On the Intersection of Local Arthur Packets for Classical Groups II

Chi-Heng Lo

Purdue University

March 11, 2022

1. Main results

2. Applications
   - L-packet of Arthur type
   - "The" local Arthur parameter

Main results

Extended multi-segments

- An extended multi-segment is given by picture as the following example.

$$\mathcal{E}_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \ominus & \oplus & \ominus & \\ & & \lhd & \rhd \end{pmatrix}_\rho.$$

- For each $\mathcal{E}$, we use rectangles to specify non-empty entries in each row, and call them the support of $\mathcal{E}$. For example,

$$\mathrm{supp}(\mathcal{E}_0) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \boxed{\ominus \quad \oplus \quad \ominus} & & & \\ & & \boxed{\lhd \quad \rhd} & \end{pmatrix}_\rho = \{[3,1]_\rho, [4,3]_\rho\}$$

- From the support of an extended multi-segment $\mathcal{E}$, we may associate a local Arthur parameter $\psi_{\mathcal{E}}$. For example,

$$\psi_{\mathcal{E}_0} = \rho \otimes S_5 \otimes S_3 + \rho \otimes S_8 \otimes S_2.$$

Extended multi-segments

$$\mathcal{E}_0 = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ & \ominus & \oplus & \ominus & \\ & & & \lhd & \rhd \end{pmatrix}_\rho .$$

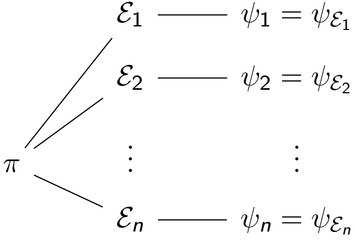
- We may associate a representation $\pi(\mathcal{E})$ (zero or irreducible) to each extended multi-segment $\mathcal{E}$. For example,

$$\pi(\mathcal{E}_0) = L(\Delta_\rho[3, -4]; \pi(1^-, 2^+, 3^-)).$$

  If $\pi(\mathcal{E})$ is nonzero, we have $\pi(\mathcal{E}) \in \Pi_{\psi_{\mathcal{E}}}$.
- Conversely, if $\pi \in \Pi_\psi$, then there exists an $\mathcal{E}$ with $\psi_{\mathcal{E}} = \psi$ and $\pi = \pi(\mathcal{E})$.

Main results

- Given a representation π. Suppose $\{\psi \mid \pi \in \Pi_\psi\} = \{\psi_1, \cdots, \psi_n\}$ ($n \geq 1$). Then we can find $\mathcal{E}_1, \cdots, \mathcal{E}_n$ such that $\pi(\mathcal{E}_i) = \pi$ and $\psi_{\mathcal{E}_i} = \psi_i$.



- Our main results answer the following problems.
  - Study the relation between $\mathcal{E}_i, \mathcal{E}_j$. Given $\mathcal{E}_1$, construct the set $\{\mathcal{E}_1, \cdots, \mathcal{E}_n\}$.
  - Determine an arbitrary π is of Arthur type or not. In the affirmative case, construct an $\mathcal{E}_i$.

Operators

- A key ingredient of our main theorem is the following four operators on extended multi-segments.

name	notation	affect on representation
Row exchange	R_i	$\pi(\mathcal{E}) = \pi(R_i(\mathcal{E}))$
Union-intersection	ui_i	$\pi(\mathcal{E}) = \pi(ui_i(\mathcal{E}))$
Aubert-Zelevinsky dual	$dual$	$\pi(dual(\mathcal{E})) = \widehat{\pi(\mathcal{E})}$
Partial dual	$dual_i^+, dual_i^-$	$\pi(\mathcal{E}) = \pi(dual_i^{\pm}(\mathcal{E}))$

where $\widehat{\pi}$ is the Aubert-Zelevinsky dual defined by

$$\widehat{\pi} := \varepsilon \sum_P (-1)^{\dim(A_P)}[\mathrm{Ind}_P^{G_n}(Jac_P(\pi))],$$

for some $\varepsilon \in \{\pm 1\}$. Aubert showed that $\widehat{\widehat{\pi}} = \pi$.

- In particular, we have $\pi(dual \circ ui_k \circ dual(\mathcal{E})) = \widehat{\widehat{\pi(\mathcal{E})}} = \pi(\mathcal{E})$.

Row Exchange

- Row exchange R_i exchanges the support of the i-th row and $i+1$-th row.

$$\begin{array}{l} \boxed{B_i \qquad\qquad\qquad\qquad A_i} \\ \qquad\quad \boxed{B_{i+1} \;\; A_{i+1}} \end{array} \quad \longleftrightarrow \quad \begin{array}{l} \qquad\qquad \boxed{B_{i+1} \;\; A_{i+1}} \\ \boxed{B_i \qquad\qquad\qquad\qquad A_i} \end{array}$$

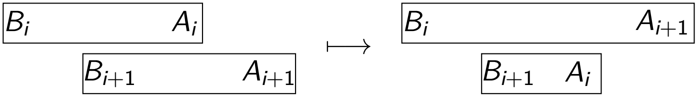
- It does not change the local Arthur parameter, but only changes the summation order.
- Consider

$$\mathcal{E} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \ominus & \oplus & \ominus & \oplus \\ & & \oplus & \end{pmatrix}_{\rho}, \; R_1(\mathcal{E}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & \ominus & \\ \lhd & \oplus & \ominus & \rhd \end{pmatrix}_{\rho}$$

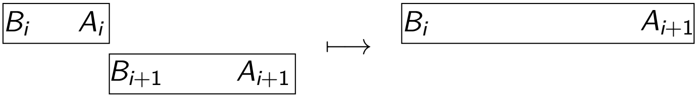
We have $\pi(\mathcal{E}) = \pi(R_1(\mathcal{E})) = L(\Delta_{\rho}[3,-4]; \pi(1^-, 2^+, 3^-))$.

Union-Intersection

- Union-intersection ui_i changes the support of the i-th and $i+1$-th rows into their union and intersection.



or



- This operator changes the local Arthur parameter.
- ui_i is not always applicable on $\mathcal{E}$. It depends on the combinatorial data of $\mathcal{E}$. Atobe's result that ui_i preserves representation gives that

$$ui_i \text{ is applicable on } \mathcal{E} \Longrightarrow \pi(\mathcal{E}) \in \Pi_{\psi_{\mathcal{E}}} \cap \Pi_{\psi_{ui_i(\mathcal{E})}}.$$

Union-Intersection

- Consider

$$\mathcal{E} = \begin{pmatrix} \overset{1}{\ominus} & \overset{2}{\oplus} & \overset{3}{} \\ & & \ominus \end{pmatrix}_{\rho}, \; ui_1(\mathcal{E}) = \begin{pmatrix} \overset{1}{\ominus} & \overset{2}{\oplus} & \overset{3}{\ominus} \end{pmatrix}_{\rho}.$$

We have $\pi(\mathcal{E}) = \pi(ui_1(\mathcal{E})) = L(\pi(1^-, 2^+, 3^-)) \in \Pi_{\psi} \cap \Pi_{ui_1(\psi)}$, where

$$\psi = \rho \otimes S_4 \otimes S_2 + \rho \otimes S_7 \otimes S_1,$$
$$ui_1(\psi) = \rho \otimes S_5 \otimes S_3.$$

Aubert-Zelevinsky Dual

- Suppose $\mathrm{supp}(\mathcal{E}) = \{[A_1, B_1]_\rho, \cdots, [A_n, B_n]_\rho\}$. Then $\mathrm{supp}(dual(\mathcal{E})) = \{[A_n, -B_n]_\rho, \cdots, [A_1, -B_1]_\rho\}$.
- Consider

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{ccccccc} \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \\ & & \lhd & \oplus & \rhd & & \\ & & & \ominus & & & \end{array} \right)_\rho \end{array}.$$

Then

$$\mathrm{supp}(\mathcal{E}) = \{[3, -3]_\rho, [1, -1]_\rho, [0, 0]_\rho\}$$
$$\mathrm{supp}(dual(\mathcal{E})) = \{[0, 0]_\rho, [1, 1]_\rho, [3, 3]_\rho\},$$

and

$$dual(\mathcal{E}) = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccc} \ominus & & & \\ & \oplus & & \\ & & & \ominus \end{array} \right)_\rho \end{array}.$$

We have $\pi(dual(\mathcal{E})) = \widehat{\pi(\mathcal{E})}$ (as proved in Atobe's work).

$dual \circ ui \circ dual$

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \\ & & \lhd & \oplus & \rhd & & \\ & & & \ominus & & & \end{pmatrix}_{\rho} \end{array}, \quad dual(\mathcal{E}) = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & & & \\ & \oplus & & \\ & & & \ominus \end{pmatrix}_{\rho} \end{array}.$$

- ui_1 is applicable on $dual(\mathcal{E})$, where

$$ui_1 \circ dual(\mathcal{E}) = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & \oplus & & \\ & & & \ominus \end{pmatrix}_{\rho} \end{array},$$

$$\mathcal{E}' := dual \circ ui_1 \circ dual(\mathcal{E}) = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \\ & & & \oplus & \ominus & & \end{pmatrix}_{\rho} \end{array}.$$

We have $\pi(\mathcal{E}) = \pi(\mathcal{E}') = L(\Delta_\rho[-3,-3]; \pi(0^-, 1^+, 2^-))$.

Partial dual

- $dual_i^+$ is applicable only if $B_i = 1/2$, and $dual_i^-$ is applicable only if $B_i = -1/2$. When it is applicable, it changes the support as follows.

$$\boxed{\tfrac{1}{2} \qquad\qquad\qquad A_i} \quad \overset{dual_i^+}{\longmapsto} \atop \underset{dual_i^-}{\longleftarrow\!\!\shortmid} \quad \boxed{\tfrac{-1}{2} \qquad\qquad\qquad A_i}$$

- Consider

$$\mathcal{E}_1 = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ & \ominus & \oplus & \\ & & & \ominus \end{pmatrix}_{\rho}, \ \mathcal{E}_2 = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \oplus & \ominus & \oplus & \\ & & & \ominus \end{pmatrix}_{\rho}.$$

We have $\mathcal{E}_2 = dual_1^+(\mathcal{E}_1)$ and $\mathcal{E}_1 = dual_1^-(\mathcal{E}_2)$, and $\pi(\mathcal{E}_1) = \pi(\mathcal{E}_2) = \pi\left(\left(\frac{1}{2}\right)^-, \left(\frac{3}{2}\right)^+, \left(\frac{5}{2}\right)^-\right)$.

Main Theorem

- We can now state our main theorem.

Theorem (Hazeltine, Liu, and L.)

Suppose that $\pi(\mathcal{E}) = \pi(\mathcal{E}') \neq 0$. *Then* $\mathcal{E}$ *and* $\mathcal{E}'$ *are related by a composition of the following four types of operators:*

- *row exchange,*
- *union-intersection,*
- *dual* $\circ$ *ui* $\circ$ *dual,*
- *partial dual*

and their inverses.

- Recently, Atobe has a logically equivalent theorem.

Canonical Form

- The main idea behind the proof is to construct a specific extended multi-segment from $\mathcal{E}$, which we call $\mathcal{E}_{can}$, the canonical form of $\mathcal{E}$.
- The construction is roughly as follows.

$$\mathcal{E} \xrightarrow{\text{all possible } ui} \mathcal{E}^{min} \xrightarrow[\text{possibly once } dual_i^-]{\text{some } dual \circ ui \circ dual} \mathcal{E}_{can}$$

Theorem (Hazeltine, Liu and L.)

- *Suppose that* $\pi(\mathcal{E}) \neq 0$. *Then*

$$\pi(\mathcal{E}) = \pi(\mathcal{E}') \Longleftrightarrow \mathcal{E}_{can} = \mathcal{E}'_{can}.$$

- *By reversing the construction of* $\mathcal{E}_{can}$, *we give a precise formula/algorithm for the set*

$$\Psi(\mathcal{E}) := \{\mathcal{E}' \mid \pi(\mathcal{E}') = \pi(\mathcal{E})\}/(\textit{row exchanges}).$$

Algorithm for representation of Arthur type

- Furthermore, we gave an algorithm

$$L\text{-data of } \pi(\mathcal{E}) \longmapsto \psi_{\mathcal{E}_{can}},$$

  which uniquely characterizes $\mathcal{E}_{can}$ by the representation theoretic properties of $\pi(\mathcal{E})$.
- Applying exactly the same algorithm to an arbitrary representation π, we have an algorithm for determining π is of Arthur type or not.

$$L\text{-data of } \pi \longmapsto \begin{cases} 0 \text{ (if some steps are not applicable)} \\ \quad \Rightarrow \pi \text{ is not of Arthur type.} \\ \psi \Rightarrow \begin{cases} \pi \notin \Pi_{\psi} & \Rightarrow \pi \text{ is not of Arthur type.} \\ \pi \in \Pi_{\psi} & \Rightarrow \pi \text{ is of Arthur type.} \end{cases} \end{cases}$$

- We remark that this algorithm is different from the one given by Atobe recently.

Example

- Consider $\pi = L(\Delta_\rho[-3,-3]; \pi(0^-, 1^+, 2^-))$. We have $\{\mathcal{E} \mid \pi(\mathcal{E}) = \pi\}/(\text{row exchanges}) = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$, where

$$\mathcal{E}_{can} = \mathcal{E}_1 = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \\ & & & \oplus & \ominus & & \end{pmatrix}_\rho \end{array}, \quad dual(\mathcal{E}_1) = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & \oplus & & \\ & & & \ominus \end{pmatrix}_\rho \end{array},$$

$$\mathcal{E}_2 = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \\ & & & \oplus & & & \\ & & & & \ominus & & \end{pmatrix}_\rho \end{array},$$

$$\mathcal{E}_3 = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \\ & & \lhd & \oplus & \rhd & & \\ & & & \ominus & & & \end{pmatrix}_\rho \end{array}, \quad dual(\mathcal{E}_3) = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & & & \\ & \oplus & & \\ & & & \ominus \end{pmatrix}_\rho \end{array}.$$

Applications

L-packet of Arthur type: Definition

- Let $\psi = \bigoplus_i \rho_i \otimes_i S_{a_i} \otimes S_{b_i}$ be a local Arthur parameter. We may associate an L-parameter ϕ_ψ by

$$\phi_\psi := \bigoplus_i \left(\bigoplus_{j=0}^{b_i-1} \rho_i |\cdot|^{\frac{b_i-1}{2}-j} \otimes S_{a_i} \right).$$

  Then the local Arthur packet Π_ψ contains the L-packet Π_{ϕ_ψ}.
- The map $\psi \mapsto \phi_\psi$ is injective. We say an L-parameter ϕ is of Arthur type if it is in the image of this map.

L-packet of Arthur type: Problem

The first application is to answer the following natural problem.

Problem

Give a criterion on the combinatorial data $\mathcal{E}$ such that $\pi(\mathcal{E}) \in \Pi_{\phi_{\psi_{\mathcal{E}}}}$.

L-packet of Arthur type: Statement

Definition

We say an extended multi-segment $\mathcal{E}$ satisfies (L) if after row exchanges, it satisfies the following conditions.

- The middle point of each row is non-decreasing.
- Every row has maximal pairs of triangles.
- Circles at the same column have the same sign.

Theorem (Hazeltine, Liu and L.)

(a) *If $\mathcal{E}$ satisfies (L), then $\pi(\mathcal{E}) \neq 0$. In this case, the L-data of $\pi(\mathcal{E})$ can be directly read from the picture of $\mathcal{E}$ (after row exchanges).*

(b) $\pi(\mathcal{E}) \in \Pi_{\phi_{\psi_{\mathcal{E}}}}$ *if and only if $\mathcal{E}$ satisfies (L).*

L-packet of Arthur type: Example

- Consider

$$\mathcal{E} = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ & \lhd & \oplus & \ominus & \rhd \\ & \lhd & \ominus & \rhd & \\ & & \oplus & & \\ & & \oplus & & \end{pmatrix}_{\rho} \xrightarrow{\text{row exchanges}} \mathcal{E}' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \color{red}{\lhd} & \color{brown}{\oplus} & \color{red}{\rhd} & \\ & \color{brown}{\oplus} & & \\ & \color{brown}{\oplus} & & \\ \color{blue}{\lhd} & \lhd & \rhd & \color{blue}{\rhd} \end{pmatrix}_{\rho}.$$

We have

$$\psi_{\mathcal{E}} = \rho \otimes S_6 \otimes S_4 + \rho \otimes S_5 \otimes S_3 + \rho \otimes S_5 \otimes S_1 + \rho \otimes S_5 \otimes S_1,$$

and

$$\pi(\mathcal{E}) = L(\Delta_{\rho}[1,-4], \Delta_{\rho}[1,-3], \Delta_{\rho}[2,-3]; \pi(2^+, 2^+, 2^+)),$$

and $\pi(\mathcal{E}) \in \Pi_{\phi_{\psi_{\mathcal{E}}}}$.

L-packet of Arthur type: Non-example

- $\Psi(\pi) = \{\mathcal{E} \mid \pi(\mathcal{E}) = \pi\}/(\text{row exchanges}) = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$, where

$$\mathcal{E}_1 = \bordermatrix{ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \cr & \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \cr & & & & \oplus & \ominus & & }_{\rho},$$

$$\mathcal{E}_2 = \bordermatrix{ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \cr & \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \cr & & & & \oplus & & & \cr & & & & & \ominus & & }_{\rho},$$

$$\mathcal{E}_3 = \bordermatrix{ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \cr & \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \cr & & & \lhd & \oplus & \rhd & & \cr & & & & \ominus & & & }_{\rho}.$$

- None of them satisfy (L), so π is not in any L-packet of Arthur type.

"The" local Arthur parameter: Problem

- A representation of Arthur type can live in multiple local Arthur packets.
- Our second application answers the following problem.

Problem

Given π *of Arthur type, specify a distinguished member* $\psi(\pi)$ *in the set* $\{\psi \mid \pi \in \Pi_\psi\}$ *with the requirement*

$$\pi \in \Pi_{\phi_\psi} \textit{ for some } \psi \Rightarrow \psi(\pi) = \psi.$$

"The" local Arthur parameter: Observation

- Let $\pi = \pi(\mathcal{E})$. To specify a distinguished member in $\{\psi \mid \pi \in \Pi_\psi\}$ is the same as to specify a distinguished member in

$$\Psi(\mathcal{E}) := \{\mathcal{E}' \mid \pi(\mathcal{E}) = \pi(\mathcal{E}')\}/(\text{row exchanges}).$$

- Since a representation lives in only one L-packet, there exists **at most one** element $\mathcal{E}'$ in $\Psi(\mathcal{E})$ satisfying (L). If such $\mathcal{E}'$ exists, then $\pi(\mathcal{E}) \in \Pi_{\phi_{\psi_{\mathcal{E}'}}}$, and the requirement implies $\psi(\pi(\mathcal{E})) = \psi_{\mathcal{E}'}$.
- If there is no such $\mathcal{E}'$, then the L-parameter of $\pi(\mathcal{E})$ is not of Arthur type. To define $\psi(\pi(\mathcal{E}))$ in this case, we need another condition weaker than (L).

Absolutely maximal

Definition

Suppose $\pi(\mathcal{E}) \neq 0$. We say $\mathcal{E}$ is absolutely maximal if

(a) No ui^{-1} is applicable on $\mathcal{E}$.

(b) No ui is applicable on $dual(\mathcal{E})$.

(c) No $dual_i^-$ is applicable on $\mathcal{E}$.

- Intuitively, if $\mathcal{E}$ is absolutely maximal, $\psi_{\mathcal{E}}$ is the most "tempered" one in the set

$$\{\psi \mid \pi(\mathcal{E}) \in \Pi_\psi\}.$$

Proposition

Suppose $\pi(\mathcal{E}) \neq 0$. *Then*

$$\mathcal{E} \text{ satisfies (L)} \Longrightarrow \mathcal{E} \text{ is absolutely maximal.}$$

"The" local Arthur parameter: Statement

- The last piece to define "the" local Arthur parameter is the following theorem.

Theorem (Hazeltine, Liu and L.)

Suppose $\pi(\mathcal{E}) \neq 0$. Then there exists ***exactly one*** *absolutely maximal member in the set*

$$\Psi(\mathcal{E}) = \{\mathcal{E}' \mid \pi(\mathcal{E}') = \pi(\mathcal{E})\}/(\textit{row exchanges}),$$

which we denote by $\mathcal{E}^{max}$, the max form of $\mathcal{E}$.

- This suggests the following definition.

Definition

Suppose $\pi = \pi(\mathcal{E})$. We define "the" local Arthur parameter of π by

$$\psi(\pi) := \psi_{\mathcal{E}^{max}}.$$

"The" local Arthur parameter: Example

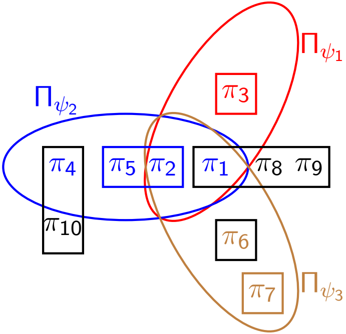
- Let ρ be the trivial representation. We consider three local Arthur parameters of $\mathrm{Sp}_{10}(F)$,

$$\psi_1 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_2 \otimes S_2,$$
$$\psi_2 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_1 \otimes S_1 + \rho \otimes S_3 \otimes S_1,$$
$$\psi_3 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_1 \otimes S_3 + \rho \otimes S_1 \otimes S_1.$$

There are seven representations in $\Pi_{\psi_1} \cup \Pi_{\psi_2} \cup \Pi_{\psi_3}$. We denote them by $\pi_1, \cdots, \pi_7$.

"The" local Arthur parameter: Example

- We visualize them in the following picture.



- Ellipses are local Arthur packets. Rectangles are *L*-packets. The color of representations indicate "the" local Arthur parameters of them. Representations and *L*-packets in black are not of Arthur type.

π_1

- $\{\mathcal{E} \mid \pi(\mathcal{E}) = \pi_1\}/(\text{row exchanges}) = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ where

$$\mathcal{E}_1 = \bordermatrix{ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \cr & \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \cr & & & & \oplus & \ominus & & }_{\rho},$$

$$\mathcal{E}_2 = \bordermatrix{ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \cr & \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \cr & & & & \oplus & & & \cr & & & & & \ominus & & }_{\rho},$$

$$\mathcal{E}_3 = \bordermatrix{ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \cr & \lhd & \lhd & \lhd & \ominus & \rhd & \rhd & \rhd \cr & & & \lhd & \oplus & \rhd & & \cr & & & & \ominus & & & }_{\rho}.$$

- Since $dual \circ ui_2 \circ dual(\mathcal{E}_3) = \mathcal{E}_1$ and $ui_2^{-1}(\mathcal{E}_1) = \mathcal{E}_2$, $\mathcal{E}_2$ is the only absolutely maximal element. Thus $\psi(\pi_1) = \psi_2$.

π_2

- $\{\mathcal{E} \mid \pi(\mathcal{E}) = \pi_2\}/(\text{row exchanges}) = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$, where

$$\mathcal{E}_1 = \begin{pmatrix} \overset{-3}{\lhd} & \overset{-2}{\lhd} & \overset{-1}{\lhd} & \overset{0}{\ominus} & \overset{1}{\rhd} & \overset{2}{\rhd} & \overset{3}{\rhd} \\ & & & \ominus & \oplus & & \end{pmatrix}_{\rho},$$

$$\mathcal{E}_2 = \begin{pmatrix} \overset{-3}{\lhd} & \overset{-2}{\lhd} & \overset{-1}{\lhd} & \overset{0}{\ominus} & \overset{1}{\rhd} & \overset{2}{\rhd} & \overset{3}{\rhd} \\ & & & \ominus & & & \\ & & & & \oplus & & \end{pmatrix}_{\rho},$$

$$\mathcal{E}_3 = \begin{pmatrix} \overset{-3}{\lhd} & \overset{-2}{\lhd} & \overset{-1}{\lhd} & \overset{0}{\ominus} & \overset{1}{\rhd} & \overset{2}{\rhd} & \overset{3}{\rhd} \\ & & \lhd & \ominus & \rhd & & \\ & & & \oplus & & & \end{pmatrix}_{\rho}.$$

- Since $dual \circ ui_2 \circ dual(\mathcal{E}_3) = \mathcal{E}_1$ and $ui_2^{-1}(\mathcal{E}_1) = \mathcal{E}_2$, $\mathcal{E}_2$ is the only absolutely maximal element. Thus $\psi(\pi_2) = \psi_2$. Also, $\mathcal{E}_2$ satisfies (L), so $\pi_2 \in \Pi_{\phi_{\psi_2}}$.

- The End.
- Thank you!