On the Intersection of Local Arthur Packets for Classical Groups I

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March 11, 2022

Notation

- We fix a non-Archimedean field F and let G_n be the split group $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n+1}(F)$.
- We let W_F be the Weil group, $\widehat{G}_n$ be the complex dual group of G_n.
- When $G_n = \mathrm{Sp}_{2n}(F)$, we have $\widehat{G}_n = \mathrm{SO}_{2n+1}(\mathbb{C})$.
- When G_n is the split group $\mathrm{SO}_{2n+1}(F)$, we have $\widehat{G}_n = \mathrm{Sp}_{2n}(\mathbb{C})$.

Notation

- For an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(F)$ $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Z}_{\geq 0}$, we let $\Delta_\rho[x, y]$ be the unique irreducible subrepresentation of $\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y$.
- For a smooth representation of G_n, π, of finite length, we let $[\pi]$ denote its semi-simplification and the socle, $soc(\pi)$, denote the maximal semi-simple subrepresentation of π.
- Let P_d be a standard parabolic subgroup of G_n with Levi subgroup isomorphic to $\mathrm{GL}_d \times G_{n-d}$ and $x \in \mathbb{R}$. The $\rho|\cdot|^x$-derivative of π, denoted $D_{\rho|\cdot|^x}(\pi)$, is a semisimple representation satisfying

$$[Jac_{P_d}(\pi)] = \rho|\cdot|^x \otimes D_{\rho|\cdot|^x}(\pi) + \sum_i \tau_i \otimes \pi_i$$

  where the sum is over all irreducible representations τ_i of $\mathrm{GL}_d(F)$ such that $\tau_i \ncong \rho|\cdot|^x$.

Arthur Packets

Theorem (Arthur)

The discrete spectrum of square integrable automorphic forms are partitioned by global Arthur packets.

- The representations in the global Arthur packets are defined as the tensor products of representations coming from local Arthur packets.
- However, unlike the global case, local Arthur packets are not necessarily disjoint.

Arthur's Construction

- Local Arthur packets are parameterized by local Arthur parameters. These are $\widehat{G}_n$-conjugacy classes of admissible homomorphisms $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}_n$ such that ψ has bounded image on W_F.
- For simplicity, assume $G_n = \mathrm{Sp}_{2n}$. Then $\widehat{G}_n = \mathrm{SO}_{2n+1}(\mathbb{C})$ and there is a $\mathrm{GL}_{2n+1}(\mathbb{C})$-conjugacy class of embeddings $\widehat{G}_n \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$.

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- Thus we may view ψ as a local Arthur parameter of GL_{2n+1}. Such ψ is necessarily selfdual.
- Local Arthur packets of GL_{2n+1} are singletons and hence we can associate a representation π_ψ of GL_{2n+1} to ψ.

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- Local Arthur packets of GL_{2n+1} are singletons and hence we can associate a representation π_ψ of GL_{2n+1} to ψ.
- Arthur associates a "multi-set" Π_ψ to ψ consisting of irreducible smooth representations of Sp_{2n} such that a linear combination of characters in Π_ψ transfers to the twisted character of π_ψ.

Intersections of Local Arthur Packets

- The intersection of local Arthur packets provides complications in various theories.
- For example, consider the non-tempered Gan-Gross-Prasad conjectures. To be precise, for both the L-packet and Arthur packet, we should consider all relevant pure inner forms of the groups involved. We call the corresponding packets the Vogan L-packet and Vogan A-packet.
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- For simplicity, let $G_1 = \mathrm{SO}_{2n+1}$ and $G_2 = \mathrm{SO}_{2n}$ and ψ_1 and ψ_2 be a relevant pair of local Arthur parameters for G_1 and G_2 respectively.
- There is an injection from local Arthur parameters to L-parameters which we denote by $\psi \mapsto \phi_\psi$. We denote the corresponding Vogan *L*-packet by Π_{ϕ_ψ}.

Conjecture (Gan, Gross, Prasad)

There exists a unique representation $\pi_1 \times \pi_2 \in \Pi_{\phi_{\psi_1}} \times \Pi_{\phi_{\psi_2}}$ *such that* $\mathrm{dim}\mathrm{Hom}_{\mathrm{SO}_{2n}}(\pi_1 \otimes \pi_2, \mathbb{C}) \neq 0$. *Moreover,* $\mathrm{dim}\mathrm{Hom}_{\mathrm{SO}_{2n}}(\pi_1 \otimes \pi_2, \mathbb{C}) = 1$.

Intersections of Local Arthur Packets

- The uniqueness property of the previous conjecture is expected since Vogan L-packets are disjoint. If we enlarge the conjecture to include Vogan A-packets, then we lose the uniqueness.
- For $G_1 = \mathrm{SO}_{2n+1}$ and $G_2 = \mathrm{SO}_{2n}$, Gan, Gross, and Prasad demonstrated a relevant pair of non-tempered local Arthur parameters ψ_1 and ψ_2 which have $\dim \mathrm{Hom}_{\mathrm{SO}_{2n}}(\pi_1 \otimes \pi_2, \mathbb{C}) = 1$ for a supercuspidal representation $\pi_1 \times \pi_2 \in \Pi_{\psi_1} \times \Pi_{\psi_2}$. Consequently, $\pi_1 \times \pi_2 \notin \Pi_{\phi_{\psi_1}} \times \Pi_{\phi_{\psi_2}}$.

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- However, there exists tempered local Arthur parameters ψ_1' and ψ_2' such that $\pi_1 \times \pi_2 \in \Pi_{\psi_1'} \times \Pi_{\psi_2'} = \Pi_{\phi_{\psi_1'}} \times \Pi_{\phi_{\psi_2'}}$.
- By the previous conjecture, there should a be a unique representation $\pi_1' \times \pi_2'$ in the Vogan L-packet with $\dim \mathrm{Hom}_{\mathrm{SO}_{2n}}(\pi_1' \otimes \pi_2', \mathbb{C}) = 1$.
- Thus, there should be at least 2 representations in the local Arthur packet $\Pi_{\psi_1} \times \Pi_{\psi_2}$ satisfying the restriction problem. The failure of uniqueness is due to the local Arthur packets having nontrivial intersections.

Mœglin's Construction

- Mœglin gave an explicit construction of Π_ψ and showed that it was multiplicity free.
- Mœglin's first reduced the general case to the good parity case. We decompose a local Arthur parameter

$$\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}$$

  where
  - ρ is an irreducible unitary supercuspidal representation of some GL_d which is identified with an irreducible bounded representation of W_F via the local Langlands correspondence for GL_d;
  - S_a is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a;
  - I_ρ is an appropriate indexing set.
- We say ψ is of good parity if every summand $\rho \otimes S_a \otimes S_b$ is self-dual and of the same type as ψ.

Mœglin's Construction

Theorem (Mœglin)

Let ψ be a local Arthur parameter. We have the decomposition

$$\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee$$

where ψ_1 is a local Arthur parameter which is not of good parity, ψ_0 is a local Arthur parameter of good parity, and $\psi_1^\vee$ denotes the dual of ψ_1. Furthermore, for $\pi \in \Pi_{\psi_0}$ the induced representation $\pi_{\psi_1} \rtimes \pi$ is irreducible, independent of choice of ψ_1, and we have

$$\Pi_\psi = \{\pi_{\psi_1} \rtimes \pi \,|\, \pi \in \Pi_{\psi_0}\}.$$

- Hence, if we know the construction of local Arthur packets of good parity, then we know the general case.

Arthur's Classification of Tempered Representations

- A local Arthur parameter ψ is tempered if $b_i = 1$ for every summand.

Theorem (Arthur)

Any irreducible tempered representation of G_n lies in Π_ψ for some tempered local Arthur parameter ψ. Moreover, if ψ_1 and ψ_2 are two non-isomorphic tempered local Arthur parameters, then

$$\Pi_{\psi_1} \cap \Pi_{\psi_2} = \emptyset.$$

Finally, if one fixes a choice of Whittaker datum for G_n and ψ is tempered, then there is a bijective map between the tempered local Arthur packet Π_ψ and the characters of the component group $\widehat{\mathcal{S}}_\psi$.

Hereinafter, we implicitly fix a choice of Whittaker datum for G_n. When ψ is tempered and of good parity, we write $\pi(\psi, \varepsilon)$ for the element of Π_ψ corresponding to $\varepsilon \in \widehat{\mathcal{S}}_\psi$ via the bijection in the above theorem.

Mœglin's Construction

- The rest of Mœglin's construction is as follows:

$$\left\{\begin{array}{c}\text{discrete}\\ \text{tempered}\end{array}\right\} \to \{\text{elementary}\} \to \left\{\begin{array}{c}\text{discrete}\\ \text{diagonal}\\ \text{restriction}\end{array}\right\} \to \{\text{good parity}\}$$

- Local Arthur packets of tempered parameters are known by the previous theorem of Arthur.
- Elementary parameters are those with $a_i = 1$ or $b_i = 1$ for every summand. To obtain elementary local Arthur packets from tempered local Arthur packets, Mœglin uses generalized Aubert involutions.
- Local Arthur parameters of discrete diagonal restriction are those for which the sets $\left[\frac{a_i+b_i}{2} - 1, \left|\frac{a_i-b_i}{2}\right|\right]$ are disjoint for any $i \in I_\rho$. To obtain these packets, Mœglin takes certain socles.
- Finally, local Arthur packets of good parity can be recovered from those of discrete diagonal restriction by taking certain derivatives.

Atobe's Construction

- The computation, in terms of the Langlands classification, of the local Arthur packets for elementary and discrete diagonal restriction cases are difficult generally.
- As a remedy, Atobe gave a refinement of Mœglin's construction:

$$\left\{\begin{array}{c}\text{discrete}\\ \text{tempered}\end{array}\right\} \to \left\{\begin{array}{c}\text{non}-\text{negative}\\ \text{discrete}\\ \text{diagonal}\\ \text{restriction}\end{array}\right\} \to \{\text{good}\,\text{parity}\}$$

- We say that a local Arthur parameter ψ is non-negative if $a_i \geq b_i$ for any $i \in I_\rho$ and every ρ.

Atobe's Reformulation

- An *extended multi-segment* for G_n is an equivalence class of multi-sets of extended segments

$$\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$$

such that

  - I_ρ is a totally ordered finite set with a fixed admissible order $>$;
  - $A_i + B_i \geq 0$ for all ρ and $i \in I_\rho$;
  - as a representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$,

$$\psi_{\mathcal{E}} = \bigoplus_\rho \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}$$

    where $(a_i, b_i) = (A_i + B_i + 1, A_i - B_i + 1)$, is a local Arthur parameter for G_n of good parity.
  - Satifies the sign condition

$$\prod_\rho \prod_{i \in I_\rho} (-1)^{[\frac{b_i}{2}] + l_i} \eta_i^{b_i} = 1.$$

Atobe's Reformulation

- Let ρ be the trivial representation. The pictograph

$$\begin{array}{ccccc} -1 & 0 & 1 & 2 & 3 \end{array}$$
$$\mathcal{E} = \begin{pmatrix} \lhd & \ominus & \oplus & \ominus & \rhd \\ & & & \lhd & \rhd \end{pmatrix}_{\rho}$$

corresponds to the extended multi-segment $\mathcal{E} = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i=1<2}$ of Sp_{26} where $A_1 = A_2 = 3$, $B_1 = -1$, $B_2 = 2$, $l_1 = l_2 = 1$, $\eta_1 = -1$, and $\eta_2 = 1$. The A_i's and B_i's denote the endpoints of the pictograph, l_i's denote the number of triangles, and η_i's denote the first sign.
The associated local Arthur parameter is

$$\psi_{\mathcal{E}} = \rho \otimes S_3 \otimes S_5 + \rho \otimes S_6 \otimes S_2.$$

Atobe's Reformulation

- Suppose $\mathcal{E}$ is an extended multi-segment such that for any ρ, if there exists $i \in I_\rho$ with $B_i < 0$, then the admissible order on I_ρ satisfies the following:

$$\text{if } B_i \geq B_j, \text{ then } i > j. \tag{P'}$$

  We first suppose that $\mathcal{E}$ satisfies
  - $B_i \geq 0$ for any $i \in I_\rho$ (i.e. $\psi_{\mathcal{E}}$ is non-negative)
  - for $i > j \in I_\rho$, $B_i > A_j$ (i.e. $\psi_{\mathcal{E}}$ is of discrete diagonal restriction).

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  In this case, Atobe has defined an irreducible representation $\pi(\mathcal{E})$.
- In general, let $t_i \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{E}' = \cup_\rho([A_i + t_i, B_i + t_i], l_i, \eta_i)_{i \in (I_\rho, >)}$ satisfies above conditions. Then we define
$$\pi(\mathcal{E}) = \circ_\rho \circ_{i \in I_\rho} \left(D_{\rho|\cdot|^{B_i+1,\dots,A_i+1}} \circ \dots \circ D_{\rho|\cdot|^{B_i+t_i,\dots,A_i+t_i}} \right) (\pi(\mathcal{E}')),$$
where if $I_\rho = \{1 < \cdots < n\}$, we write $\circ_{i \in I_\rho} D_i = D_n \circ \cdots \circ D_1$.
- In this case, either $\pi(\mathcal{E})$ is irreducible or zero (Bin Xu gives an explicit condition on $\mathcal{E}$ which determines if $\pi(\mathcal{E}) \neq 0$).

Langlands Classification

- The Langlands classification for G_n states that any irreducible smooth representation π of G_n is a unique irreducible subrepresentation of $\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi'$ where
  - ρ_i is an irreducible unitary supercuspidal representation of GL_{d_i},
  - $x_1 + y_1 \leq x_2 + y_2 \leq \cdots \leq x_r + y_r < 0$,
  - and π' is a tempered representation.

  We write $\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi')$.

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  We write $\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi')$.
- In particular, when $\psi = \bigoplus_{i=1}^{n} \rho \otimes S_{a_i} \otimes S_1$ is a tempered local Arthur parameter, for $\varepsilon \in \widehat{\mathcal{S}}_{\psi}$, we write $\varepsilon(\rho \otimes S_{a_i}) = \epsilon_i \in \{\pm 1\}$. Let $\pi' = \pi(\psi, \varepsilon) \in \Pi_{\psi}$ be the representation corresponding to ε via Arthur's theorem and $x_i = \frac{a_i - 1}{2}$ for $i \in I_{\rho}$. Then we write

$$\pi(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}) = \pi(\psi, \varepsilon).$$

- We will use this notation in several examples.

Atobe's Reformulation

- Let ρ be the trivial representation. Consider

$$\mathcal{E} = \begin{pmatrix} & -1 & 0 & 1 & 2 & 3 \\ & \lhd & \ominus & \oplus & \ominus & \rhd \\ & & & & \lhd & \rhd \end{pmatrix}_{\rho}.$$

Then we shift to

$$\mathcal{E}' = \begin{pmatrix} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ & \lhd & \ominus & \oplus & \ominus & \rhd & & \\ & & & & & & \lhd & \rhd \end{pmatrix}_{\rho}.$$

We compute $\pi(\mathcal{E}') = L(\Delta_\rho[0,-4], \Delta_\rho[5,-6]; \pi(1^-, 2^+, 3^-))$ and $\pi(\mathcal{E}) = L(\Delta_\rho[1,-3], \Delta_\rho[2,-3]; \pi(0^-, 1^+, 2^-))$.

Atobe's Reformulation

Theorem (Atobe)

Let ψ be a local Arthur parameter of good parity and $\Psi(\psi)$ be the set of extended multi-segments $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ such that $\psi_\mathcal{E} = \psi$ and if $B_i < 0$ for some $i \in I_\rho$, then the admissible order satisfies (P'). Then

$$\Pi_\psi = \{\pi(\mathcal{E}) | \mathcal{E} \in \Psi(\psi)\} \setminus \{0\}.$$

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$$\Pi_\psi = \{\pi(\mathcal{E}) | \mathcal{E} \in \Psi(\psi)\} \setminus \{0\}.$$

- Thus, the construction data $\mathcal{E}$ determines the local Arthur packet.

Intersection of Local Arthur Packets

- Suppose that $\pi(\mathcal{E}) \neq 0$. Atobe recently defined operators on extended multi-segments which classify the set $\{\mathcal{E}' \mid \pi(\mathcal{E}) = \pi(\mathcal{E}')\}$.
- Jointly with Liu and Lo, we give a different set of operators on extended multi-segments which classify the set $\{\mathcal{E}' \mid \pi(\mathcal{E}) = \pi(\mathcal{E}')\}$.

Theorem (Atobe; H., Liu, and Lo)

We can determine all the local Arthur packets to which a fixed representation belongs. Consequently, we can determine all the local Arthur packets which intersect a fixed local Arthur packet.

Applications

Theorem (H., Liu, and Lo)

1. *Given any local Arthur parameter ψ, we give a formula to count the number of tempered representations inside Π_ψ and describe their L-data.*
2. *We prove the enhanced Shahidi conjecture. That is, a local Arthur packet Π_ψ contains a generic member if and only if ψ is tempered.*
3. *We determine all $\mathcal{E}$ such that $\pi(\mathcal{E})$ is in the L-packet associated with $\psi_{\mathcal{E}}$.*
4. *For a representation π of Arthur type, we give a conjectural definition of "the" local Arthur parameter $\psi(\pi)$ of π, such that*
    1. $\pi \in \Pi_{\psi(\pi)}$.
    2. *If π is in the L-packet associated with some local Arthur parameter ψ, then $\psi(\pi) = \psi$.*

- The End.
- Thank you!