GENERIC REPRESENTATIONS FOR QUASI-SPLIT SIMILITUDE GROUPS

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1. Introduction

The following is drawn from *The generic dual of p-adic groups and local Langlands parameters*[1], joint with Baiying Liu.

Let F be a p-adic field of characteristic 0.

We focus on classical and similitude groups whose standard Levi factors have the form

$$M = GL_{n_1} \times \cdots \times GL_{n_k} \times G_{n_0},$$

where G_{n_0} is a lower rank group of the same type, and whose Weyl group acts as permutations and sign changes.

[1]Legal disclaimer: paper in preparation; corrections possible.

1.1. **Groups considered.** For quasi-split groups, fix a quadratic extension $E = F(\sqrt{\varepsilon})$. We let G_n be one of the following groups:

- classical groups: G_n is SO_{2n+1}, Sp_{2n}, SO_{2n}, U_{2n+1}, U_{2n}, or SO^*_{2n+2}
- similitude groups: G_n is GSp_{2n}, GSO_{2n}, $GSpin_{2n+1}$, $GSpin_{2n}$, GU_{2n+1}, GU_{2n}, GSO^*_{2n+2}, or $GSpin^*_{2n+2}$

Parabolic subgroups and basic structure of these groups may be found in [Tad1], [Gol], [Asg], [H-S], [Xu] among other places.

1.2. **Goals.**

- Classify generic representations (with an eye toward applications as in [J-S], [Liu], [J-L])
- Fill in the gaps in the tools needed.
- Uniformize arguments across the different groups considered (as in [Tad3], [M-T], [ACS])

2. Generic representations

2.1. **Generic essentially discrete series.** Generic discrete series were classified for $G_n = SO_{2n+1}$ and Sp_{2n} in [Mui] and for SO_{2n} in [J-L].

Notation.

- $\nu = |\cdot|$ *composed with determinant or similitude character*
- $\times, \rtimes$ *denote (normalized) parabolic induction*
- π *essentially tempered if* $\exists\ \varepsilon(\pi) \in \mathbb{R}$ *such that* $\nu^{-\varepsilon(\pi)}(\pi)$ *is tempered*
- $\beta = \begin{cases} \frac{1}{2}\varepsilon \text{ if } G_n = GSpin_{2n+1}, GSpin_{2n} \text{ with } n = 0, \\ \varepsilon \text{ if } G_n = GSpin^*_{2n+2} \text{ or } G_n = GSpin_{2n+1}, GSpin_{2n} \text{ with } n > 0, \\ 0 \text{ if } G_n \text{ is not a general spin group.} \end{cases}$
- c=*usual outer automorphism–for* $SO_{2n}, GSO_{2n}, GSpin_{2n}$ *acts nontrivially on the simple roots; for* $SO^*_{2n+2}, GSO^*_{2n+2}, GSpin^*_{2n+2}$, *acts trivially on the* F*-data but nontrivially on the maximal quasi-split torus*

We recall the following combination of results from [Zel] and [Jac]:

Theorem (general linear groups)**.** *Let τ be an irreducible unitary supercuspidal representation of a general linear group. Let*

$$[\nu^m\tau, \nu^n\tau] = \{\nu^m\tau, \nu^{m+1}\tau, \dots, \nu^n\tau\}$$

(with $m - n \in \mathbb{Z}$). We define $\delta([\nu^m\tau, \nu^n\tau])$ to be the (unique) irreducible quotient of $\nu^m\tau \times \nu^{m+1}\tau \times \cdots \times \nu^n\tau$. Then $\delta([\nu^m\tau, \nu^n\tau])$ is essentially square-integrable, and every irreducible essentially square-integrable representation of a general linear groups has this form. Further, every such representation is generic.

Theorem (generic essentially discrete series)**.** *Let* $\Delta_i = [\nu^{-a_i}\tau_i, \nu^{b_i}\tau_i]$, $1 \leq i \leq k$, *where* τ_i *is an irreducible unitary supercuspidal representation of a general linear group. Assume that if* $i < j$ *has* $\tau_i \cong \tau_j$, *then* $a_i < b_i < a_j < b_j$. *Let* $\sigma^{(e0)}$ *be an irreducible supercuspidal generic representation of* $G_{n'}(F)$ *and assume that for each* i, *one of the following holds (necessarily exclusive):*

(1) $\nu^{1+\beta}\tau_i \rtimes \sigma^{(e0)}$ *is reducible, in which case* $a_i \in \beta + (\mathbb{Z} \setminus \{0\})$ *and* $a_i \geq \beta - 1$;

(2) $\nu^{\frac{1}{2}+\beta}\tau_i \rtimes \sigma^{(e0)}$ *is reducible, in which case* $a_i \in \beta - \frac{1}{2} + \mathbb{Z}_{\geq 0}$;

(3) $\nu^{\beta}\tau_i \rtimes \sigma^{(e0)}$ *is reducible, in which case* $a_i \in \beta + \mathbb{Z}_{\geq 0}$;

(4) $\nu^{x}\tau_i \rtimes \sigma^{(e0)}$ *is irreducible for all* $x \in \mathbb{R}$

(a) $\check{\tau}_i \cong \tau_i$ *but* $c^{d(\tau)} \cdot \sigma^{(e0)} \ncong \sigma^{(e0)}$ *for* SO_{2n}, SO^*_{2n+2},

(b) $\check{\tau}_i \cong \tau_i$ *but* $\omega_\tau \sigma^{(e0)} \ncong \sigma^{(e0)}$ *for* $GSp_{2n}, GU_{2n+1}, GU_{2n}$,

(c) $\check{\tau}_i \cong \tau_i$ *but* $\omega_\tau(c^{d(\tau)} \cdot \sigma^{(e0)}) \ncong \sigma^{(e0)}$ *for* GSO_{2n}, GSO^*_{2n+2},

(d) $\nu^{-2\beta}\omega_{\sigma^{(e0)}}\check{\tau}_i \cong \tau_i$ *but* $c^{d(\tau)} \cdot \sigma^{(e0)} \ncong \sigma^{(e0)}$ *for* $GSpin_{2n}, GSpin^*_{2n+2}$,

in which case $a_i \in \beta + \mathbb{Z}_{\geq 0}$. *(This case does not occur for* SO_{2n+1}, Sp_{2n}, U_{2n+1}, U_{2n}, $GSpin_{2n+1}$.*)*

Then, if π *is the generic subquotient of* $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma^{(e0)}$*,* π *is essentially square-integrable. Conversely, any generic irreducible essentially square-integrable* π *of a group* $G_n(F)$ *is of this form (with* $\Delta_1, \ldots, \Delta_k$ *unique up to permutation), and further*

$$\pi \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma^{(e0)}.$$

Note that the cusidal reducibility values in (1)–(3) follow from [Sha].

2.2. **Generic essentially tempered representations.**

PROPOSITION. *Let $\tau_1, \ldots, \tau_c$ generic irreducible unitary supercuspidal representations of general linear groups and $\sigma^{(e2)}$ a generic irreducible essentially square-integrable representation of $G_n(F)$. Let $\Psi_1, \ldots, \Psi_c$ be segments of the form $\Psi_i = [\nu^{\frac{-k_i+1}{2}}\tau_i, \nu^{\frac{k_i-1}{2}}\tau_i]$. Then the generic component*

$$\sigma^{(et)} \hookrightarrow \nu^{\beta}\delta(\Psi_1) \times \cdots \times \nu^{\beta}\delta(\Psi_c) \rtimes \sigma^{(e2)}$$

is a generic essentially tempered representation, where $\beta = \beta(\sigma^{(e2)})$. Further, any generic essentially tempered representation may be realized this way (with inducing representation unique up to Weyl conjugation).

This follows directly from a result of Harish-Chandra (cf. Proposition III.4.1 [Wa]).

2.3. **Generic admissible representations.**

NOTATION.

- *For* $\Sigma = [\nu^a\xi, \nu^b\xi]$, *set* $\check{\Sigma} = [\nu^{-b}\check{\xi}, \nu^{-a}\check{\xi}]$ *(so* $\delta(\Sigma)^\vee = \delta(\check{\Sigma})$*), where* $\check{}$ *=contragredient composed with Galois conjugation for unitary and general unitary groups and contragredient otherwise.*
- $\omega'_{\sigma^{(et)}} = \begin{cases} \omega_{\sigma^{(et)}} \text{ (central character) for general spin groups,} \\ 1 \text{ otherwise.} \end{cases}$

Note that by the standard module conjecture ([H-O]), the Langlands quotient of a generic standard module is generic if and only if the standard module induces irreducibly.

Theorem (the Langlands classification). [2] *Let* $\delta_1, \ldots, \delta_k$ *be essentially square integrable representations of general linear groups and and* T *an essentially tempered representation of* $G_n(F)$ *satisfying*

$$\varepsilon(\delta_1) \geq \cdots \geq \varepsilon(\delta_k) > \beta(T).$$

Then

$$\delta_1 \times \cdots \times \delta_k \rtimes T$$

contains a unique irreducible quotient $L(\delta_1 \otimes \cdots \otimes \delta_k \otimes T)$. *Further, any irreducible admissible representation may be written in this form, with the data unique up to permutations among representations of general linear groups having the same central exponents.*

See [B-W], [Sil], [Kon], etc., for the general result; [Tad1] (Sp_{2n} and GSp_{2n}), [Jan1] (SO_{2n+1} and SO_{2n}), [KimW] ($GSpin_{2n+1}$, $GSpin_{2n}$, $GSpin^*_{2n}$).

[2] Note that this formulation requires an artifice for SO_{2n}, $GSpin_{2n}$, GSO_{2n} (also used later).

Theorem. *Suppose* $\delta(\Sigma_i) = \nu^{x_i}\delta_i$, $i = 1, 2, \cdots, f$ *with* $x_1 \geq x_2 \geq \cdots \geq x_f > \beta$. *Then, the representation*

$$\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_j) \rtimes \sigma^{(et)}$$

is irreducible if and only if $\{\Sigma_j\}_{j=1}^{f}$ *and* $\sigma^{(et)}$ *satisfy the following properties:*

(1) $\delta(\Sigma_i) \times \delta(\Sigma_j)$ *and* $\delta(\Sigma_i) \times \omega'_{\sigma^{(et)}}\delta(\check{\Sigma}_j)$ *are irreducible for all* $1 \leq i \neq j \leq f$*; and*

(2) $\delta(\Sigma_i) \rtimes \sigma^{(et)}$ *is irreducible for all* $1 \leq i \leq f$.

The reducibility for (1) is known from [Zel]*; for (2) we write* $\sigma^{(et)}$ *as in the proposition above. Then* $\delta(\Sigma) \rtimes \sigma^{(et)}$ *is irreducible if and only if the following hold:*

(1) $\delta(\Sigma) \times \nu^{\beta}\delta(\Psi_j)$ *and* $\omega'_{\sigma^{(e2)}}\delta(\check{\Sigma}) \times \nu^{\beta}\delta(\Psi_j)$ *are irreducible for all* $1 \leq j \leq c$*; and*

(2) $\delta(\Sigma) \rtimes \sigma^{(e2)}$ *is irreducible.*

To understand when the second condition above holds, write $\sigma^{(e2)}$ as in the theorem above. Then, $\delta(\Sigma) \rtimes \sigma^{(e2)}$ is irreducible if and only if the following hold:

(1) $\delta(\Sigma)\times\delta(\Delta_i)$ *and* $\omega'_{\sigma^{(e0)}}\delta(\check{\Sigma})\times\delta(\Delta_i)$ *are irreducible for all* $i = 1, 2, \cdots, k'$; *and*

(2) *either (a)* $\delta(\Sigma) \rtimes \sigma^{(e0)}$ *is irreducible, or (b)* $\delta(\Sigma) = \delta([\nu^{1+\beta}\xi, \nu^{b+\beta}\xi])$, *with* $\nu^{1+\beta}\xi\rtimes\sigma^{(e0)}$ *reducible and there is some* i *having* $\delta(\Delta_i) = \delta([\nu^{1+\beta}\xi, \nu^{b_i+\beta}\xi])$ *and* $b_i \geq b$.

Finally, for the second condition above, we have $\delta(\Sigma) \rtimes \sigma^{(e0)}$ *is irreducible if and only if one of the following hold: for* $\Sigma = [\nu^{-a}\xi, \nu^{b}\xi]$, *we have*

(1) $\xi \ncong \check{\xi}$; *or*

(2) $\xi \cong \check{\xi}$ *and the following: (i) if* $\nu^x\xi \rtimes \sigma^{(e0)}$ *is reducible for some (necessarily unique)* $x = \alpha \geq 0$, *then* $\pm\alpha \notin \{-a, -a+1, \cdots, b\}$; *(ii) if* $\nu^x\xi \rtimes \sigma^{(e0)}$ *is irreducible for all* $x \geq 0$, *then* $a \notin \mathbb{Z}_{\geq 0}$.

3. Tools

The main results needed:

- results on general linear groups
- cuspidal reducibility conditions
- standard module conjecture
- the Langlands classification and Casselman criterion
- formula for twisting an induced representation by characters
- μ^* structure formula for calculating Jacquet modules

3.1. **Twisting by characters.** The formulas below are analogues of that for GSp_{2n} in [S-T].

LEMMA. *Let χ be a character of $F^\times$ and identify χ with a character of $G_n(F)$ via composition with the similitude character.*

(1) *For* $G_n = GSpin_{2n+1}$, $\chi(\pi \rtimes \theta) \cong \begin{cases} \chi\pi \rtimes \chi\theta \text{ if } n > 0 \\ \chi\pi \rtimes \chi^2\theta \text{ if } n = 0. \end{cases}$

(2) *For* $G_n = GSp_{2n}$, GSO^*_{2n+2}, GU_{2n+1}, GU_{2n}, $\chi(\pi \rtimes \theta) \cong \pi \rtimes \chi\theta$.

(3) *For* $G_n = GSO_{2n}$, $\chi(\pi \rtimes \theta) \cong \pi \rtimes \chi\theta$ $(n \neq 1)$.

(4) *For* $G_n = GSpin_{2n}$, $\chi(\pi \rtimes \theta) \cong \begin{cases} \chi\pi \rtimes \chi\theta \text{ if } n > 1, \\ \chi\pi \rtimes \chi^2\theta \text{ if } n = 0. \end{cases}$

(5) *For* $G_n = GSpin^*_{2n+2}$, $\chi(\pi \rtimes \theta) \cong \chi\pi \rtimes \chi\theta$.

3.2. μ^* **structure formula.** This has been done in [Tad2] (SO_{2n+1}, Sp_{2n}, and GSp_{2n}), [M-T] (U_{2n+1} and U_{2n}), [J-L] (SO_{2n}), [KimY1] ($GSpin_{2n+1}$), [KimY2] ($GSpin_{2n}$), [K-M] (GU_{2n}). The corresponding result for general linear groups is in [Zel]. Note that this also uses the artifice for SO_{2n}, $GSpin_{2n}$, GSO_{2n}.

Definition. *For general linear groups,*

$$m^* = \sum_{k=0}^{n} r_{M_k,G},$$

where $M_k = GL_k \times GL_{n-k}$. *For* G_n, *we set*

$$\mu^* = \sum_{k=0}^{n} r_{M_k,G},$$

where $M_k = GL_k \times G_{n-k}$.

Let

$$N^* = (\check{} \otimes m^*)_D \circ s \circ m^*,$$

where $s : \tau_1 \otimes \tau_2 \mapsto \tau_2 \otimes \tau_1$ and

$$(\check{} \otimes m^*)_D(\tau_1 \otimes \tau_2) = \begin{cases} \check{\tau}_1 \otimes m^*(\tau_2) \otimes e \text{ if } \tau_1 \text{ a representation of } GL_{even}, \\ \check{\tau}_1 \otimes m^*(\tau_2) \otimes c \text{ if } \tau_1 \text{ a representation of } GL_{odd}. \end{cases}$$

THEOREM. [3]

$$\mu^*(\tau \rtimes \pi) = N^*(\tau) \tilde{\rtimes} \mu^*(\pi),$$

where

$$(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes d) \tilde{\rtimes} (\rho \otimes \sigma)$$
$$= \begin{cases} (\rho_1 \times \rho_2 \times \rho) \otimes (\rho_3 \rtimes \sigma) \text{ for } G_n = SO_{2n+1}, Sp_{2n}, U_{2n+1}, U_{2n}, \\ (\omega_\sigma \rho_1 \times \rho_2 \times \rho) \otimes (\rho_3 \rtimes \sigma) \text{ for } G_n = GSpin_{2n+1}, \\ (\rho_1 \times \rho_2 \times \rho) \otimes (\rho_3 \rtimes \omega_{\check{\rho}_1} \sigma) \text{ for } G_n = GSp_{2n}, GU_{2n+1}, GU_{2n}, \\ (\rho_1 \times \rho_2 \times \rho) \otimes d(\rho_3 \rtimes \sigma) \text{ for } G_n = SO_{2n}, SO^*_{2n+2}, \\ (\rho_1 \times \rho_2 \times \rho) \otimes d(\rho_3 \rtimes \omega_{\check{\rho}_1} \sigma) \text{ for } G_n = GSO_{2n}, GSO^*_{2n+2}, \\ (\omega_\sigma \rho_1 \times \rho_2 \times \rho) \otimes d(\rho_3 \rtimes \sigma) \text{ for } G_n = GSpin_{2n}, GSpin^*_{2n+2}. \end{cases}$$

[3] Thanks to [Arc] for helping us realize the role of c in the quasi-split case.

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