(\mathbf{G}, \phi, \iota \in I)$ and the facet \mathcal{F} . Then we have

an inclusion $M \cap K_{\iota, \mathcal{F}} \subset K_{\iota_M, \pi(\mathcal{F})}$ and an equality $M \cap K_{\iota, \mathcal{F}}^\dagger = K_{\iota_M, \pi(\mathcal{F})}^\dagger$. So our functor π can be upgraded to

$$\pi : [\mathcal{B}'_{\iota, \bullet/e} / (M \cap K_\iota)] \longrightarrow [\mathcal{B}'(\mathbf{M}, F)_{\iota_M, \bullet/e} / K_{\iota_M}].$$

Now, the Heisenberg–Weil coefficient system \mathcal{W}_{ι_M} provided by Proposition 3.7.5, is a functor from $[\mathcal{B}'(\mathbf{M}, F)_{\iota_M, \bullet/e} / K_{\iota_M}]$ to $R\text{-Mod}$, so we may and will compose it with π to get a coefficient system on $[\mathcal{B}'_{\iota, \bullet/e} / (M \cap K_\iota)]$ that we still denote by \mathcal{W}_{ι_M} . In the next paragraphs, we start from \mathcal{W}_{ι_M} to construct a Heisenberg–Weil coefficient system \mathcal{W}_ι on $[\mathcal{B}'_{\iota, \bullet/e} / K_\iota]$.

We set $K_{\iota_M, \mathcal{F}}^? := M \cap K_{\iota, \mathcal{F}}^?$ for $? = \dagger, +, \emptyset$. We have $K_{\iota_M, \mathcal{F}}^\dagger = K_{\iota_M, \pi(\mathcal{F})}^\dagger$, $K_{\iota_M, \mathcal{F}}^+ = K_{\iota_M, \pi(\mathcal{F})}^+$, and $K_{\iota_M, \pi(\mathcal{F})}^\circ \subset K_{\iota_M, \mathcal{F}} \subset K_{\iota_M, \pi(\mathcal{F})}$.

3.7.7 Construction of a Heisenberg–Weil coefficient system 1: modules. — We choose a parabolic subgroup \mathbf{P} of \mathbf{G} whose Levi subgroup is \mathbf{M} and we denote its unipotent radical by \mathbf{U} and follow the notation from 2.8.7 and 2.8.8, in particular $\bar{\mathbf{P}}$ denotes the opposite parabolic subgroup of \mathbf{P} with respect to \mathbf{M} with unipotent radical $\bar{\mathbf{U}}$, and $U_{\iota, \mathcal{F}}^\dagger = K_{\iota, \mathcal{F}}^\dagger \cap \mathbf{U}$, $U_{\iota, \mathcal{F}}^+ = K_{\iota, \mathcal{F}}^+ \cap \mathbf{U}$, $\bar{U}_{\iota, \mathcal{F}}^\dagger = K_{\iota, \mathcal{F}}^\dagger \cap \bar{\mathbf{U}}$, and $\bar{U}_{\iota, \mathcal{F}}^+ = K_{\iota, \mathcal{F}}^+ \cap \bar{\mathbf{U}}$. In the setting here, we have $K_{\iota, \mathcal{F}} = U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}} \bar{U}_{\iota, \mathcal{F}}^\dagger$. Indeed, this follows from $K_{\iota, \mathcal{F}}^\dagger = U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}}^\dagger \bar{U}_{\iota, \mathcal{F}}^\dagger$ and $G_{\iota, \mathcal{F}} = M_{\iota_M, \mathcal{F}}$. Recall from Lemma 2.8.9 i) that $\check{\phi}_{\iota, \mathcal{F}}^+$ is the character on $K_{\iota, \mathcal{F}}^+ = U_{\iota, \mathcal{F}}^+ K_{\iota_M, \mathcal{F}}^+ \bar{U}_{\iota, \mathcal{F}}^+$ whose restriction to $U_{\iota, \mathcal{F}}^+ \bar{U}_{\iota, \mathcal{F}}^+$ is trivial and whose restriction to $K_{\iota_M, \mathcal{F}}^+$ is $\check{\phi}_{\iota_M, \mathcal{F}}^+$.

For every $\mathcal{F} \in \mathcal{B}'_{\iota, \bullet/e}$, we extend $\mathcal{W}_{\iota_M, \mathcal{F}}$ to a representation of $U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}} \bar{U}_{\iota, \mathcal{F}}^+$ by requiring $U_{\iota, \mathcal{F}}^\dagger$ and $\bar{U}_{\iota, \mathcal{F}}^+$ to act trivially. Then we set

$$\mathcal{W}_{\iota, \mathcal{F}} := \text{ind}_{U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}} \bar{U}_{\iota, \mathcal{F}}^+}^{K_{\iota, \mathcal{F}}} \mathcal{W}_{\iota_M, \mathcal{F}},$$

which comes with the two $U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}} \bar{U}_{\iota, \mathcal{F}}^+$ -equivariant maps corresponding to the identity via Frobenius reciprocity:

$$i_{\mathcal{F}} : \mathcal{W}_{\iota_M, \mathcal{F}} \hookrightarrow \mathcal{W}_{\iota, \mathcal{F}} \quad \text{and} \quad pr_{\mathcal{F}} : \mathcal{W}_{\iota, \mathcal{F}} \twoheadrightarrow \mathcal{W}_{\iota_M, \mathcal{F}}.$$

Note that $pr_{\mathcal{F}} \circ i_{\mathcal{F}}$ is the identity and $i_{\mathcal{F}} \circ pr_{\mathcal{F}}$ is the projection $e_{U_{\iota, \mathcal{F}}^\dagger}$ on the submodule on which $U_{\iota, \mathcal{F}}^\dagger$ acts trivially.

Lemma. — $\mathcal{W}_{\iota, \mathcal{F}}|_{K_{\iota, \mathcal{F}}^\dagger}$ is a Heisenberg representation for the triple $(K_{\iota_M, \mathcal{F}}^\dagger, K_{\iota_M, \mathcal{F}}^+, \check{\phi}_{\iota_M, \mathcal{F}}^+)$

Proof. We have $\mathcal{W}_{\iota, \mathcal{F}}|_{K_{\iota, \mathcal{F}}^\dagger} = \text{ind}_{U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}}^\dagger \bar{U}_{\iota, \mathcal{F}}^+}^{K_{\iota, \mathcal{F}}^\dagger} \mathcal{W}_{\iota_M, \mathcal{F}}|_{K_{\iota_M, \mathcal{F}}^\dagger}$ with $\mathcal{W}_{\iota_M, \mathcal{F}}|_{K_{\iota_M, \mathcal{F}}^\dagger}$ a Heisenberg representation for the triple $(K_{\iota_M, \mathcal{F}}^\dagger, K_{\iota_M, \mathcal{F}}^+, \check{\phi}_{\iota_M, \mathcal{F}}^+)$. Since the restriction of $\check{\phi}_{\iota, \mathcal{F}}^+$ to $U_{\iota, \mathcal{F}}^+$ and $\bar{U}_{\iota, \mathcal{F}}^+$ is trivial and to $K_{\iota_M, \mathcal{F}}^+$ is $\check{\phi}_{\iota_M, \mathcal{F}}^+$, we can prove that $\mathcal{W}_{\iota, \mathcal{F}}|_{K_{\iota, \mathcal{F}}^\dagger}$ is a Heisenberg representation for the triple $(K_{\iota, \mathcal{F}}^\dagger, K_{\iota, \mathcal{F}}^+, \check{\phi}_{\iota, \mathcal{F}}^+)$ as in Lemma 3.5.5 iii)(a). Namely, up to twisting by an invertible R -module, we may assume that $\mathcal{W}_{\iota_M, \mathcal{F}}|_{K_{\iota_M, \mathcal{F}}^\dagger}$ is induced from some Lagrangian $W \subset K_{\iota_M, x}^\dagger / K_{\iota_M, x}^+$, in which case, $\mathcal{W}_{\iota, \mathcal{F}}|_{K_{\iota, \mathcal{F}}^\dagger}$ is induced from the Lagrangian $U_{\iota, x}^\dagger / U_{\iota, x}^+ \oplus W \subset K_{\iota, x}^\dagger / K_{\iota, x}^+$. \square

3.7.8 Construction of a Heisenberg–Weil coefficient system 2: action and face maps. —

Let $\mathcal{F}, \mathcal{F}' \in \mathcal{B}'_{\iota, \bullet/e}$ and $g \in G_{\iota, \mathcal{F}, \mathcal{F}'} \subset M$, then $gU_{\iota, \mathcal{F}}^\dagger g^{-1} = U_{\iota, g\mathcal{F}}^\dagger$, $gK_{\iota_M, \mathcal{F}} g^{-1} = K_{\iota_M, g\mathcal{F}}$, $g\bar{U}_{\iota, \mathcal{F}}^+ g^{-1} = \bar{U}_{\iota, g\mathcal{F}}^+$ and $gK_{\iota, \mathcal{F}} g^{-1} = K_{\iota, g\mathcal{F}}$. Hence $g\mathcal{W}_{\iota_M, \mathcal{F}}$ induces a unique $K_{\iota, \mathcal{F}}$ -equivariant isomorphism $g\mathcal{W}_{\iota, \mathcal{F}} : \mathcal{W}_{\iota, \mathcal{F}} \xrightarrow{\sim} g^*\mathcal{W}_{\iota, g\mathcal{F}}$, which agrees with the action of g on $\mathcal{W}_{\iota, \mathcal{F}}$ if $g \in G_{\iota, \mathcal{F}}$.

Lemma. — For each pair of e -facets $\mathcal{F}, \mathcal{F}' \in \mathcal{B}'_{\iota, \bullet/e}$ with $\bar{\mathcal{F}} \supseteq \mathcal{F}'$, there exists a unique generator $\beta_{\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{F}'} \in \text{Hom}_{R(K_{\iota, \mathcal{F}}^\dagger \cap K_{\iota, \mathcal{F}'}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{W}_{\iota, \mathcal{F}'})$ such that $\text{pr}_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} = \beta_{\mathcal{W}_{\iota_M, \mathcal{F}}, \mathcal{F}'}$.

Proof. Let \mathcal{F}_0 and \mathcal{F}'_0 be two e -facets whose closures contain \mathcal{F} and \mathcal{F}' , respectively, and such that

$$U_{\iota, \mathcal{F}_0}^\dagger = U_{\iota, \mathcal{F}}^\dagger, \quad \bar{U}_{\iota, \mathcal{F}_0}^+ = \bar{U}_{\iota, \mathcal{F}}^+ = \bar{U}_{\iota, \mathcal{F}_0}^+, \quad U_{\iota, \mathcal{F}'_0}^\dagger = U_{\iota, \mathcal{F}'}^\dagger, \quad \bar{U}_{\iota, \mathcal{F}'_0}^+ = \bar{U}_{\iota, \mathcal{F}'}^+ = \bar{U}_{\iota, \mathcal{F}'_0}^+,$$

and

$$K_{\iota_M, \mathcal{F}_0} = K_{\iota_M, \mathcal{F}}, \quad K_{\iota_M, \mathcal{F}_0}^\dagger = K_{\iota_M, \mathcal{F}}^\dagger, \quad K_{\iota_M, \mathcal{F}'_0} = K_{\iota_M, \mathcal{F}'}, \quad K_{\iota_M, \mathcal{F}'_0}^\dagger = K_{\iota_M, \mathcal{F}'}^\dagger.$$

This can be achieved by letting \mathcal{F}_0 be the e -facet containing $x + \varepsilon\lambda$ for x a point in \mathcal{F} , $\varepsilon > 0$ sufficiently small and λ an appropriate cocharacter of the center of \mathbf{M} (recall that the maximal split torus of the center of \mathbf{M} is contained in \mathbf{M}_ι). The e -facet \mathcal{F}'_0 can be obtained analogously. Then we have

$$(3.7.8a) \quad K_{\iota, \mathcal{F}_0}^\dagger = U_{\iota, \mathcal{F}_0}^\dagger K_{\iota_M, \mathcal{F}_0}^\dagger \bar{U}_{\iota, \mathcal{F}_0}^+ = U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}}^\dagger \bar{U}_{\iota, \mathcal{F}}^+ \text{ and } \mathcal{W}_{\iota, \mathcal{F}_0} = \mathcal{W}_{\iota_M, \mathcal{F}_0} \simeq \mathcal{W}_{\iota_M, \mathcal{F}}$$

$$(3.7.8b) \quad K_{\iota, \mathcal{F}'_0}^\dagger = U_{\iota, \mathcal{F}'_0}^\dagger K_{\iota_M, \mathcal{F}'_0}^\dagger \bar{U}_{\iota, \mathcal{F}'_0}^+ = U_{\iota, \mathcal{F}'}^\dagger K_{\iota_M, \mathcal{F}'}^\dagger \bar{U}_{\iota, \mathcal{F}'}^+ \text{ and } \mathcal{W}_{\iota, \mathcal{F}'_0} = \mathcal{W}_{\iota_M, \mathcal{F}'_0} \simeq \mathcal{W}_{\iota_M, \mathcal{F}'}$$

Applying Lemma 3.5.6 to points $(x, x', x'') \in \mathcal{F}_0 \times \mathcal{F} \times \mathcal{F}'$ and to points $(x, x', x'') \in \mathcal{F}_0 \times \mathcal{F}' \times \mathcal{F}'_0$ yields an isomorphism arising from the composition:

$$(3.7.8c) \quad \begin{aligned} & \text{Hom}_{R(K_{\iota, \mathcal{F}_0}^\dagger \cap K_{\iota, \mathcal{F}}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}_0}, \mathcal{W}_{\iota, \mathcal{F}}) \otimes \text{Hom}_{R(K_{\iota, \mathcal{F}}^\dagger \cap K_{\iota, \mathcal{F}'}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{W}_{\iota, \mathcal{F}'}) \otimes \\ & \otimes \text{Hom}_{R(K_{\iota, \mathcal{F}'}^\dagger \cap K_{\iota, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}'}, \mathcal{W}_{\iota, \mathcal{F}'_0}) \xrightarrow{\sim} \text{Hom}_{R(K_{\iota, \mathcal{F}_0}^\dagger \cap K_{\iota, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}_0}, \mathcal{W}_{\iota, \mathcal{F}'_0}) \end{aligned}$$

and likewise

$$(3.7.8d) \quad \begin{aligned} & \text{Hom}_{R(K_{\iota_M, \mathcal{F}_0}^\dagger \cap K_{\iota_M, \mathcal{F}}^\dagger)}(\mathcal{W}_{\iota_M, \mathcal{F}_0}, \mathcal{W}_{\iota_M, \mathcal{F}}) \otimes \text{Hom}_{R(K_{\iota_M, \mathcal{F}}^\dagger \cap K_{\iota_M, \mathcal{F}'}^\dagger)}(\mathcal{W}_{\iota_M, \mathcal{F}}, \mathcal{W}_{\iota_M, \mathcal{F}'}) \otimes \\ & \otimes \text{Hom}_{R(K_{\iota_M, \mathcal{F}'}^\dagger \cap K_{\iota_M, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{\iota_M, \mathcal{F}'}, \mathcal{W}_{\iota_M, \mathcal{F}'_0}) \xrightarrow{\sim} \text{Hom}_{R(K_{\iota_M, \mathcal{F}_0}^\dagger \cap K_{\iota_M, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{\iota_M, \mathcal{F}_0}, \mathcal{W}_{\iota_M, \mathcal{F}'_0}). \end{aligned}$$

Since \mathcal{W}_{ι_M} is a Heisenberg–Weil coefficient system, all the Hom-space on the left hand side of (3.7.8d) are free R -modules of rank 1, and hence so is $\text{Hom}_{R(K_{\iota_M, \mathcal{F}_0}^\dagger \cap K_{\iota_M, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{\iota_M, \mathcal{F}_0}, \mathcal{W}_{\iota_M, \mathcal{F}'_0})$.

We will show that the same applies to all the Hom-spaces in (3.7.8c).

First, we have $\text{Hom}_{R(K_{\iota, \mathcal{F}_0}^\dagger \cap K_{\iota, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}_0}, \mathcal{W}_{\iota, \mathcal{F}'_0}) = \text{Hom}_{R(K_{\iota_M, \mathcal{F}_0}^\dagger \cap K_{\iota_M, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{\iota_M, \mathcal{F}_0}, \mathcal{W}_{\iota_M, \mathcal{F}'_0})$, which we just saw is a free R -module of rank 1.

Next, recalling that $(\mathcal{W}_{\iota, \mathcal{F}})|_{K_{\iota, \mathcal{F}}^\dagger} = \text{ind}_{U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}}^\dagger \bar{U}_{\iota, \mathcal{F}}^+}^{K_{\iota, \mathcal{F}}^\dagger} \mathcal{W}_{\iota_M, \mathcal{F}}$ we obtain from (3.7.8a) that

$$\begin{aligned} \text{Hom}_{R(K_{\iota, \mathcal{F}_0}^\dagger \cap K_{\iota, \mathcal{F}}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}_0}, \mathcal{W}_{\iota, \mathcal{F}}) & \simeq \text{Hom}_{R(U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}}^\dagger \bar{U}_{\iota, \mathcal{F}}^+)}(\mathcal{W}_{\iota_M, \mathcal{F}}, \mathcal{W}_{\iota, \mathcal{F}}) \\ & \simeq \text{Hom}_{R(K_{\iota, \mathcal{F}}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{W}_{\iota, \mathcal{F}}), \end{aligned}$$

which is a free R -module of rank 1 by Lemmas 3.7.7 and 3.5.2. Note that a generator of $\text{Hom}_{R(K_{l,\mathcal{F}_0}^\dagger \cap K_{l,\mathcal{F}}^\dagger)}(\mathcal{W}_{l,\mathcal{F}_0}, \mathcal{W}_{l,\mathcal{F}})$ is given by $i_{\mathcal{F}} \circ \beta_{\mathcal{W}_{l_M}, \mathcal{F}_0, \mathcal{F}}$, where we view the isomorphism¹⁰ $\beta_{\mathcal{W}_{l_M}, \mathcal{F}_0, \mathcal{F}} \in \text{Hom}_{R(K_{l_M, \mathcal{F}_0}^\dagger \cap K_{l_M, \mathcal{F}}^\dagger)}(\mathcal{W}_{l_M, \mathcal{F}_0}, \mathcal{W}_{l_M, \mathcal{F}}) = \text{Hom}_{R(K_{l_M, \mathcal{F}}^\dagger)}(\mathcal{W}_{l_M, \mathcal{F}_0}, \mathcal{W}_{l_M, \mathcal{F}})$ also as an isomorphism in $\text{Hom}_{R(U_{l, \mathcal{F}}^\dagger K_{l_M, \mathcal{F}}^\dagger \bar{U}_{l, \mathcal{F}}^+)}(\mathcal{W}_{l_M, \mathcal{F}_0}, \mathcal{W}_{l_M, \mathcal{F}})$. Similarly, we have

$$\begin{aligned} \text{Hom}_{R(K_{l, \mathcal{F}'}^\dagger \cap K_{l, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{l, \mathcal{F}'}, \mathcal{W}_{l, \mathcal{F}'_0}) &\simeq \text{Hom}_{R(U_{l, \mathcal{F}'}^\dagger K_{l_M, \mathcal{F}'}^\dagger \bar{U}_{l, \mathcal{F}'}^+)}(\mathcal{W}_{l, \mathcal{F}'}, \mathcal{W}_{l_M, \mathcal{F}'},) \\ &\simeq \text{Hom}_{R(K_{l, \mathcal{F}'}})(\mathcal{W}_{l, \mathcal{F}'}, \mathcal{W}_{l, \mathcal{F}'},), \end{aligned}$$

which is a free R -module of rank 1, and a generator of $\text{Hom}_{R(K_{l, \mathcal{F}'}^\dagger \cap K_{l, \mathcal{F}'_0}^\dagger)}(\mathcal{W}_{l, \mathcal{F}'}, \mathcal{W}_{l, \mathcal{F}'_0})$ is given by $\beta_{\mathcal{W}_{l_M}, \mathcal{F}', \mathcal{F}'_0} \circ pr_{\mathcal{F}'}$.

So we have shown that all the Hom-spaces in (3.7.8c) other than the invertible module $\text{Hom}_{R(K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}})(\mathcal{W}_{l, \mathcal{F}}, \mathcal{W}_{l, \mathcal{F}'})$ are free R -modules of rank 1. It follows that the latter is also free of rank 1 and, combining the isomorphisms (3.7.8c) and (3.7.8d), that there exists a unique generator $\beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'}$ of $\text{Hom}_{R(K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}})(\mathcal{W}_{l, \mathcal{F}}, \mathcal{W}_{l, \mathcal{F}'})$ such that

$$\beta_{\mathcal{W}_{l_M}, \mathcal{F}', \mathcal{F}'_0} \circ pr_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} \circ \beta_{\mathcal{W}_{l_M}, \mathcal{F}_0, \mathcal{F}} = \beta_{\mathcal{W}_{l_M}, \mathcal{F}', \mathcal{F}'_0} \circ \beta_{\mathcal{W}_{l_M}, \mathcal{F}, \mathcal{F}'} \circ \beta_{\mathcal{W}_{l_M}, \mathcal{F}_0, \mathcal{F}}.$$

Since $\beta_{\mathcal{W}_{l_M}, \mathcal{F}', \mathcal{F}'_0}$ and $\beta_{\mathcal{W}_{l_M}, \mathcal{F}_0, \mathcal{F}}$ are isomorphisms, this implies that there exists a unique generator $\beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'}$ of $\text{Hom}_{R(K_{l, \mathcal{F}}^\dagger \cap K_{l, \mathcal{F}'}})(\mathcal{W}_{l, \mathcal{F}}, \mathcal{W}_{l, \mathcal{F}'})$ such that $pr_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} = \beta_{\mathcal{W}_{l_M}, \mathcal{F}, \mathcal{F}'}$. \square

For each pair of e-facets $\mathcal{F}, \mathcal{F}' \in \mathcal{B}'_{l, \bullet/e}$ with $\bar{\mathcal{F}} \supseteq \mathcal{F}'$, set $\beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'}$ to be as in Lemma 3.7.8.

3.7.9 Construction of a Heisenberg–Weil coefficient system 3: morphisms. — We first observe that the face maps have the following desired properties that will allow us to use them to define a desired Heisenberg–Weil coefficient system.

Lemma. — *Let $\mathcal{F}, \mathcal{F}' \in \mathcal{B}'_{l, \bullet/e}$ with $\bar{\mathcal{F}} \supseteq \mathcal{F}'$.*

- i) *If $\mathcal{F}'' \in \mathcal{B}'_{l, \bullet/e}$ with $\bar{\mathcal{F}}' \supseteq \mathcal{F}''$, then we have $\beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} = \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}''}$.*
- ii) *If $g \in G_l$, then $g_{\mathcal{W}_{l, \mathcal{F}'}} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} = \beta_{\mathcal{W}_{l, g\mathcal{F}, g\mathcal{F}'}} \circ g_{\mathcal{W}_{l, \mathcal{F}}}$.*
- iii) *The map $\beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'}$ is $K_{l, \mathcal{F}} \cap K_{l, \mathcal{F}'}$ -equivariant.*

Proof. i) By Lemma 3.7.8 and since \mathcal{W}_{l_M} is a Heisenberg–Weil coefficient system, it suffices to show that $pr_{\mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''} \circ i_{\mathcal{F}'} \circ pr_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} = pr_{\mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}}$. Using that $i_{\mathcal{F}'} \circ pr_{\mathcal{F}'} = e_{U_{l, \mathcal{F}'}}^\dagger$, $pr_{\mathcal{F}''} = pr_{\mathcal{F}''} \circ i_{\mathcal{F}''} \circ pr_{\mathcal{F}''} = pr_{\mathcal{F}''} \circ e_{U_{l, \mathcal{F}''}}^\dagger$, $U_{\mathcal{F}'}^\dagger \subseteq U_{\mathcal{F}''}^\dagger$, and that $\beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''}$ is $U_{\mathcal{F}'}^\dagger \cap U_{\mathcal{F}''}^\dagger = U_{\mathcal{F}''}^\dagger$ -equivariant, we have

$$\begin{aligned} pr_{\mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''} \circ i_{\mathcal{F}'} \circ pr_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} &= pr_{\mathcal{F}''} \circ e_{U_{l, \mathcal{F}''}}^\dagger \circ \beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''} \circ e_{U_{l, \mathcal{F}'}}^\dagger \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} \\ &= pr_{\mathcal{F}''} \circ e_{U_{l, \mathcal{F}''}}^\dagger \circ e_{U_{l, \mathcal{F}'}}^\dagger \circ \beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} \\ &= pr_{\mathcal{F}''} \circ e_{U_{l, \mathcal{F}''}}^\dagger \circ \beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}} \\ &= pr_{\mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}'}, \mathcal{F}''} \circ \beta_{\mathcal{W}_{l, \mathcal{F}}, \mathcal{F}'} \circ i_{\mathcal{F}}. \end{aligned}$$

¹⁰Note that if we choose \mathcal{F}_0 such that \mathcal{F} and \mathcal{F}_0 map to the same facet in $\mathcal{B}'_{l_M, \bullet/e}$, then $\beta_{\mathcal{W}_{l_M}, \mathcal{F}_0, \mathcal{F}}$ is just the identity.

ii) We have

$$\begin{aligned} pr_{\mathcal{F}'} \circ g_{\mathcal{W}_{\iota, \mathcal{F}'}} \circ \beta_{\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}} \circ g_{\mathcal{W}_{\iota, \mathcal{F}}}^{-1} \circ i_{g_{\mathcal{F}}} &= g_{\mathcal{W}_{\iota_M, \mathcal{F}'}} \circ pr_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}} \circ i_{g_{\mathcal{F}}} \circ g_{\mathcal{W}_{\iota_M, \mathcal{F}}}^{-1} \\ &= g_{\mathcal{W}_{\iota_M, \mathcal{F}'}} \beta_{\mathcal{W}_{\iota_M, \mathcal{F}, \mathcal{F}'}} \circ g_{\mathcal{W}_{\iota_M, \mathcal{F}}}^{-1} = \beta_{\mathcal{W}_{\iota_M, g_{\mathcal{F}}, g_{\mathcal{F}'}}}, \end{aligned}$$

from which the claim follows by the definition of $\beta_{\mathcal{W}_{\iota, g_{\mathcal{F}}, g_{\mathcal{F}'}}}$.

iii) Since $K_{\iota, \mathcal{F}} \cap K_{\iota, \mathcal{F}'} = (K_{\iota_M, \mathcal{F}} \cap K_{\iota_M, \mathcal{F}'}) (K_{\iota, \mathcal{F}}^\dagger \cap K_{\iota, \mathcal{F}'}^\dagger)$, it suffices to show that $\beta_{\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}}$ is $K_{\iota_M, \mathcal{F}} \cap K_{\iota_M, \mathcal{F}'}$ -equivariant. Let $k \in K_{\iota_M, \mathcal{F}} \cap K_{\iota_M, \mathcal{F}'}$, then

$$pr_{\mathcal{F}'} \circ k \beta_{\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}} k^{-1} \circ i_{\mathcal{F}} = k \circ pr_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}} \circ i_{\mathcal{F}} \circ k^{-1} = k \beta_{\mathcal{W}_{\iota_M, \mathcal{F}, \mathcal{F}'}} k^{-1} = \beta_{\mathcal{W}_{\iota_M, \mathcal{F}, \mathcal{F}'}} ,$$

from which the claim follows by the definition of $\beta_{\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}}$. \square

Analogous to 3.7.5 we can complete the definition of the functor $[\mathcal{B}'_{\iota, \bullet/e}/K_\iota] \rightarrow R\text{-Mod}$ that sends \mathcal{F} to $\mathcal{W}_{\iota, \mathcal{F}}$ as follows. For $\mathcal{F}, \mathcal{F}' \in \mathcal{B}'_{\iota, \bullet/e}$, the functor sends an element $g \in \text{Hom}(\mathcal{F}, \mathcal{F}') = \{g \in K_{\iota, \mathcal{F}} G_\iota, \overline{g\mathcal{F}} \supseteq \mathcal{F}'\}$ to

$$\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}(g) := g_{\mathcal{W}, g^{-1}\mathcal{F}'} \circ \beta_{\mathcal{F}, g^{-1}\mathcal{F}'}$$

Corollary. — We have $\mathcal{W}_{\iota, \mathcal{F}', \mathcal{F}''}(h) \circ \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}(g) = \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}''}(hg)$ for all $g \in \text{Hom}(\mathcal{F}, \mathcal{F}')$ and $h \in \text{Hom}(\mathcal{F}', \mathcal{F}'')$.

Proof. The proof is the same as the proof of Lemma 3.7.5 using Lemma 3.7.9. \square

Proof of Theorem 3.3.7. By Corollary 3.7.9 the maps $\mathcal{F} \mapsto \mathcal{W}_{\iota, \mathcal{F}}$ and $g \in \text{Hom}(\mathcal{F}, \mathcal{F}') \mapsto \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}(g)$ define a functor $[\mathcal{B}'_{\iota, \bullet/e}/K_\iota] \rightarrow R\text{-Mod}$. By Lemma 3.7.7 the action of $K_{\iota, \mathcal{F}}^\dagger$ on $\mathcal{W}_{\iota, \mathcal{F}}$ through $\mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}}$ is a Heisenberg representation, and by construction, i.e., Lemma 3.7.8, each face map $\beta_{\mathcal{F}, \mathcal{F}'} = \mathcal{W}_{\iota, \mathcal{F}, \mathcal{F}'}(1)$ is a generator of $\text{Hom}_{R(K_{\iota, \mathcal{F}}^\dagger \cap K_{\iota, \mathcal{F}'}^\dagger)}(\mathcal{W}_{\iota, \mathcal{F}}, \mathcal{W}_{\iota, \mathcal{F}'})$. \square

The following proposition is not needed for the construction of the equivalence itself, but is a result of independent interest and allows us to deduce Corollary 3.7.11 and Corollary 4.1.2.

3.7.10 Proposition. — $\mathcal{W}_{\iota, \mathcal{F}}$ is a twisted $RK_{\iota, \mathcal{F}}$ -Heisenberg–Weil representation whose restriction to $K_{\iota, \mathcal{F}}^+$ is $\check{\phi}_{\iota, \mathcal{F}}^+$ -isotypic.

Proof. Let $\mathcal{W}'_{\iota, \mathcal{F}}$ denote the twisted $RK_{\iota, \mathcal{F}}$ -Heisenberg–Weil representation attached to the Heisenberg representation $\mathcal{W}_{\iota, \mathcal{F}}|_{K_{\iota, \mathcal{F}}^\dagger}$, we need to show that $\mathcal{W}_{\iota, \mathcal{F}} \simeq \mathcal{W}'_{\iota, \mathcal{F}}$. Let \mathcal{C} be an e-facet of maximal dimension whose closure contains \mathcal{F} and such that $U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}} \bar{U}_{\iota, \mathcal{F}}^+ = U_{\iota, \mathcal{C}}^\dagger K_{\iota_M, \mathcal{C}} \bar{U}_{\iota, \mathcal{C}}^+$. Then by the above observation we have $\mathcal{W}_{\iota_M, \mathcal{F}} \simeq \mathcal{W}_{\iota, \mathcal{C}}$ viewed as representations of $U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}} \bar{U}_{\iota, \mathcal{F}}^+ = K_{\iota, \mathcal{C}}$ via appropriate inflation when needed. Combined with the restriction of $i_{\mathcal{F}}$ to $K_{\iota, \mathcal{F}}^\dagger$, we obtain a $K_{\iota, \mathcal{F}}^\dagger \cap K_{\iota, \mathcal{C}}^\dagger$ -equivariant homomorphism $\mathcal{W}_{\iota_M, \mathcal{F}} \simeq \mathcal{W}_{\iota, \mathcal{C}} \rightarrow \mathcal{W}'_{\iota, \mathcal{F}}$, which by Proposition 3.6.11 is also $K_{\iota, \mathcal{F}} \cap K_{\iota, \mathcal{C}}$ -equivariant. Since $K_{\iota, \mathcal{F}} \cap K_{\iota, \mathcal{C}} = K_{\iota, \mathcal{C}} = U_{\iota, \mathcal{F}}^\dagger K_{\iota_M, \mathcal{F}} \bar{U}_{\iota, \mathcal{F}}^+$, Frobenius reciprocity provides us with a non-trivial $K_{\iota, \mathcal{F}}$ -morphism $\mathcal{W}_{\iota, \mathcal{F}} \rightarrow \mathcal{W}'_{\iota, \mathcal{F}}$, which is an isomorphism of $R(K_{\iota, \mathcal{F}}^\dagger)$ -modules by Lemma 3.5.5.iii), hence an isomorphism of $R(K_{\iota, \mathcal{F}})$ -modules. \square

3.7.11 Corollary. — Given a point $x \in \mathcal{B}'_l$ and any twisted $RK_{l,x}$ -Heisenberg–Weil representation $\kappa_{l,x}$ whose restriction to $K_{l,x}^+$ is $\check{\phi}_{l,x}^+$ -isotypic, there exists a Heisenberg–Weil coefficient system \mathcal{W}_l on $\mathcal{B}'_{l,\bullet/e}$ such that $\kappa_{l,x} \simeq \mathcal{W}_{l,\mathcal{F}}$, where \mathcal{F} denotes the e -facet that contains x .

Proof. Since by Definition 3.6.8 a twisted $RK_{l,x}$ -Heisenberg–Weil representation is determined by the underlying Heisenberg representation of $K_{l,x}^+$, it suffices by Proposition 3.7.10 to prove that any Heisenberg representation of $K_{l,x}^+$ whose restriction to $K_{l,x}^+$ is $\check{\phi}_{l,x}^+$ -isotypic can occur as the restriction of $\mathcal{W}_{l,\mathcal{F}}$ for some Heisenberg–Weil coefficient system \mathcal{W}_l . Let \mathcal{W}'_l be some Heisenberg–Weil coefficient system as constructed above where we choose \mathcal{C}_0 and \mathcal{S} in 3.7.2 so that \mathcal{S} contains the image of \mathcal{F} in $\mathcal{B}'_{l_M,\bullet/e}$. Then, by 3.5.4, we have $\kappa_{l,x}|_{K_{l,x}^+} \simeq \mathcal{W}'_{l,\mathcal{F}}|_{K_{l,x}^+} \otimes_R R'$ where R' denotes the invertible R -module $\text{Hom}_{K_{l,x}^+}(\mathcal{W}'_{l,\mathcal{F}}|_{K_{l,x}^+}, \kappa_{l,x}|_{K_{l,x}^+})$. Replacing all the twisted Heisenberg–Weil representations in the basic choices in 3.7.2 by their R -tensor product with R' , we obtain from the above construction a new Heisenberg–Weil coefficient system \mathcal{W}_l with the property that $\kappa_{l,x}|_{K_{l,x}^+} \simeq \mathcal{W}'_{l,\mathcal{F}}|_{K_{l,x}^+} \otimes_R R' \simeq \mathcal{W}_{l,\mathcal{F}}|_{K_{l,x}^+}$. \square

3.7.12 Uniqueness – proof of Proposition 3.3.8. — Let \mathcal{W}_l and \mathcal{W}'_l be two Heisenberg–Weil coefficient systems as in Definition 3.3.5. For any e -facet $\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}$, the R -modules $\mathcal{W}_{l,\mathcal{F}}$ and $\mathcal{W}'_{l,\mathcal{F}}$ are both equipped with an action of $K_{l,\mathcal{F}}^+$ that turns them into Heisenberg representations for $(K_{l,\mathcal{F}}^+, K_{l,\mathcal{F}}^+, \check{\phi}_{l,\mathcal{F}}^+)$. By Lemma 3.5.2, the R -module $H_{\mathcal{F}} := \text{Hom}_{RK_{l,\mathcal{F}}^+}(\mathcal{W}_{l,\mathcal{F}}, \mathcal{W}'_{l,\mathcal{F}})$ is invertible. We equip the collection $(H_{\mathcal{F}})_{\mathcal{F} \in \mathcal{B}'_{l,\bullet/e}}$ with the following structure of a smooth G_l -equivariant coefficient system whose transition maps are all isomorphisms.

- If \mathcal{F}' is an e -facet contained in the closure $\overline{\mathcal{F}}$, then from part i) of Proposition 3.3.6, we get an R -linear map $H_{\mathcal{F}} \xrightarrow{\gamma_{\mathcal{F},\mathcal{F}'}} H_{\mathcal{F}'}$, $\alpha_{\mathcal{F}} \mapsto \alpha_{\mathcal{F}'}$, characterized by the property that $\alpha_{\mathcal{F}'} \circ \beta_{\mathcal{W}_{l,\mathcal{F},\mathcal{F}'}} = \beta_{\mathcal{W}'_{l,\mathcal{F},\mathcal{F}'}} \circ \alpha_{\mathcal{F}}$. This characterization implies the transitivity $\gamma_{\mathcal{F}'',\mathcal{F}'} \circ \gamma_{\mathcal{F}',\mathcal{F}} = \gamma_{\mathcal{F}'',\mathcal{F}}$ for $\mathcal{F}'' \subset \overline{\mathcal{F}'}$, due to the same transitivity properties for the maps $\beta_{\mathcal{W}_{l,-}}$ and $\beta_{\mathcal{W}'_{l,-}}$. Conversely, from point ii) of Proposition 3.3.6, we get an R -linear map $H_{\mathcal{F}'} \rightarrow H_{\mathcal{F}}$, $\alpha_{\mathcal{F}'} \mapsto \alpha_{\mathcal{F}}$ characterized by the same equality as above. In particular this map is an inverse of $\gamma_{\mathcal{F},\mathcal{F}'}$.
- If $g \in G_l$, we get an isomorphism of R -modules $g_{\mathcal{F}} : H_{\mathcal{F}} \xrightarrow{\sim} H_{g_{\mathcal{F}}}$ by sending $\alpha_{\mathcal{F}}$ to $g_{\mathcal{W}'_{l,\mathcal{F}}} \circ \alpha_{\mathcal{F}} \circ (g_{\mathcal{W}_{l,\mathcal{F}}})^{-1}$. The equalities $h_{g_{\mathcal{F}}} \circ g_{\mathcal{F}} = (hg)_{\mathcal{F}}$ for any h, g and \mathcal{F} follow from the same type of equalities for \mathcal{W}_l and \mathcal{W}'_l . Compatibility with the face maps $\gamma_{\mathcal{F},\mathcal{F}'}$ again follows from the same compatibilities for \mathcal{W}_l and \mathcal{W}'_l . Finally, since $\alpha_{\mathcal{F}}$ is $K_{l,\mathcal{F}}^+$ -equivariant, we have $g_{\mathcal{F}} = \text{Id}$ whenever $g \in G_{l,\mathcal{F},0+}$.

It follows that $(H_{\mathcal{F}}, g_{\mathcal{F}}, \gamma_{\mathcal{F},\mathcal{F}'})_{\mathcal{F},\mathcal{F}'}$ defines a depth-0 object in $\text{Coef}_R([\mathcal{B}'_{l,\bullet/e}/G_l])$, that we can also see as an object in $\text{Coef}_R([\mathcal{B}'_{l,\bullet/e}/\overline{G}_l])$ and then in $\text{Coef}_R([\mathcal{B}'_{l,\bullet/e}/K_l])$, as in 3.4.1. By construction, the evaluation maps $\mathcal{W}_{l,\mathcal{F}} \otimes_R H_{\mathcal{F}} \xrightarrow{\sim} \mathcal{W}'_{l,\mathcal{F}}$ induce an isomorphism $\mathcal{W}_l \otimes_R H \xrightarrow{\sim} \mathcal{W}'_l$ in $\text{Coef}_R([\mathcal{B}'_{l,\bullet/e}/K_l])$.

Let us now set $L := \text{colim}_{\mathcal{B}'_{l,\bullet/e}} H$. Since all face maps of H are isomorphisms, L is an invertible R -module and all canonical maps $H_{\mathcal{F}} \xrightarrow{\text{can}_{\mathcal{F}}} L$ are isomorphisms. The G_l -equivariant structure induces a smooth action of G_l on L , given by a character $\theta : G_l \rightarrow R^\times$, which has depth 0 since it is trivial on each subgroup $G_{l,\mathcal{F},0+}$. Let us denote by L_θ the R -module L

with its action of G_ι through θ . We see that the collection of maps $(\text{can}_{\mathcal{F}})_{\mathcal{F} \in \mathcal{B}'_{\iota, \bullet/e}}$ provides an isomorphism from H to the constant G_ι -equivariant system associated to L_θ . In other words, we have constructed an isomorphism $\mathcal{W}_\iota \otimes_R L_\theta \xrightarrow{\sim} \mathcal{W}'_\iota$. \square

4 Sample applications

4.1 Reduction to depth zero for Hecke algebras of types

4.1.1 The setup. — Let $R = \mathcal{C}$ be an algebraically closed field of characteristic different from p , e.g., the complex numbers. Let $((\vec{\mathbf{G}}, \mathbf{M}^0), \vec{r}, x, (K_{M^0}, \rho_{M^0}), \vec{\psi})$ be a \mathbf{G} -datum as in [AFMO24b, Definition 4.1.1] that follows [KY17, 7.2]. Contrary to [AFMO24b] and [KY17] we do not record the embeddings of the extended Bruhat–Tits buildings of the twisted Levi subgroups $\mathbf{M}^i \subseteq \mathbf{G}^i \subseteq \mathbf{G}$, where \mathbf{M}^i is the centralizer in \mathbf{G}^i of the maximal split torus $\mathbf{A}_{\mathbf{M}^0}$ in the center of \mathbf{M}^0 , as part of the datum because here we fix such embeddings and then choose $x \in \mathcal{B}(\mathbf{M}^0) \subseteq \mathcal{B}(\mathbf{G})$ so that all the conditions of a \mathbf{G} -datum are satisfied. From such a \mathbf{G} -datum, the construction in [AFMO24b] that follows the construction of types by Kim and Yu ([KY17]) but includes a twist by the quadratic character of [FKS23], provides us with a compact, open subgroup $K_x \subset G$ and a representation ρ_x thereof. Moreover, $((\mathbf{G}^0, \mathbf{M}^0), x, (K_{M^0}, \rho_{M^0}))$ is a depth-zero datum to which there is attached a compact, open subgroup $K_x^0 \supseteq K_{M^0}$ of G^0 with representation ρ_x^0 whose restriction to K_{M^0} is ρ_{M^0} , see [AFMO24a, 5.1] or [AFMO24b, 4.1]. Note that K_x^0 is contained in G_x^0 and contains the parahoric subgroup $G_{x,0}^0$. We write $\mathcal{H}(G, K_x, \rho_x) := \text{End}_G(\text{ind}_{K_x}^G \rho_x)$ and $\mathcal{H}(G, K_x^0, \rho_x^0) := \text{End}_G(\text{ind}_{K_x^0}^{G^0} \rho_x^0)$ for the Hecke algebras attached to (K_x, ρ_x) and (K_x^0, ρ_x^0) . If $\mathcal{C} = \mathbb{C}$, then (K_x^0, ρ_x^0) is a depth-zero type for G^0 and (K_x, ρ_x) is a type for G . If, in addition, K_{M^0} is chosen to equal M_x^0 , then these types describe single Bernstein blocks. In that case, the Bernstein blocks are equivalent to the category of right unital complex $\mathcal{H}(G, K_x, \rho_x)$ -modules and $\mathcal{H}(G, K_x^0, \rho_x^0)$ -modules, respectively. If p does not divide the order of the Weyl group of \mathbf{G} , then every Bernstein block admits a type of this form ([Fin21]).

4.1.2 Corollary. — Suppose that p is odd and not a torsion prime of $\hat{\mathbf{G}}$, nor of \mathbf{G} . We have an isomorphism of \mathcal{C} -algebras $\mathcal{H}(G, K_x, \rho_x) \simeq \mathcal{H}(G^0, K_x^0, \rho_x^0)$.

Proof. We denote by \bar{x} the image of x in \mathcal{B}'_ι . The type (K_x, ρ_x) only depends on the product $\psi := \prod \psi_i|_{G^0}$ and not on the single characters ψ_i . Applying the construction of the characters from directly before [Fin21, Lemma 7.3] (there the characters are called ϕ_i , here we call them ψ'_i) to an irreducible subrepresentation of the restriction $\rho_x|_{K_x^+ K_{\iota, \bar{x}}^+}$ of ρ_x to $K_x^+ K_{\iota, \bar{x}}^+ \subseteq K_x$ (using that the roots of unity are divisible rather than Pontryagin duality to extend characters) and using [Fin21, Lemma 7.3], we obtain \mathbf{G}^{i+1} -generic characters ψ'_i of G^i that are trivial on the derived subgroup of G^i such that $\prod \psi_i|_{G_{x,0+}^0} = \prod \psi'_i|_{G_{x,0+}^0}$. Setting $\psi''_i = \psi'_i$ for $i > 0$ and $\psi''_0 = \psi'_0 \delta$ for an appropriate character δ of G^0 that is trivial on $G_{x,0+}^0$, we obtain that $\prod \psi_i|_{G^0} = \prod \psi''_i|_{G^0}$, and hence the types constructed from $((\vec{\mathbf{G}}, \mathbf{M}^0), \vec{r}, x, (K_{M^0}, \rho_{M^0}), \vec{\psi})$ and $((\vec{\mathbf{G}}, \mathbf{M}^0), \vec{r}, x, (K_{M^0}, \rho_{M^0}), \vec{\psi}'')$ agree. Moreover, also the pairs $(K_x^+, \hat{\psi}_x)$ attached to the three truncated Yu data $(\vec{\mathbf{G}}, \vec{\psi}, x)$, $(\vec{\mathbf{G}}, \vec{\psi}', x)$ and $(\vec{\mathbf{G}}, \vec{\psi}'', x)$ by Yu as in 2.8.4 agree. According to Lemma 2.8.5, there exists a pair (ϕ, ι) consisting of a wild inertia parameter ϕ and

$\iota \in I \subseteq I_\phi$ such that $\mathbf{G}_\iota^0 = \mathbf{G}^0$, $K_{\iota, \bar{x}}^+ = K_x^+$, and $\check{\phi}_{\iota, \bar{x}}^+ = \hat{\psi}_x$. Now the group K_x^0 is contained in $G_{\iota, \bar{x}}$ and by the construction of the types $K_x = K_x^0 K_{\iota, \bar{x}}^\dagger$ and $\rho_x = \rho_x^0 \otimes \kappa_{\iota, x}|_{K_x}$ for a twisted $\mathcal{C}K_{\iota, \bar{x}}$ -Heisenberg–Weil representation whose restriction to $K_{\iota, x}^+ = K_{\iota, \bar{x}}^+$ is $\hat{\psi}_x = \check{\phi}_{\iota, \bar{x}}^+$ -isotypic. Let $\rho := \text{ind}_{K_x^0}^{G_{\iota, \bar{x}}} \rho_x^0$, and let \mathcal{W}_ι be a Heisenberg–Weil coefficient system on $\mathcal{B}'_{\iota, \bullet/e}$ such that $\kappa_{\iota, x} = \mathcal{W}_{\iota, \mathcal{F}}$, where \mathcal{F} denotes the e -facet that contains \bar{x} , which exists by Corollary 3.7.11. By Corollary 3.4.14, under the equivalence $\mathcal{I}_{\mathcal{W}_\iota}$ of categories between $\text{Rep}_R^1(G_\iota)$ and $\text{Rep}_R^{\phi, I}(G)$, the representation $\text{ind}_{K_x^0}^{G_\iota} \rho_x^0 = \text{ind}_{G_{\iota, \bar{x}}}^{G_\iota} \rho$ gets sent to a representation isomorphic to $\text{ind}_{K_{\iota, \bar{x}}}^G(\mathcal{W}_{\iota, \mathcal{F}} \otimes_{\mathcal{C}} \rho) = \text{ind}_{K_{\iota, \bar{x}}}^G(\kappa_{\iota, x} \otimes_{\mathcal{C}} \text{ind}_{K_x}^{K_{\iota, \bar{x}}} \rho_x^0) = \text{ind}_{K_x}^G(\kappa_{\iota, x} \otimes \rho_x^0) \simeq \text{ind}_{K_x}^G \rho_x$, where we used that $K_{\iota, \bar{x}} = G_{\iota, \bar{x}} K_{\iota, \bar{x}}^\dagger$ and $K_x = K_x^0 K_{\iota, \bar{x}}^\dagger$. Hence $\mathcal{H}(G, K_x^0, \rho_x^0) = \text{End}_G(\text{ind}_{K_x^0}^{G_\iota} \rho_x^0) \simeq \text{End}_G(\text{ind}_{K_x}^G \rho_x) = \mathcal{H}(G, K_x, \rho_x)$. \square

This result was previously proven in [AFMO24b, Theorem 4.4.1] (under the assumption that K_{M^0} is normalized by $N_{G^0}(M^0)(F)_{[x]_{M^0}}$ in the notation of *loc. cit.*), where a more explicit isomorphism of Hecke algebras has been obtained.

4.2 Projective generators, and connection with a result of Chinello

4.2.1 Projective generators. — We return to the general setting that R may be any commutative $\mathbb{Z}[\mu_{4p^\infty}, \frac{1}{\sqrt{p}}]$ -algebra, and we fix a pair (ϕ, I) and an embedding $\iota \in I$ as above. Let \mathcal{C}_0 be a chamber of $\mathcal{B}'(\mathbf{G}_\iota, F)$ (for the usual Bruhat–Tits polysimplicial structure), and let \bar{S} be a set of representatives of G_ι -orbits of vertices of $\mathcal{B}'(\mathbf{G}_\iota, F)$ that is contained in the closure of \mathcal{C}_0 . For each $\bar{x} \in \bar{S}$, pick an e -vertex x in $\mathcal{B}'_{\iota, \bullet/e}$ above \bar{x} , and denote by S the subset of $\mathcal{B}'_{\iota, \bullet/e}$ formed by these lifts. The depth-0 category $\text{Rep}_R^1(G_\iota)$ is generated by the direct sum $\bigoplus_{x \in S} \text{ind}_{G_{\iota, x, 0+}}^{G_\iota}(R)$. Hence, according to Theorem 3.1.1, the RG -module

$$\mathcal{I}_{\mathcal{W}_\iota} \left(\bigoplus_{x \in S} \text{ind}_{G_{\iota, x, 0+}}^{G_\iota}(R) \right) \simeq \bigoplus_{x \in S} \text{ind}_{K_{\iota, x}^\dagger}^G(\mathcal{W}_{\iota, x}|_{K_{\iota, x}^\dagger})$$

is a projective generator of $\text{Rep}_R^{\phi, I}(G)$. Recall from our explicit construction of $\{\mathcal{W}_{\iota, x}\}$ that the restriction $\mathcal{W}_{\iota, x}|_{K_{\iota, x}^\dagger}$ is just a Heisenberg representation for $(K_{\iota, x}^\dagger, K_{\iota, x}^+, \check{\phi}_{\iota, x}^+)$, whose definition is quite simple and does not need the more subtle theory of Heisenberg–Weil representations. However, the isomorphism of R -algebras

$$\text{End}_{RG_\iota} \left(\bigoplus_{x \in S} \text{ind}_{G_{\iota, x, 0+}}^{G_\iota}(R) \right) \xrightarrow{\sim} \text{End}_{RG} \left(\bigoplus_{x \in S} \text{ind}_{K_{\iota, x}^\dagger}^G(\mathcal{W}_{\iota, x}|_{K_{\iota, x}^\dagger}) \right)$$

induced by $\mathcal{I}_{\mathcal{W}_\iota}$ a priori uses the full force of Theorem 3.3.7.

4.2.2 Example: $\mathbf{G} = \text{GL}_n$. — Suppose $\mathbf{G} = \text{GL}_n$. Following [Dat17, 2.4.6], the centralizer $C_{\mathbf{G}}(\phi)$ together with its canonical outer action of W_F is of the form $\prod_{i=1}^r \text{Ind}_{W_{F_i}}^{W_F} \text{GL}_{e_i}$ for some tamely ramified extensions F_i/F of degree d_i satisfying $\sum_{i=1}^r e_i d_i = n$. In other words, we have $\mathbf{G}_\phi = \prod_{i=1}^r \text{Res}_{F_i|F} \text{GL}_{e_i}$. Accordingly, we have $\mathbf{S}_\phi = \prod_{i=1}^r \text{Res}_{F_i|F} \text{GL}_1$. Moreover, there is a unique $\mathbf{G}(F)$ -conjugacy class I of F -rational Levi-center-embeddings $\mathbf{S}_\phi \hookrightarrow \mathbf{G}$, so we may suppress I from the discussion, and all \mathbf{G}_ι , $\iota \in I$, are isomorphic to \mathbf{G}_ϕ .

- The left vertical arrow is provided by this paper. Again, *several choices are possible*. Assuming that $\text{Pic}(R) = \{0\}$, e.g., $R = \overline{\mathbb{Z}}[\frac{1}{p}]$ or $R = \overline{\mathbb{F}}_\ell$ for $\ell \neq p$, Propositions 3.3.8 and 3.4.12 imply that the set of equivalences $\text{Rep}_R(G_{\phi,I}) \xrightarrow{\sim} \text{Rep}_R^{\phi,I}(G)$ constructed in this paper is a torsor under the group of characters $G_{\phi,I} \rightarrow R^\times$ that are trivial on $(G_{\phi,I})_{x,0+}$ for all $x \in \mathcal{B}(\mathbf{G}_{\phi,I}, F)$. Actually, among the choices made to construct these equivalences (including, for example, the choice of a $\iota \in I$), the only significant one is the choice of a character $\check{\varphi}_0 : G_\iota \rightarrow R^\times$ that extends each $(\check{\phi}_{\iota,x}^+)|_{G_{\iota,x,0+}}$, made in the definition of Heisenberg–Weil representations 3.6.6. In diagram (4.3.2), we have denoted by $\mathcal{I}_{\check{\varphi}_0}$ the product of the equivalences associated with the Heisenberg–Weil coefficient systems stemming from this choice.
- The right vertical arrow is essentially provided by [DHKM25]. By Lemma 2.2.5, there exists an extension $\varphi_0 : W_F \rightarrow {}^L\mathbf{G}(\mathbb{C})$ of ϕ such that $\varphi_0(W_F)$ normalizes a pinning ε_ϕ of $C_{\hat{\mathbf{G}}}(\phi)$. It is not clear if we can choose φ_0 such that $\varphi_0(W_F) \subset {}^L\mathbf{G}(\overline{\mathbb{Z}}[\frac{1}{p}])$ but the argument of [Dat25, Thm. 4.6] shows that we can find one such that $\varphi_0(W_F) \subset {}^L\mathbf{G}(\overline{\mathbb{Z}}[\frac{1}{pN}])$, whenever the center $Z(\hat{\mathbf{G}})$ is smooth over $\overline{\mathbb{Z}}[\frac{1}{pN}]$. For example, one can set $N = |\pi_1(\mathbf{G}_{\text{der}})|$. *Once φ_0 is chosen*, the map $(c, w) \mapsto c\varphi_0(w)$ provides an L -homomorphism

$${}^L\varphi_0 : {}^L\mathbf{G}_\phi = C_{\hat{\mathbf{G}}}(\phi) \rtimes_{\text{Ad}_{\varphi_0}} W_F \rightarrow {}^L\mathbf{G}.$$

By construction, the induced map on 1-cocycles $Z^1(W_F^0, \hat{\mathbf{G}}_\phi)_R \xrightarrow{\sim} Z^1(W_F^0, \hat{\mathbf{G}})_R$ restricts to an isomorphism $Z^1(W_F^0/P_F, \hat{\mathbf{G}}_\phi)_R \xrightarrow{\sim} Z^1(W_F^0, \hat{\mathbf{G}})_{\phi,R}$ for any $\overline{\mathbb{Z}}[\frac{1}{pN}]$ -algebra R , where the right hand side is the moduli space of extensions of ϕ to W_F^0 . It induces in turn an isomorphism of R -stacks

$$\left(Z^1(W_F^0/P_F, \hat{\mathbf{G}}_\phi) / \hat{\mathbf{G}}_\phi \right)_R \xrightarrow{\sim} \left(Z^1(W_F^0, \hat{\mathbf{G}}) / \hat{\mathbf{G}} \right)_{[\phi],R},$$

where the right hand side denotes the summand associated with ϕ . The symbol $\mathcal{I}_{\varphi_0}^{\text{spec}}$ of the above diagram is the associated equivalence on Ind.Coh .

How to make these choices ? From the above discussion, each one of the solid arrows of the above diagram depends on choices. In order to get the “correct” dashed arrow, one should carefully make these choices. Here are our expectations, that we hope to verify in future work.

- The choice of $I_0 \in \mathcal{I}_\phi$ will depend on a choice of a Whittaker datum (U, ψ) in \mathbf{G} . Denoting by $\Gamma_{U,\psi}$ the associated Gelfand–Graev $\overline{\mathbb{Z}}[\frac{1}{p}]$ -representation of G , the set I_0 should be the unique I in \mathcal{I}_ϕ such that the (ϕ, I) -component $(\Gamma_{U,\psi})^{\phi,I}$ of $\Gamma_{U,\psi}$ is non-zero. Moreover, $\mathcal{I}_{\check{\varphi}_0}^{-1}((\Gamma_{U,\psi})^{\phi,I_0})$ should be the depth-0 summand of a Gelfand–Graev representation of \mathbf{G}_{ϕ,I_0} , that should be used to normalize the tame categorical local Langlands correspondence for \mathbf{G}_{ϕ,I_0} (the lower map in our diagram above).
- There is a priori no best choice for φ_0 nor $\check{\varphi}_0$, but we expect that over $R = \overline{\mathbb{Z}}[\frac{1}{pN}]$, for any choice of φ_0 , there should be a unique $\check{\varphi}_0$ such that the equivalence $\mathcal{I}_{\check{\varphi}_0}$ is compatible with Langlands functoriality along ${}^L\varphi_0$ for irreducible complex representations. Before explaining what we mean, here is a natural way of producing a map $\varphi_0 \mapsto \check{\varphi}_0$. Suppose given a *tamely ramified* L -embedding ${}^L\mathbf{G}_\phi \hookrightarrow {}^L\mathbf{G}$, i.e., an embedding that extends both

the inclusion $C_{\hat{\mathbf{G}}}(\phi) \subset \hat{\mathbf{G}}$ and the identity on $\{1\} \times P_F$. Writing such an embedding as $(c, w) \mapsto c \cdot \psi(w)$ for some $\psi : W_F \rightarrow {}^L\mathbf{G}$ whose image normalizes the pinning ε_ϕ of $C_{\hat{\mathbf{G}}}(\phi)$, we can then write $\varphi_0 = \hat{\varphi}_0 \cdot \psi$ for some $\hat{\varphi}_0 \in Z^1(W_F, Z(C_{\hat{\mathbf{G}}}(\phi)))$ that extends ϕ , as in (the proof of) Lemma 2.2.4. Applying Borel’s procedure to $\hat{\varphi}_0$ then provides a $\check{\varphi}_0$. We expect that a good choice of ψ , associated to suitable χ -data as in [Kal21a, §6.1], will provide the “correct” matching between φ_0 and $\check{\varphi}_0$, in the sense that $\mathcal{I}_{\check{\varphi}_0}$ induces the functorial lifting along ${}^L\varphi_0$ of the L -packets associated to supercuspidal Langlands parameters in [Kal21b]. In any case, once ψ is chosen (or, equivalently, the L -embedding ${}^L\mathbf{G}_\phi \hookrightarrow {}^L\mathbf{G}$), the dashed arrow that makes Diagram (4.3.2) commutative is independent of the choice of a matching pair $(\varphi_0, \check{\varphi}_0)$.

4.3.3 The example of GL_n . — In the case $\mathbf{G} = \mathrm{GL}_n$, several points in the above discussion simplify drastically. For one, the sets \mathcal{I}_ϕ are always singletons, so that there is no need to choose a base point there. Moreover, when $R = \overline{\mathbb{Q}}_\ell$, the tame categorical Langlands correspondence for \mathbf{G}_ϕ is available from Zhu’s work, because \mathbf{G}_ϕ is a product of restrictions of scalars of GL_{n_i} ’s. Therefore, the bottom map of Diagram (4.3.2) is available for all $\phi \in \Phi(P_F, \mathrm{GL}_n)$ (regardless of ϕ having abelian image, actually).

Moreover, since $\hat{\mathbf{G}}_\phi = C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of GL_n , the projection map $N_{\hat{\mathbf{G}}}(C_{\hat{\mathbf{G}}}(\phi)) \rightarrow N_{\hat{\mathbf{G}}}(C_{\hat{\mathbf{G}}}(\phi))/C_{\hat{\mathbf{G}}}(\phi)$ has natural sections given by permutation matrices. Using again that \mathbf{G}_ϕ is a product of restrictions of scalars of GL_{n_i} , this provides a “natural” way of choosing an embedding ${}^L\mathbf{G}_\phi \rightarrow {}^L\mathbf{G}$ and a ψ as in the last bullet point of Section 4.3.1. So, associated to this choice of ψ , we get an embedding $D\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}^\phi(\mathrm{GL}_n(F)) \hookrightarrow \mathrm{Ind.Coh}(\mathrm{Hom}(W_F^0, \mathrm{GL}_n)/\mathrm{GL}_n)_{[\phi], \overline{\mathbb{Q}}_\ell}$ that makes diagram (4.3.2) commutative. However, this embedding generally needs to be “rectified”. In the work of Bushnell and Henniart, this rectification is given by precomposing the left vertical arrow of (4.3.2) by twisting with a certain tamely ramified character of G_ϕ , called a “rectifier”. Alternatively, one could also modify the chosen embedding ${}^L\mathbf{G}_\phi \rightarrow {}^L\mathbf{G}$ using appropriate χ -data. The dictionary between both approaches is explained in [Tam16] and [OT21], in a slightly different setting, but we expect that their arguments can be adapted to our setting.

Corollary. — *Assuming that $p > n$, there is an embedding*

$$D\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(F)) \hookrightarrow \mathrm{Ind.Coh}(\mathrm{Hom}(W_F^0, \mathrm{GL}_n)/\mathrm{GL}_n)_{\overline{\mathbb{Q}}_\ell}.$$

Proof. Since $p > n$, the centralizer of any $\phi \in \Phi(P_F, \mathrm{GL}_n)$ is a Levi subgroup, so, after making choices as explained above, it only remains to sum over all $\phi \in \Phi(P_F, \mathrm{GL}_n)$. \square

Such an embedding has already been constructed in [BZCHN24]. It would be interesting to compare both constructions.

Over $\overline{\mathbb{F}}_\ell$. As of writing this section, the tame CLLC is not available over $\overline{\mathbb{F}}_\ell$. However, Zhu’s paper contains a *unipotent* version, at least when $\ell > n$, that we introduced in the end of the introduction. In Corollary 1.1.4 of [Dat18], a “reduction-to-unipotent” result for depth-0 blocks of $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\mathrm{GL}_n(F))$ is proven, with a pattern very similar to what we are doing in this paper, except that we start from tamely ramified “inertia parameters”. We refer to Section 1.1 of [Dat18] and also Paragraph 1.2.4 of [Dat17] for the details. Combining that result with our reduction-to-depth-0 result, we obtain the following

Corollary. — Assuming that $p > n$ and $\ell > n$, there is an embedding

$$D\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\mathrm{GL}_n(F)) \hookrightarrow \mathrm{Ind.Coh}(\mathrm{Hom}(W_F^0, \mathrm{GL}_n)/\mathrm{GL}_n)_{\overline{\mathbb{F}}_\ell}.$$

Proof. Same as the last corollary, using Corollary 1.1.4 of [Dat18] on top of our equivalences. \square

Again, some choices are involved, and more work is needed to exhibit the “correct” embedding.

Selected notation

\mathcal{B} , 28	I_x , 38
$\mathcal{B}'_{\bullet/e}$, 57	$K_{\ell, \mathcal{F}}$, 60
$\mathcal{B}'_{\ell, \bullet/e}$, 59	$K_{\ell, x}$, 31, 56
$\beta_{\mathcal{F}, cF'}$, 57	$K_{\ell, x}^\circ$, 31
$\beta_{\mathcal{V}, \mathcal{F}, cF'}$, 57	$K_{\ell, x}^\dagger$, 31, 56
\mathcal{B}_ℓ , 28	$K_{\ell, \mathcal{F}}^\dagger$, 60
\mathcal{B}' , 42	$K_{\ell, x}^+$, 28, 56
\mathcal{B}'_ℓ , 43	$K_{\ell, \mathcal{F}}^+$, 60
$\text{Coef}_R^1(\mathcal{B}'_{\ell, \bullet/e}/G_\ell)$, 59	$K_{\phi, \ell, x}^+$, 31
$\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)$, 57	$K_{\ell, x}^{++}$, 28
$\text{Coef}_R^{\phi, I}(\mathcal{B}'_{\bullet/e}/G)^{\text{cart}}$, 58	\mathbf{L}_ϕ , 17
$\text{Coef}_R(\mathcal{B}'_{\bullet/e}/G)$, 57	\mathcal{L}_ϕ , 17
C_x , 81	$\mathcal{L}_{\phi, \varepsilon_\phi}$, 17
e , 56	$\hat{\mathbf{L}}_\phi$, 17
$e_{\ell, x}$, 31	$\check{\varphi}_i$, 27
$e_{I, x}$, 39	$\check{\phi}_{\ell, x}^+$, 28
$e_{\phi, \ell, x}$, 31	$\check{\phi}_{\ell, \mathcal{F}}^+$, 60
$e_{\phi, I, x}$, 39	ψ_i , 28
e_x , 39	$\text{Rep}_R^{\phi, I}(G)$, 42
$g_{\mathcal{F}}$, 57	\mathbf{S} , 11, 12
$G_{\mathcal{F}\mathcal{F}_\ell}$, 65	$\hat{\mathbf{S}}$, 11, 12
$\hat{\mathbf{G}}_i$, 11	$\hat{\mathbf{S}}_{\{i\}}$, 12
\mathbf{G}_ℓ , 11, 13	$\mathbf{S}_{\{i\}}$, 11, 12
$G_{\ell, \mathcal{F}}$, 60	\mathbf{S}_ϕ , 18, 24
\mathbf{G}_ϕ , 21	$\hat{\mathbf{S}}_\phi$, 24
$g_{\mathcal{V}, \mathcal{F}}$, 57	V_x , 81
I , 38	
I_ϕ , 18	

Selected terminology

Heisenberg representation, 74	14
Heisenberg triple, 73	twisted $RK_{\ell, x}$ -Heisenberg–Weil representation, 84
Levi-center-embedding, 11	
relevant F -rational Levi-center-embedding,	wild inertia parameter, 10

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