Study of parity sheaves arising from graded Lie algebras

Tamanna Chatterjee

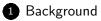
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Overview



2 Motivation

3 Parity complex

- 4 Conjectures
- 5 Main results
- 6 Example of conjecture 2

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• Let G be a connected, reductive, algebraic group over \mathbb{C} and \mathfrak{g} be the Lie algebra of G.

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- Let G be a connected, reductive, algebraic group over $\mathbb C$ and $\mathfrak g$ be the Lie algebra of G.
- We fix a cocharacter map, $\chi: \mathbb{C}^{\times} \to G$ and define,

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the centralizer of $\chi(\mathbb{C}^{\times})$.

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• For $n \in \mathbb{Z}$, define,

$$\mathfrak{g}_n = \{x \in \mathfrak{g} | Ad(\chi(t))x = t^n x, \forall t \in \mathbb{C}^{\times}\}.$$

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 \bullet This defines a grading on $\mathfrak{g},$

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n.$$

Clearly, $\mathfrak{g}_0 = \text{Lie}(G_0)$ and G_0 acts on \mathfrak{g}_n .

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- We can think of Local systems(locally constant sheaves) on a space X as the representations of the fundamental group, $\pi_1(X)$.
- Intersection cohomology complexes are some important objects in $D_{H}^{b}(X)$, the equivariant derived category on X.
- For example, intersection cohomology complexes in $D_G^b(\mathcal{N}_G)(\mathcal{N}_G)$ is the nilpotent cone) are in bijection with the pairs (C, \mathcal{F}) , where C is a G-orbit and \mathcal{F} , a local system on C.

Example of \mathcal{IC} 's on Sp_4

Table: Orbits in \mathfrak{sp}_4

•	orbits:	$\mathcal{O}[4]$	$\mathcal{O}[2^2]$	$\mathcal{O}[2,1^2]$	$\mathcal{O}[1^4]$
	π_1 :	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

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This means we have 7 simple \mathcal{IC} 's in $D^b_{Sp_4}(\mathcal{N}_{Sp_4})$.

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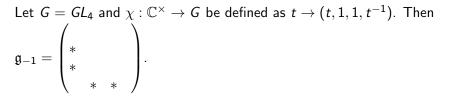
 Intersection cohomology complexes on D^b_{G0}(g_n) are IC(O, L), where O ⊂ g_n is G₀-orbit and L is a local system on O. Denote the set of all collection of these pairs by 𝒴(g_n).

\mathcal{IC} 's in $D^b_{G_0}(\mathfrak{g}_n)$ for $G = GL_4$.

Let $G = GL_4$ and $\chi : \mathbb{C}^{\times} \to G$ be defined as $t \to (t, 1, 1, t^{-1})$.



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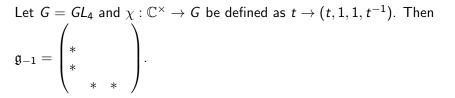


Table: Table for representatives of non-zero G_0 -orbits and their π_1

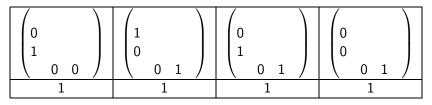
$ \left(\begin{array}{ccc} 0 & & \\ 1 & & \\ & 0 & 0 \end{array}\right) $	$\left(\begin{array}{ccc}1&&\\0&&\\&0&1\end{array}\right)$	$ \left(\begin{array}{ccc} 0 & & \\ 1 & & \\ & 0 & 1 \end{array}\right) $	$ \left(\begin{array}{ccc} 0 & & \\ 0 & & \\ & 0 & 1 \end{array}\right) $
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Let
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 and $\chi : \mathbb{C}^{\times} \to G$ be defined as $t \to (t, 1, 1, t^{-1})$. Then
 $\mathfrak{g}_{-1} = \begin{pmatrix} * & & \\ * & & \\ & * & * \end{pmatrix}$.

Table: Table for representatives of non-zero G_0 -orbits and their π_1



Including the zero orbit and what we have in the table, we have 5 simple \mathcal{IC} 's in $D^b_{G_0}(\mathfrak{g}_{-1})$.

• We consider a diagram for a Levi subgroup *L* contained in a parabolic *P* with *U*_{*P*}, the unipotent radical of *P*.

$$\mathcal{N}_{L} \xleftarrow{\pi_{P}} \mathcal{N}_{L} + \mathfrak{u}_{P} \xrightarrow{e_{P}} G \times^{P} (\mathcal{N}_{L} + \mathfrak{u}_{P}) \xrightarrow{\mu_{P}} \mathcal{N}_{G}$$
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• $\operatorname{Res}_{P}^{G}: D_{G}^{b}(\mathscr{N}_{G}, \Bbbk) \to D_{L}^{b}(\mathscr{N}_{L}, \Bbbk)$ is defined by $\operatorname{Res}_{P}^{G}(\mathcal{F}) = \pi_{P!}e_{P}^{*}\mu_{P}^{*}\operatorname{For}_{L}^{G}(\mathcal{F}).$

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- $\operatorname{Ind}_{P}^{G}: D_{L}^{b}(\mathscr{N}_{L}, \Bbbk) \to D_{G}^{b}(\mathscr{N}_{G}, \Bbbk)$ is defined by, $\operatorname{Ind}_{P}^{G}(\mathcal{F}) := \mu_{P!}(e_{P}^{*}\operatorname{For}_{P}^{G})^{-1}\pi_{P}^{*}(\mathcal{F}).$

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- $\operatorname{Res}_{P}^{G}$ is left adjoint to $\operatorname{Ind}_{P}^{G}$.

Cuspidal pairs

Let \$\mathcal{I}(G)\$ denote the collection of all pairs (C, \$\mathcal{E})\$, where C is a G-orbit and \$\mathcal{E}\$ is a G-equivariant local system on \$C\$.

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Definition

A pair $(C, \mathcal{E}) \in \mathscr{I}(G)$, is called cuspidal if $\operatorname{Res}_P^G \mathcal{IC}(C, \mathcal{E}) = 0$ for any proper parabolic P and a Levi factor $L \subset P$.

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• For nilpotent cones, how to get a $\mathcal{IC}(C, \mathcal{F})$ from the induction of some cuspidal pair have been studied in "Springer theory".

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Induction and restriction in the graded setting.

• The diagram below defines induction and restriction.

$$\mathfrak{l}_n \xleftarrow{\pi} \mathfrak{p}_n \xrightarrow{e} G_0 \times^{P_0} \mathfrak{p}_n \xrightarrow{\mu} \mathfrak{g}_n$$

 $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}: D_{L_{0}}^{b}(\mathfrak{l}_{n}) \to D_{G_{0}}^{b}(\mathfrak{g}_{n}),$

 $\operatorname{\mathsf{Res}}^{\mathfrak{g}}_{\mathfrak{p}}: D^b_{G_0}(\mathfrak{g}_n) \to D^b_{L_0}(\mathfrak{l}_n).$

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- Res^g_p is left adjoint to Ind^g_p.

Lusztig's work in characteristic 0.

Definition (Cuspidal on \mathfrak{g}_n)

 $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$ will be called cuspidal if there exists a cuspidal pair $(\mathcal{C}, \mathcal{E}) \in \mathscr{I}(\mathcal{G})$, such that $\mathcal{C} \cap \mathfrak{g}_n = \mathcal{O}$ and $\mathcal{L} = \mathcal{E}|_{\mathcal{O}}$.

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Theorem (Lusztig)

In characteristic 0, for any $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$, there exists a Levi subgroup L contained in a parabolic subgroup P with a cuspidal pair $(\mathcal{O}_L, \mathcal{L}') \in \mathscr{I}(\mathfrak{l}_n)$ so that, $\mathcal{IC}(\mathcal{O}, \mathcal{L})$ appears as direct summand of $\mathrm{Ind}_p^{\mathfrak{g}} \mathcal{IC}(\mathcal{O}_L, \mathcal{L}')$.

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- This theorem is not true when the characteristic of the field of sheaf coefficients is positive.
- Following the pattern from other works in modular representation theory, often the appropriate replacement for "semisimple complex" or \mathcal{IC} 's is "parity complex".

• Parity complexes are first introduced by Juteau, Mautner and Williamson in their paper "Parity sheaves".

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- Parity complexes are first introduced by Juteau, Mautner and Williamson in their paper "Parity sheaves".
- Parity sheaves are classified as the class of constructible complexes on some stratified varieties, where the strata satisfy some cohomology vanishing properties.

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- We denote the parity sheaf associated to the pair $(\mathcal{O}, \mathcal{L})$ by $\mathcal{E}(\mathcal{O}, \mathcal{L})$, where $\mathcal{E}(\mathcal{O}, \mathcal{L})|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}].$

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• Unlike for \mathcal{IC} 's, $\mathcal{E}(\mathcal{O}, \mathcal{L})$ does not exist automatically.

Cohomology vanishing

Assumption

The field \Bbbk is algebraically closed and characteristic I of \Bbbk is a "pretty good" prime for G.

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Proposition (Juteau-Mautner-Williamson)

Let C be a nilpotent orbit in \mathcal{N}_G and $\mathcal{L} \in \text{Loc}_{f,G}(C, \mathbb{k})$, then $H^*_G(\mathcal{L})$ vanishes in odd degrees.

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Proposition (C)

Let \mathcal{O} be a G_0 -orbit in \mathfrak{g}_n and $\mathcal{L} \in \text{Loc}_{f,G_0}(\mathcal{O}, \Bbbk)$, then $H^*_{G_0}(\mathcal{L})$ vanishes in odd degrees.

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Cohomology vanishing needed for parity sheaves to make sense.

Definition (Clean)

A pair $(C, \mathcal{E}) \in \mathscr{I}(G, \Bbbk)$ is called *I*-clean if the corresponding $\mathcal{IC}(C, \mathcal{E})$ has vanishing stalks on $\overline{C} - C$. Similarly, a pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \Bbbk)$ is called *I*-clean if the corresponding $\mathcal{IC}(\mathcal{O}, \mathcal{L})$ has vanishing stalks on $\overline{\mathcal{O}} - \mathcal{O}$.

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Conjecture (1)

If the characteristic I of \Bbbk is a "pretty good" prime for G, then every 0-cuspidal pair $(C, \mathcal{E}) \in \mathscr{I}(G)$ is I-clean.

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• This already holds when the characteristic does not divide the order of the Weyl group of the group *G*.

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- This already holds when the characteristic does not divide the order of the Weyl group of the group *G*.
- If every irreducible factor of the root system of *G* is either of type *A*, *B*₄, *C*₃, *D*₅ or of exceptional types then also the conjecture holds.

Existence of parity for cuspidal pairs

Assuming the conjecture is true.

Theorem (C)

Under the assumption on the characteristic of \Bbbk , any cuspidal pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \Bbbk)$ is clean.

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Theorem (C)

Under the assumption on the characteristic of \Bbbk , any cuspidal pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \Bbbk)$ is clean.

Corollary

For any cuspidal pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \mathbb{k}), \ \mathcal{IC}(\mathcal{O}, \mathcal{L}) = \mathcal{E}(\mathcal{O}, \mathcal{L}).$ Therefore parity sheaf exists for cuspidal pairs.

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Conjecture (2)

Let P be a parabolic subgroup of G and L be its Levi subgroup. For a pair $(C, \mathcal{E}) \in \mathscr{I}(L)^{cusp}$, $\operatorname{Ind}_{P}^{G} \mathcal{IC}(C, \mathcal{E})$ is a parity complex.

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• In characteristic 0, the proof follows from the decomposition theorem.

• In positive characteristic, the result is still unknown.

Conjecture (2)

Let P be a parabolic subgroup of G and L be its Levi subgroup. For a pair $(C, \mathcal{E}) \in \mathscr{I}(L)^{cusp}$, $\operatorname{Ind}_{P}^{G} \mathcal{IC}(C, \mathcal{E})$ is a parity complex.

- In characteristic 0, the proof follows from the decomposition theorem.
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- In the last section of our paper we have calculated $\operatorname{Ind}_{P}^{G} \mathcal{IC}(C, \mathcal{E})$ for SL_4 and Sp_4 , where the conjecture is true.
- In our next paper we are trying to prove this conjecture for classical groups.

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Main results

Assuming both the conjectures are true.

Theorem (C)

For any pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$, there exists a parabolic subgroup P with L, its Levi subgroup L and $(\mathcal{O}_L, \mathcal{L}') \in \mathscr{I}(\mathfrak{l}_n)^{cusp}$ such that $\mathcal{E}(\mathcal{O}, \mathcal{L})$ occurs as direct summand of $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{E}(\mathcal{O}_L, \mathcal{L}'))$.

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Let P be a parabolic subgroup of G with a Levi factor L, the induction functor sends parity complexes to parity complexes.

Calculation of Ind_P^G for $G = Sp_4$.

• $L = GL_1 \times Sp_2$ is a Levi subgroup. $(\mathcal{O}_{prin}, \mathcal{L}) \in \mathscr{I}(L)^{cusp}$.

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Table: Stalks of $\operatorname{Ind}_{P}^{G} \mathcal{IC}(\mathcal{O}_{prin}, \mathcal{L})$

	dim	<i>O</i> [4]	$O[2^2]$	$\mathcal{O}[2,1^2]$ rank 1	$\mathcal{O}[1^4]$
•	-2			rank 1	
	-2 -3 -4				
	-4			rank 1	
	-5				
	-6				
	-7				
	-8				
	-9				
	-10	rank 1			

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Hence the parity condition is satisfied.

Thank you for your attention!

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