On the geometric connected components of unramified local Shimura varieties (Work in progress)

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Local Shimura datum

- $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{un}}$. σ lift of Frobenious. $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$.
- Formalism conjectured by Rapoport-Viehman. Materialized by Scholze-Weinstein.
- p-adic shtuka datum is a triple: (G, b, μ).
- $G/\mathbb{Q}_p$ reductive group.
- $b \in G(\breve{\mathbb{Q}}_p)$.
- $\mu \in \{\mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}\}/G$ conjugacy class of cocharacters.
- If μ is minuscule $\implies$ (G, b, μ) is local Shimura datum.
- In analogy with Shimura datum (G, ν).
- Just as $\nu \rightsquigarrow$ Hodge strcture. $(b, \mu) \rightsquigarrow$ "p-adic Hodge structure".
- $b \rightsquigarrow$ "isocrystal". $\mu \rightsquigarrow$ "filtration".

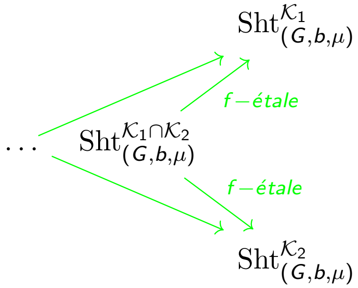
Some gadgets

We can associate:

1. $E = E(\mu)$ reflex field (of conj. class).
   - Consider the Weil group W_E.
2. Reductive group J_b.
   - σ-centralizer of b.
   - $J_b(\mathbb{Q}_p) = \{g \in G(\breve{\mathbb{Q}}_p) \mid g^{-1} b \sigma(g) = b\}$
   - Inner form of a Levi subgroup.
   - $J_b(\mathbb{Q}_p)$ is "Automorphism group of isocrystal".
3. A p-adic period domain $\mathscr{F}\ell_\mu / \mathbb{C}_p$.
   - Some geometric object of p-adic analytic geometry. (Scholze's diamonds).
   - If μ is minuscule $\mathscr{F}\ell_\mu = G/P_\mu$. In particular, it is a rigid-analytic space.
   - Open subset $\mathscr{F}\ell_\mu^b \subseteq \mathscr{F}\ell_\mu$. Called admissible locus.
   - Commes equipped with $J_b(\mathbb{Q}_p) \times W_E$-action.

Local Shimura Varieties/moduli of p-adic shtukas.

- Tower parametrized by compact open subgroups $\mathcal{K} \subseteq G(\mathbb{Q}_p)$:
- $J_b(\mathbb{Q}_p) \times W_E$-equivariant tower of analytic spaces:



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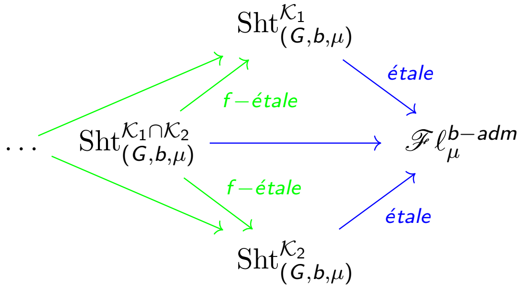

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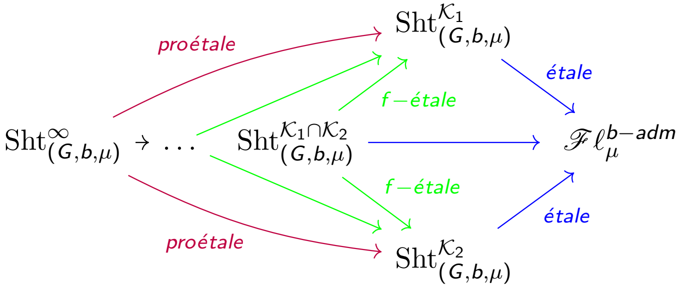


- $\mathrm{Sht}^{\infty}_{(G,b,\mu)} \to \mathscr{F}\ell^{b-adm}_{\mu}$. A $G(\mathbb{Q}_p)$ proétale Galois cover.
- $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ acts on $\mathrm{Sht}^{\infty}_{(G,b,\mu)}$. Geometric incarnation of LLC and JLC. (Kottwitz conjecture).

Example

- If $G = GL_n$.
- $b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 1 \\ p & 0 & 0 & \dots & 0 \end{pmatrix}$, $\mu = (1, 0, \dots, 0)$,
- Then $\mathrm{Sht}_{(GL_n, b, \mu)}$ is the Lubin-Tate tower. Used on Harris-Taylor's proof of LLC.
- $\mathscr{F}\ell^b_\mu = \mathbb{P}^{n-1}$.
- $\mathrm{Sht}^{GL_n(\mathbb{Z}_p)}_{(GL_n, b, \mu)} = \mathcal{M}_\eta$ it is the Raynaud generic fiber of a formal scheme.
- $\mathcal{M}$ is deformation space of 1-dimensional formal p-divisible group of height n.

Main Goal

- Describe connected components. $\pi_0(\mathrm{Sht}^{\infty}_{(G,b,\mu)}) \Leftrightarrow \pi_0(\mathit{Tower})$.
- Understand $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$-action.
- Consequences:
  1. Explicit computation of $H^0(\mathrm{Sht}_{(G,b,\mu)})$.
  2. $H^0(\mathrm{Sht}_{(G,b,\mu)})$ acts on rest of cohomology through "cup product". Allows twists by characters.
  3. Expresses geometrically: LLC and JLC are compatible with character twists.

Case of tori

If G is a torus:

- $\mathscr{F}\ell_\mu = \{*\}$.
- $\mathrm{Sht}^{\infty}_{(G,b,\mu)} \cong G(\mathbb{Q}_p) \times \{*\}$. Free $G(\mathbb{Q}_p)$-torsor.
- $J_b = G$ canonically. Take $j \in J_b(\mathbb{Q}_p)$ and $\gamma \in W_E$ then:

$$j \cdot_{J_b} y = j^{-1} \cdot_G y$$

$$\gamma \cdot_{W_E} y = [Nm \circ \mu \circ Art_E(\gamma)] \cdot_{G(\mathbb{Q}_p)} y$$

$$W_E \xrightarrow{Art_E} E^\times \xrightarrow{\mu} G(E) \xrightarrow{Nm} G(\mathbb{Q}_p)$$

Main Result

Theorem (G.)

Assume G unramified, (b,μ) HN-irreducible and G^{der} is simply connected, then $det : (G,b,\mu) \to (G^{ab}, det(b), det(\mu))$ gives

$$\det : \mathrm{Sht}^{\infty}_{(G,b,\mu)} \to \mathrm{Sht}^{\infty}_{(G^{ab},det(b),det(\mu))}$$

and induces a $G(\mathbb{Q}_p) \times \mathrm{J}_b(\mathbb{Q}_p) \times W_E$-equivariant bijection

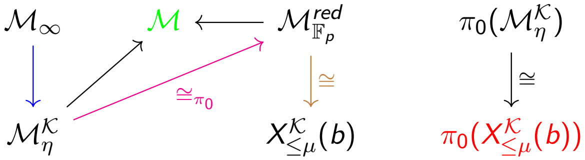
$$\pi_0(\det) : \pi_0(\mathrm{Sht}_{(G,b,\mu)}) \xrightarrow{=} \pi_0(\mathrm{Sht}_{(G^{ab},det(b),det(\mu))})$$

- Canonical map $G(\mathbb{Q}_p) \times \mathrm{J}_b(\mathbb{Q}_p) \times W_E \to G^{ab}(\mathbb{Q}_p) \times G^{ab}(\mathbb{Q}_p) \times W_E$.

Remarks

1. De Jong (1994): Lubin-Tate case over $\mathbb{Q}_p$.
2. Strauch(2006): Lubin-Tate case over arbitrary local field (including ramification).
3. M. Chen(2012): Computation for tori. Introduced determinant map.
4. M. Chen(2014): Unramified Rapoport-Zink space of EL or PEL type.
5. New for local Shimura varieties associated to exceptional reductive groups. We handle μ non-minuscule. We handle G^{der} non-simply connected.
6. Central strategy is the same, but techniques and difficulties are very different.

Road map for de Jong + M. Chen



Blue From infinite to hyperspecial level.

- Group theoretic techniques.
- M. Chen's work on "generic" crystalline representations.

Green Theory of Rapoport-Zink spaces.

Magenta Formal smoothness of unramified RZ spaces.

Brown Dieudonne Theory

Red Computation of connected components of minuscule affine Deligne Lusztig varieties . (Chen-Kisin-Viehmann).

Integral models

- Main difficulty: What are integral models for diamond? We make an attack to this question.
- Solve analogy:

$$\textit{Rigid analytic spaces} \rightsquigarrow \textit{Diamonds}$$
$$\textit{Formal schemes} \rightsquigarrow ?$$

- First approximation to "?": Kimberlites

Road map for us

$$
\begin{array}{ccccccc}
\mathrm{Sht}^{\infty}_{(G,b,\mu)} & & \mathrm{Sht}^{\mathcal{K}}_{(\mathcal{G},b,\mu)} & \longleftarrow & (\mathrm{Sht}^{\mathcal{K}}_{(\mathcal{G},b,\mu)})^{\mathrm{red}} & & \pi_0(\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}) \\
\downarrow & \nearrow & & \nearrow_{\cong_{\pi_0}} & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{Sp}} \\
\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)} & & & & X^{\mathcal{K}}_{\leq\mu}(b) & & \pi_0(X^{\mathcal{K}}_{\leq\mu}(b))
\end{array}
$$

Blue: Adapt Chen's strategy to diamonds.

Green: Scholze-Weinstein propose a model.

Magenta: Specialization maps for kimberlites. Prove SW are kimberlites.

Brown: Reduction of kimberlites.

Red: Computation of connected components of general ADLV. (Chen-Kisin-Viehmann, He-Zhou, Nie).

Main technical theorem

Theorem (G.)

There is a continuous specialization map
$\mathrm{Sp} : |\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}| \to |X^{\mathcal{K}}_{\leq\mu}(b)|$, *it satisfies the following properties:*

- *Continuous and a quotient map.*
- $\mathrm{J}_b(\mathbb{Q}_p)$*-equivariant.*
- *Induces a bijection* $\pi_0(\mathrm{Sp}) : \pi_0(\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}) \cong \pi_0(X^{\mathcal{K}}_{\leq\mu}(b))$.

This is the end

Thanks!!!