Ramification of supercuspidal parameters

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September 27, 2020

Outline

Review of V. Lafforgue's global results

Let p be a prime number, $k = \mathbb{F}_q$ a finite field of characteristic p, Y be a smooth projective curve over k, $\ell \neq p$ a prime. Let G be a split semisimple group over $K = k(Y)$, $\mathcal{A}_0(G)$ the set of cuspidal automorphic representations of $G(\mathbf{A}_K)$, $\mathcal{G}^{ss}(G)$ the set of semisimple homomorphisms

$$\rho_\ell : Gal(K^{sep}/K) \to \hat{G}(\overline{\mathbb{Q}}_\ell).$$

Theorem (VL)

There is a map

$$\mathcal{L} : \mathcal{A}_0(G) \to \mathcal{G}^{ss}(G)$$

with the following property: if v is a place of K and $\Pi \in \mathcal{A}_0(G)$ is a cuspidal automorphic representation such that Π_v is unramified, then $\mathcal{L}(\Pi)$ is unramified at v, and the semisimplification $\mathcal{L}^{ss}(\Pi) \mid_{W_{K_v}}$ is the Satake parameter of Π_v.

Local parameters

Theorem (Genestier-Lafforgue)

With the above hypotheses, let w be any place of K. Then

$$\mathcal{L}_w(\Pi_w) := [\mathcal{L}(\Pi)\,|_{W_{K_w}}]^{ss}$$

depends only on K_w and Π_w (not on the globalizations K and Π). Moreover, $\mathcal{L}_w$ is compatible with parabolic induction in the obvious sense.

In particular, if $F = k((t))$ is an equal characteristic local field and π is an irreducible representation of $G(F)$, we can define the semisimple homomorphism

$$\mathcal{L}(\pi) : W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell).$$

Weights

Let $\sigma : \hat{G} \to GL(N)$ be any representation, S the set of primes where Π is ramified. Then

$$\mathcal{L}(\Pi)_\sigma := \sigma \circ \mathcal{L}(\Pi) : Gal(K^{sep}/K) \to GL(N, \overline{\mathbb{Q}}_\ell)$$

corresponds to a semi-simple ℓ-adic local system $L(\Pi)_\sigma$ on $Y \setminus |S|$.

By Deligne's Weil II, each irreducible summand of $L(\Pi)_\sigma$ is punctually pure (up to twist by a character of $Gal(\bar{k}/k)$). It follows that for any w, the eigenvalues of $\sigma \circ \mathcal{L}_w(\Pi_w)(Frob_w)$ are Weil q-numbers of various weights (up to the twist, which we ignore).

Say a representation π of $G(F)$ is *pure* if for some (equivalently, for any) faithful σ, all the eigenvalues of $\sigma \circ \mathcal{L}(\pi)(Frob_q)$ have the same weight.

What about supercuspidals?

If $G \neq GL(n)$, not all supercuspidals are pure. Here is our main theorem. Recall that G is split semisimple.

Theorem (GHLS)

Let π be a pure supercuspidal representation of $G(F)$. Suppose π is compactly induced from a compact open subgroup of G. Suppose moreover that $q > 3$. Then $\mathcal{L}(\pi)$ is not unramified.

Henceforward we assume p does not divide the order of the Weyl group $W(G)$, so we can apply Fintzen's theorem. Because the local parametrization $\mathcal{L}$ is compatible with parabolic induction, we immediately conclude

Corollary

Let π be a pure representation of $G(F)$. Suppose Suppose $\mathcal{L}(\pi)$ is unramified. Then π is an irreducible constituent of an unramified principal series.

Incorrigible representations

Suppose there is a good notion of local and global *cyclic stable base change* over function fields, as treated in Labesse's book. We define an *incorrigible representation* of $G(F)$ to be a supercuspidal representation π such that, for any sequence $F \subset F_1 \subset \cdots \subset F_r$ of cyclic Galois extensions, the base change of π to F_r (which is an L-packet) contains a supercuspidal member.

Corollary

Under the above hypotheses, no pure supercuspidal representation is incorrigible.

For $G = GL(n)$ this was proved by Henniart, using his numerical correspondence, and again by Scholze, using nearby cycles. In retrospect, this, together with the existence of a canonical parametrization, is a key step in any proof of the local Langlands correspondence for $GL(n)$.

Globalization

Let $Y = \mathbb{P}^1$, $K = k(t)$. Choose a Borel $B \subset G$, B_- an opposite Borel. Let $I_0 \subset G(K_0)$ (resp. $I_{\infty,+} \subset G(K_\infty)$ denote the Iwahori corresponding to B (resp. the pro-unipotent radical of the Iwahori corresponding to B_-). We construct a cuspidal automorphic representation Π of $G(\mathbf{A}_K)$ such that

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k) \subset \mathbb{P}^1(\bar{k})$, $\Pi_z \xrightarrow{\sim} \pi$;

(b) For $x \notin |\mathbb{P}^1(k)|$, Π_x is unramified

(c) $\Pi_\infty^{I_{\infty,+}} \neq 0$.

(d) $\Pi_0^{I_{0,+}}$ contains a vector transforming under a certain character χ_k of $I_0/I_{0,+}$.

Application of purity

Suppose $\mathcal{L}(\pi)$ is unramified. By the purity hypothesis, for any faithful $\sigma \in Rep(\hat{G})$, $\mathcal{L}(\Pi)_\sigma$, which is a priori an ℓ-adic local system on $\mathbb{P}^1 \setminus |\mathbb{P}^1(k)|$, extends to a punctually pure local system on $\mathbb{G}_m$. (There is no unipotent monodromy at the points in $\mathbb{G}_m(k)$.) Moreover, our hypotheses imply that the ramification at 0 and ∞ is *tame*. Thus it is a sum of local systems induced from characters of finite order of the tame fundamental group of $\mathbb{G}_m$. Of course, $\mathcal{L}(\Pi)_z = \mathcal{L}(\pi)$ for every $z \in \mathbb{G}_m(k)$. By varying the character χ_k, we obtain a contradiction.

Suppose for simplicity π is compactly induced from $U = G(\mathcal{O}_F)$. Let φ_π be a matrix coefficient of π supported in U, $\varphi_\pi(1) = 1$.
We construct Poincaré series on $G(\mathbf{A}_K)$ as in the Gan-Lomelí paper

$$P_\varphi(g) = \sum_{\gamma \in G(K)} \varphi(\gamma \cdot g), g \in G(\mathbf{A}_K)$$

where $\varphi = \prod_x \varphi_x$ with

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k)$, $\varphi_z = \varphi_\pi$;
(b) For $x \notin |\mathbb{P}^1(k)|$, $\varphi_x = 1_{G(\mathcal{O}_x)}$;
(c) $\varphi_\infty = 1_{I_{\infty,+}}$;
(d) $\varphi_0 = \chi_k : I_0/I_0(p) \to \mathbb{C}^\times$

The support conditions imply $P_\varphi(1) = 1$ and then P_φ generates the desired Π.

In general

The assumption that π is compactly induced from $G(\mathcal{O}_F)$ allows us to choose the local groups at 0 and ∞ very simply. In general one shows they can be chosen to guarantee $P_\varphi(1) = \varphi(1) = 1$ by an argument on the Bruhat-Tits building of G.

The case of reductive G reduces easily to the semisimple case. We have not looked seriously at non-split G.

Wild ramification

Theorem

Suppose π is a pure supercuspidal compactly induced from an open compact subgroup that is **small** *in an appropriate sense. Then $\mathcal{L}(\pi)$ is wildly ramified.*

An example of small open compact subgroup is the principal congruence subgroup $G(\mathcal{O}_F)_+ \subset G(\mathcal{O}_F)$. Arguing as before, one gets a non-vanishing Poincaré series that is unramified outside ∞, thus a local system on $\mathbb{A}^1$. By the previous theorem, $\mathcal{L}(\pi)$ is ramified, and since there are no tamely ramified local systems on $\mathbb{A}^1$, the ramification must be wild.
There are more general "small" open compacts – any pro-p open compact is "small" – but the general argument is more subtle.