Coefficients of the local character expansion are motivic

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Harish-Chandra's local character expansion

- G – connected reductive group,
- $\mathfrak{g}$ – its Lie algebra,
- F – non-Archimedean local field (could be char. 0 or char. p),
- π – irreducible admissible representation of $G(F)$,
- θ_π – the character of π (thought of as a function on $G(F)^{\mathrm{reg}}$; will define).

Theorem

For Y close to 0 in $\mathfrak{g}$, (in finite characteristic, assume p large enough),

$$\theta_\pi(exp(Y)) = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(Y)$$

– an equality of functions defined on the set of regular semisimple elements. The sum is over nilpotent orbits.

Broad questions:

- What can be said about $c_{\mathcal{O}}$?
- To what extent are the both sides 'independent of F'?

Observation (M. Assem, Waldspurger) :

With the right choice of measures, the coefficients $c_{\mathcal{O}}$ are rational numbers, in many cases, e.g.:

- $c_{\{0\}}(\pi) = \frac{(-1)^{\ell}}{d(St)} d(\pi)$, where $d(\pi)$ is the formal degree (H-C, Rogawski);
- Can normalize measures so that $c_{\mathcal{O}}(\pi)$ is a natural number (if non-zero) for every $\mathcal{O}$ of maximal dimension, for every smooth irrep π. (Moeglin-Waldspurger)

Conjecture: The coefficients $c_{\mathcal{O}}$ are rational for any admissible representation of a connected reductive p-adic group. (Note: sufficient to prove for *supercuspidal* representations).

Two main goals:

- Prove this conjecture (using new information about a different local character expansion)
- Do it 'uniformly in p', namely, prove that $c_{\mathcal{O}}$ are 'motivic'.

Plan for the talk:

- Define 'motivic'
- Define characters and orbital integrals.
- Construct a lot of the data in the 'motivic' framework.
- Outline the proof.
- (if there's time) discuss general results about motivic functions.

Denef-Pas Language

Formulas are allowed to have variables of three sorts:

two functions for variables of the valued field sort

- F: valued field sort, $(+, \times, '0','1', \mathrm{ac}(\cdot), \mathrm{ord}(\cdot)\ );$
- $\mathbb{Z}$: value sort, $(+, '0', '1', \equiv_n, n \geq 1)$ (Presburger language);
- k: residue field sort, (language of rings: $+, \times, '0', '1'$)

ord

ac

Formulas are built from arithmetic operations, quantifiers, and symbols $\mathrm{ord}(\cdot)$ and $\mathrm{ac}(\cdot)$.

Example

$\phi(y) =$ '$\exists x, y = x^2$ $\wedge (y \neq 0)$', (here y is a *free* and x is a *bound* variable) or, equivalently,

$$\phi(y) = \text{'}\mathrm{ord}(y) \equiv 0 \mod 2 \wedge \exists \underset{\neq 0}{\xi} : \mathrm{ac}(y) = \xi^2\text{'}.$$

A subset of $F^m \times k^n \times \mathbb{Z}^r$ is called definable if it can be defined by '$\phi(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n, z_1, \ldots, z_r)$ is true', where ϕ is a formula in the language.

Definable sets and functions

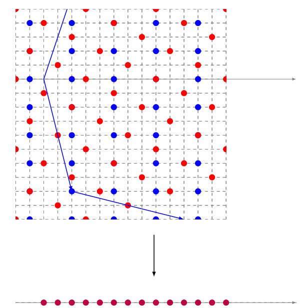
Main point: 'stratify' definable subsets of F^n into disjoint unions of families of (p-adic) balls whose centres are parametrized by a constructible subset of k^r, and whose radii are indexed by a definable subset of $\mathbb{Z}^m$.

Then integration over such sets would reduce to counting points on varieties over the finite residue field, and summation of geometric series.

Example

Definable subsets of $\mathbb{Z}^n$ in Presburger language:

- For $n = 1$: finite unions of points and (one-sided) arithmetic progressions
- generally: finite unions of periodic subsets cut off by hyperplanes, and singletons



projection of a union of two definable subsets of $\mathbb{Z}^2$ onto a def. subset of $\mathbb{Z}$

Specialization

A definable function is a function $f : F^a \times k^b \times \mathbb{Z}^c \to F^n \times k^m \times \mathbb{Z}^r$ whose graph is a definable set.

Example

Definable functions:

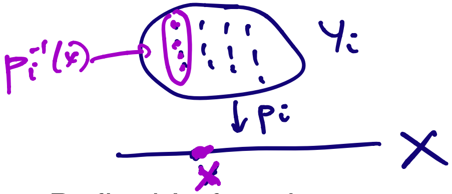
- Polynomials with $\mathbb{Z}$-coefficients on F^n;
- piecewise-linear functions from $\mathbb{Z}^n$ to $\mathbb{Z}$;
- $\mathrm{ord}(f(x_1, .., x_n))$, f - such a polynomial, is a definable function from F^n to $\mathbb{Z}$;
- Characteristic function of a definable set.

For a given field:

(F, ϖ) (ϖ - a uniformizer) gives a structure for Denef-Pas language: we allow the variables of the corresponding sorts to range over F and k, the residue field of F, respectively.

- The function $\mathrm{ord}(\cdot)$ specializes to the usual valuation;
- the function $\mathrm{ac}(\cdot)$ specializes to the first nonzero coefficient of the ϖ-adic expansion.
- Any formula in the language of rings with n free variables ranging over the residue field specializes to the number of the points in k^n that satisfy it.
- We introduce a special symbol $\mathbb{L}$ that specializes to $q = \#k$.

Because of specialization our results are only valid for large p!



Motivic functions

Definable functions are still far from what we need to have a chance of describing representation theory.
A motivic function on a definable subset X of F^n has the form:

$$f(x) = \sum_{i=1}^{N} \underbrace{\mathbb{L}^{\alpha_i(x)}}_{q^{\alpha_i(x)}} \underbrace{[p_i^{-1}(x)]}_{\# p_i^{-1}(x)} \underbrace{\Big(\prod_{j=1}^{N'} \beta_{ij}(x)\Big)}_{= \prod \beta_{ij}(x)} \underbrace{\Big(\prod_{\ell=1}^{N''} \frac{1}{1-\mathbb{L}^{a_{i\ell}}}\Big)}_{\prod \frac{1}{1-q^{a_i}}},$$

specialization

where:

- $a_{i\ell}$ are negative integers;
- $\alpha_i : X \to \mathbb{Z}$ with $i = 1, \dots N$, and $\beta_{ij} : X \to \mathbb{Z}$ with $i = 1 \dots, N$, $j = 1, \dots, N'$ are $\mathbb{Z}$-valued definable functions;
- Y_i are definable sets such that $Y_i \subset k^{r_i} \times X$ for some $r_i \in \mathbb{Z}$, and $p_i : Y_i \to X$ is the coordinate projection.

Motivic functions take values in $\mathbb{Z}\left[\mathbb{L}, \mathbb{L}^{-1}, (1-\mathbb{L}^{-i})^{-1}, i \geq 1\right]$.

Example

Motivic constant: a motivic function on a single point; has the form $\beta\mathbb{L}^{\alpha}\prod_{i=1}^{N}(1-q^{a_i})^{-1}[Y]$, where $a_i, \beta, \alpha \in \mathbb{Z}$; Y - a constructible set over the residue field. Given F, specializes to $\beta q^{\alpha}\prod_{i=1}^{N}(1-q^{a_i})^{-1}\#Y(k)$.
(actually, as above, a finite sum of such terms).

Integration

The point: we want a class of functions on F^n that contains characteristic functions of definable sets, and functions of the form $|f(x_1,..x_n)|$, where f is a polynomial. The other terms come from the need to make this class of functions stable under integration.

Theorem

(R. Cluckers and F. Loeser + I. Halupczok + J.G.)
The class of motivic functions is closed under integration:

$$\int_{F^m} f(x_1,\dots,x_m,\overline{y},\overline{z})\,dx_1\dots dx_k = h(x_{k+1},\dots,x_m,\overline{y},\overline{z})$$

whenever $f(\overline{x},\overline{y},\overline{z})\in L^1(dx_1\dots dx_k)$, where h is again a motivic function.

Motivic exponential functions

There is a way to define "motivic exponential functions" that specialize to functions of the form

$$F(x) = \sum_i a_i(x)\psi(h_i(x)),$$

where a_i are motivic functions and h_i are definable (with values in the valued field).

To get a specialization of a motivic exponential function, we need to specify F, ϖ, and a level zero additive character ψ of F. (hence, the name "exponential").

The class of motivic exponential functions is preserved under (motivic) integration, and Fourier transform, as well as under taking pointwise limits, and L^p-limits.

Characters and orbital integrals

- $\pi : G(F) \to \mathrm{End}(V)$ – an irreducible *admissible* representation; V is a $\mathbb{C}$-vector space (usually, infinite-dimensional).
- The distribution character is represented on G^{rss} by a locally constant, locally integrable function; that is, for $f \in C_c^\infty(G)$:

$$\Theta_\pi(f) := \mathrm{Tr} \int_G f(g)\pi(g)\, dg = \int_G \theta_\pi(g) f(g)\, dg.$$

  where θ_π is locally constant on G^{rss} (and can be extended by zero outside).
- Orbital integral:

$$\mu_X(f) = \int_{G/C_G(X)} f(gXg^{-1}) d^*g,$$

  where $C_G(X) = \{g \in G \mid gXg^{-1} = X\}$.
  If F has positive characteristic, we have to assume it is sufficiently large so that the orbital integral is a distribution on $C_c^\infty(\mathfrak{g})$.
  Normalization of d^*g: for every reductive group M that arises as a centralizer (i.e. $C_G(X) = M(F)$ for some $X \in \mathfrak{g}(F)$), fix the canonical measure dm on M (B. Gross). Then we get the G-invariant measure $d^*g := dg/dm$ on the adjoint orbit of X. This gives a way to normalize the measures on the orbits uniformly, and these measures are 'rational-valued'. (There are finitely may possibilities for the root datum of M^0).

Fourier transform; exponential map

- For $f \in C_c^\infty(\mathfrak{g})$,
$$\hat{f}(X) := \int_{\mathfrak{g}(F)} f(Y)\psi(\langle X, Y\rangle)dY,$$
where $\langle , \rangle$ is a G-invariant non-degenerate bilinear form on $\mathfrak{g}$.
- for μ – a distribution on $C_c^\infty(\mathfrak{g})$, $\hat{\mu}(f) := \mu(\hat{f})$.
- $\hat{\mu}_X$ is represented by a locally constant, locally integrable function $\hat{\mu}_X$.
- "exp : $\mathfrak{g}(F) \to G(F)$", a $G(F)$-equivariant map; will need it to preserve Moy-Prasad filtrations and be definable. Its existence is a mild extra hypothesis.
- There is a finite list of nilpotent orbits $\mathcal{N}$ (if finite characteristic p, assume p is large).

H.-C. local character expansion

Theorem

For every admissible representation π there exist constants $c_{\mathcal{O}}$, $\mathcal{O} \in \mathcal{N}$, such that near the origin in $\mathfrak{g}(F)$,

$$\theta_\pi(\exp Y) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}}(Y).$$

(Harish-Chandra and Howe in characteristic zero; Waldspurger; DeBacker in large positive characteristic, with explicit neighbourhood of the identity $\mathfrak{g}_{r+}$, depending on the depth r of π).

We only consider *supercuspidal* representations, and put two depth r representations in the same equivalence class if their characters coincide on $\mathfrak{g}_{r+}$.
Intermediate goal: Prove that this is an equality of motivic exponential functions.
However, just that would not be enough for rationality of $c_{\mathcal{O}}$, because of the character ψ. We will use a similar equality of *distributions* with some carefully chosen test functions to prove that $c_{\mathcal{O}}$ are motivic constants.

The parametrizing data

First, we need to define what we mean by θ_π in a field-independent way.

Fix r. First, $(\mathbf{G}, \mathfrak{g}, \mathfrak{g}_{r+}, \mathcal{N})$ are constructed as fibres in a definable family:

- *fixed choices*: Root system, Galois group of an extension that splits G, *alcove*, ...
- A definable set Z of cocycles defining G, $\mathfrak{g}$;
- $\mathcal{N}$, $\mathfrak{g}_{r+}$ can be constructed as fibres in a definable family over Z. (T. Hales, JG).

Every group appears infinitely many times (do not identify isomorphic fibres), and there are some "garbage fibres". The orbital integrals and their Fourier transforms are motivic (exponential) functions on this definable set.

Next, need the data to parametrize our 'equivalence classes' of depth-r representations (i.e. restrictions of their characters to $\mathfrak{g}_{r+}$).

J.-K. Yu's construction

We assume that p is large.

Supercuspidal representations π of G_z are parametrized by J.-K. Yu's data $(\vec{G}, \vec{\phi}, \pi_0, x)$, where $\vec{G} = (G^0, \dots G^d)$ is a sequence of twisted Levi subgroups with $G^d = G$, ϕ_i is a character of G^i, π_0 is a depth-zero supercuspidal representation of G^0, and x is a point in the reduced building of G.

The point x can be treated as a *fixed choice*.

The groups G^i are encoded by appropriate cocycle spaces Z^i.

The representation π_0 is a depth-zero representation of G^0; it is obtained from a representation ρ_0 of a group G^0 over the residue field (next slide).

The things in red are not part of motivic framework; but we (intend to) prove that they do not contribute anything non-motivic near the identity.

Depth-zero representations:

$$
\begin{tikzcd}
G^0 & \pi_0 \\
K^0 \arrow[d, "\text{mod } \varpi"'] & \tilde{\rho}_0 \arrow[u, "\text{induction}"'] \\
\mathrm{G}^0(k) & \color{red}{\rho_0} \arrow[u, "\text{inflation}"']
\end{tikzcd}
$$

Only need the character or ρ_0 restricted to *unipotent* elements, and that is ‘motivic’ (comes from Lusztig).

Assem's proof of rationality

Recall the equality of distributions:

$$\Theta_\pi(f) = \sum_{\mathcal{O}} c_{\mathcal{O}} \mu_{\mathcal{O}}(\hat{f}).$$

1. Assem proves that orbital integrals are rational for rational-valued test functions, using Igusa theory.
2. finds a collection of test functions such that with the ordering of nilpotent orbits by dimension, the matrix of $c_{\mathcal{O}}$'s is upper-triangular, and
3. It is somehow also known that $\Theta_\pi(f)$ is rational for these test functions.

(3) works for depth-zero π (the test functions are Gelfand-Graev characters: Barbasch-Moy), and for the π where Murnaghan-Kirillov theory gives $\theta_\pi = d_\pi \hat{\mu}_X$.

Kim-Murnaghan character expansion

J.-L. Kim and F. Murnaghan associate to the G-datum $(\vec{G}, x_0, \vec{\phi})$ an element Γ in $\mathfrak{g}$ and prove that, if $\mathcal{O}^G(\Gamma)$ is the set of adjoint orbits of G on $\mathfrak{g}$ whose closures contain Γ then there is an indexed family $(c_{\mathcal{O}}(\pi))_{\mathcal{O}\in\mathcal{O}^G(\Gamma)}$ of complex numbers such that

$$\theta_\pi(\exp(Y)) = \sum_{\mathcal{O}\in\mathcal{O}^G(\Gamma)} c_{\mathcal{O}}(\pi)\hat{\mu}^G_{\mathcal{O}}(Y)$$

for $Y \in \mathfrak{g}_{(r/2)+}$, where r is the depth of π.

Theorem (L. Spice, in progress)

$$\theta_{\pi_0}(\exp(Y)) = \sum_{\mathcal{O}_0\in\mathcal{O}^{G^0}(0)} c_{Ad(G)(\Gamma+\mathcal{O}^0)}(\pi)\hat{\mu}^{G^0}_{\mathcal{O}^0}(Y)$$

for $Y \in \mathfrak{g}^0_{0+}$.

That is, the coefficients in the local character expansion of π_0 are the same as the corresponding coefficients in the asymptotic character expansion of π. Since $C_G(\Gamma) = G^0$ the 'fattening map' $\mathcal{O}^{G^0}(0) \to \mathcal{O}^G(\Gamma)$ that sends $\mathcal{O}^0$ to $Ad(G)(\Gamma + \mathcal{O}^0)$ is surjective, so that the asymptotic character expansion of π contains precisely the same information as the local character expansion of π^0.

Putting it together

To prove that $c_{\mathcal{O}}$ are motivic constants (in particular, rational):

- Prove that the Fourier transforms of Gelfand-Graev characters are 'motivic'.
- Conclude that in the depth-zero case, $c_{\mathcal{O}}$'s are a collection of ratios of motivic functions of all the parameters, as in Assem's proof.
- Use L. Spice's theorem to reduce the general case to depth zero.

Remark. We never construct representations in the 'motivic' world (this cannot be done without multiplicative characters of the field, as a minimum); but from the fixed choices and the parameter spaces attached to the fixed choices, we construct *both* the (Fourier transforms of) nilpotent orbital integrals, and the orbital integrals that appear on the left-hand-side via Murnaghan-Kirillov theory. Then we just use the fact that orbital integrals are 'motivic distributions'.

Some more model theory

Lemma (Cluckers-Halupczok)

Consider $f(x) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ an 'exponential polynomial' – a function of the form

$$f(x) = \sum_{i=1}^{r} c_i x^{a_i} b_i^x,$$

where $b_i > 0$ are distinct, $a_i, c_i \in \mathbb{R}$ (for simplicity we take $a_i \in \mathbb{Z}$). Then the number of zeroes of f is bounded by a constant that depends only on r.

This Lemma follows from a deep theorem due to Alex Wilkie.

Now, a motivic function on $\mathbb{Z}$ has the form:

$$f(x) = \sum_{i=1}^{N} c_i \left(\prod_{j=1}^{N_j} \alpha_{ij}(x) \right) p^{\beta_i(x)},$$

where, as before,

- $\alpha_{ij} : \mathbb{Z} \to \mathbb{Z}$, $\beta_i : \mathbb{Z} \to \mathbb{Z}$ are definable functions;
- $c_i \in \mathbb{R}$ are constants.

(The domain could be $\mathbb{Z}^n$ but we do not need it now).

Key point: By the Lemma the number of zeroes of such a function (if it is not identically zero) is bounded independently of everything, including p.

Properties of motivic functions: uniformity in p

Theorem (Cluckers-Halupczok)

Let $F(x, y)$ be a motivic function on $X \times Y$. Then $\{x : F(x, y) is\_\_\_$ as a function of $y\}$ is the zero locus of some motivic function on X.

Here ___ can be: 'integrable', 'bounded', 'in L^2', 'approaching 0', etc.

In particular, as we apply it to a family $\{f_n\}$ $(n \in \mathbb{Z})$ of motivic functions (say, on $G(\mathbb{Q}_p)$), we get a motivic function on $\mathbb{Z}$, and the "key point" applies. It follows that, for example:

- suppose we know that $f_n(x) \to 0$ as $x \to 0$ in all $\mathbb{Q}_p$. Then given ϵ, the N you need does not depend on p.
- (Uniformity of asymptotic expansions) Suppose H_1, H_2 are motivic (exponential) functions, say, on an affine space, and for every p-adic field F, $(H_1)_F = (H_2)_F$ on a neighbourhood of 0. Then this neighbourhood is of the same radius for all F of sufficiently large residue characteristic, and this statement can be transferred to F of positive characteristic. In particular, this applies to Shalika germ expansions. (Not new for the local character expansion, of course).

Further Properties: uniformity of bounds

Some examples:

- If a motivic (exponential) function H is bounded for every value of a parameter $\lambda \in \mathbb{Z}^n$, then it is bounded by the expression of the form $q^{a+b\|\lambda\|}$.
- In particular, one gets uniform bounds for orbital integrals on the spherical Hecke algebra (and now we could get them for characters, as well).
- (Harish-Chandra, Herb). Let ω be a compact subset of $\mathfrak{g}(F)$. Then

$$\sup_{\omega} |D(X)|^{1/2}|D(Y)|^{1/2}|\hat{\mu}_X(Y)| < \infty.$$

- It follows from (1): If ω_m is a family of definable compact sets, then

$$\sup_{\omega_m} |D(X)|^{1/2}|D(Y)|^{1/2}|\hat{\mu}_X(Y)| \leq q_F^{a+b\|m\|}.$$

Stronger bounds for characters are known by other methods for large p (J.-L. Kim, E. Lapid, ...), but maybe not for completely general families of test functions.