Maximal Unramified Tori in Classical Groups

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Notation

- $[k : \mathbb{Q}_p] < \infty$
- $K \subseteq \overline{k}$ = maximal unramified extension of k.
- $\mathfrak{o}, \mathfrak{p}, \varpi$
- $\mathbb{F}_q = \mathfrak{o}/\mathfrak{p}$
- Fr = topological generator of $\mathrm{Gal}(K/k) \longleftrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$
- G = reductive algebraic group defined over k

Motivation

While all maximal k-tori in G are conjugate over the algebraic closure, they are not all $G(k)$-conjugate.

A natural question then is what are the $G(k)$-conjugacy classes of maximal k-tori?

Many important representations emerge from pairs (T, θ), where T is a maximal k-torus and θ is a complex character of T. Understanding these representations thus requires us to understand the maximal k-tori in G, and one way to do this is to parameterize the $G(k)$-conjugacy classes.

Motivation

A parameterization of the $G(k)$-conjugacy classes of all maximal k-tori has remained elusive, but in the case of unramified tori (k-tori which split over K), there are known results.

- For the classical groups Sp_{2n}, SO_n, and unramified U_n, Waldspurger gives a parameterization in terms of triples of partitions (μ_0, μ', μ''). He constructs a regular semisimple element in the Lie algebra by constructing an endomorphism of a k-algebra whose structure is governed by the parts of the partitions.
- For general reductive groups, DeBacker gives a parameterization in terms of Bruhat-Tits theory.

Our goal for this talk is to give a comparison of the two parameterizations in the case of Sp_{2n}.

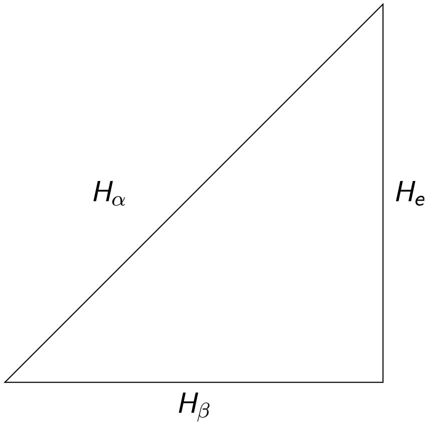
The DeBacker Parameterization 1

For simplicity, I will assume that G is k-split. We fix a maximal k-split torus S of G.

Since we are looking at the $G(k)$-conjugacy classes, it suffices to work within the closure of a fixed alcove lying in the apartment of S in the Bruhat-Tits building of G.

The walls of such an alcove are given by hyperplanes H_{α_i}, where $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ forms a simple system for the roots of S in G, and by another hyperplane H_e, where e is the highest root of S in G with respect to the partial order defined by Δ.

An Alcove for Sp_4



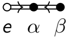
The DeBacker Parameterization 2

The facets in the fundamental alcove are in bijection with the proper subdiagrams of the extended Dynkin diagram.

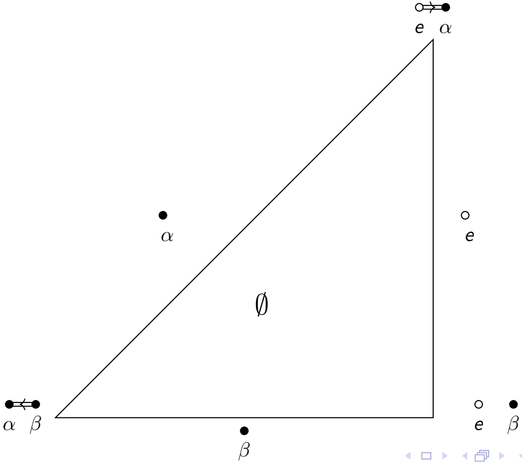
In particular, given a subdiagram, the corresponding facet is the one which vanishes at the hyperplanes H_α, where α is a root in $\Delta \cup \{e\}$ so that the vertex associated to α in the extended Dynkin diagram lies in our subdiagram.

The Facets in an Alcove of Sp_4

The extended Dynkin diagram of Sp_4 is:



Our facets are the labeled:



The DeBacker Parameterization 3

Given a facet F in our alcove, we let G_F denote the parahoric subgroup of G associated to F, and we let G_F^+ denote the pro-unipotent radical of G_F.

Then $\mathsf{G}_F := G_F/G_F^+$ can be thought of as a reductive group over $\mathbb{F}_q$, and the rational conjugacy classes of maximal $\mathbb{F}_q$-tori in G_F are parameterized by conjugacy classes in the Weyl group W_F of G_F.

By Bruhat-Tits, a maximal $\mathbb{F}_q$-torus in G_F can be lifted to a maximal unramified torus of G, and DeBacker shows that all maximal unramified tori arise in this way.

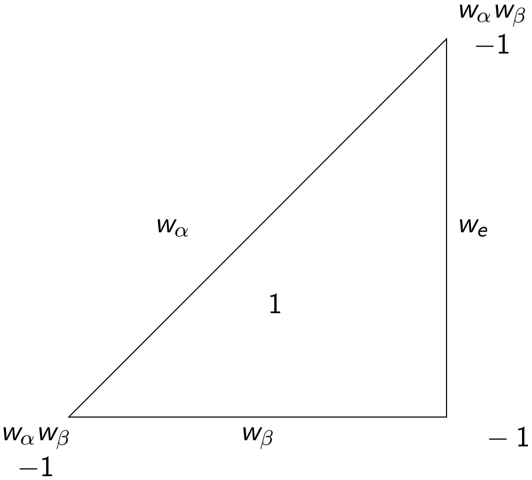
The DeBacker Parameterization 4

However, some tori can occur in the reductive quotients of multiple facets. To deal with this, we restrict to the conjugacy classes of elliptic Weyl group elements, i.e. those that do not lie in any proper parabolic subgroup.

We also define an equivalence relation on facets in the alcove.

DeBacker then shows that the maximal unramified tori are in bijective correspondence with equivalence classes of pairs (F, w), where F is a facet in our alcove and w is elliptic in the Weyl group W_F of G_F.

The DeBacker Parameterization for Sp_4



The Waldspurger Parameterization 1

We let $G = \mathrm{Sp}_{2n}$ for the remainder of the talk.

- Take a finite set of unramified extensions $\{k_i^{\#}/k\}_{i \in I}$ so that $\sum_{i \in I}[k_i^{\#} : k] = n$.
- For each $k_i^{\#}$, take a 2-dimensional $k_i^{\#}$-algebra k_i so that either
  - k_i is a field extension of k or
  - $k_i = k_i^{\#}[x]/(x^2 - \gamma^2)$, where $\gamma \in k_i^{\#}$ satisfies $k_i^{\#} = k[\gamma^2]$
- Let τ_i be the unique non-trivial $k_i^{\#}$-automorphism of k_i.

The Waldspurger Parameterization 2

For each algebra k_i, choose a_i and c_i in k_i so that:

- $\tau_i(a_i) = -a_i$ and $\tau_i(c_i) = -c_i$
- a_i generates k_i as a k-algebra
- for $i \neq j$, there does not exist a k-isomorphism from k_i to k_j sending a_i to a_j.

Then we let V be the k-algebra given by taking the direct sum of the k_i, and we define the symplectic form on V by

$$\langle v, v' \rangle = \sum_{i \in I} [k_i : k]^{-1} \mathrm{Trace}_{k_i/k}(\tau_i(v) v' c_i).$$

After fixing a suitable symplectic basis of V, a regular semisimple element for an unramified torus is given by the matrix of the k-endomorphism defined on each summand of V by multiplication by a_i.

The Waldspurger Parameterization 3

If $x_i = [k_i^\# : k]$, we place x_i in one of three partitions:

- μ_0 if k_i is not a field
- μ' if k_i is a field and c_i has even valuation in k_i
- μ'' if k_i is a field and c_i has odd valuation in k_i

Then the conjugacy classes of maximal unramified tori are parameterized by triples of partitions (μ_0, μ', μ'') so that $S(\mu_0) + S(\mu') + S(\mu'') = n$, where $S(\mu)$ is the sum of the terms in the partition.

Main Result

Given a triple of partitions (μ_0, μ', μ''), we need to find a facet F in our alcove and a Weyl group element w in W_F associated to the corresponding unramified torus in the DeBacker parameterization.
For the facet, we consider a facet associated to the following subdiagram of the extended Dynkin diagram:

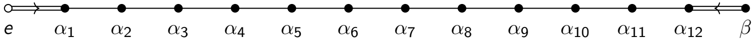
- We take a subdiagram of type $C_{S(\mu'')}$ containing the vertex associated to the highest root e.
- We take a subdiagram of type $C_{S(\mu')}$ containing the vertex associated to the long root β in the simple system associated to our alcove.
- For each $x_j \in \mu_0$ so that $x_j > 1$, we take a subdiagram of type $A_{x_j - 1}$. (Different choices give different facets in the equivalence class.)

For the Weyl group element, we take:

- An odd cycle of length x_j for each x_j in μ' or μ''.
- An even cycle of length x_m for each x_m in μ_0.

Example

Consider the triple $(\mu_0, \mu', \mu'') = (\{3,2,1\}, \{2,2,1\}, \{1,1\})$ associated to a torus in Sp_{26}.
Then the extended Dynkin diagram is



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A facet associated to this triple is

The Weyl group element decomposes as three odd cycles of length 1, two odd cycles of length 2, an even cycle of length 3, and an even cycle of length 2.

The Main Result for Sp_4