Unitary Representations of GL_n and the geometry of Langlands parameters

Adam Brown[†]

[†]Institute of Science and Technology Austria

[†] *abrown@ist.ac.at*

Outline

$G = GL_n$

Outline

$G = GL_n$

$\mathcal{K}\mathrm{Rep}G(k)$

Outline

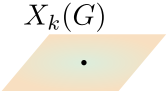
$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$$\mathcal{K}\mathrm{Rep}G(k) \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G)$$

Outline

$G = GL_n \quad {}^\vee G = G(\mathbb{C})$

$\mathcal{K}\mathrm{Rep}G(k) \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G)$

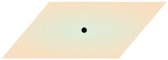


Outline

$G = GL_n \quad {}^\vee G = G(\mathbb{C})$

$X_k(G)$

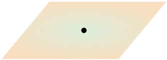
$\mathcal{K}\mathrm{Rep}G(k) \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \quad {}^\vee G \circlearrowright$



Outline

$G = GL_n \quad {}^\vee G = G(\mathbb{C})$

$X_k(G)$

$\mathcal{K}\mathrm{Rep}G(k) \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \quad {}^\vee G \circlearrowright$ 

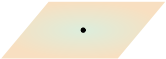
$\mathcal{K}\mathrm{Rep}G(k')$

Outline

$$G = GL_n \qquad {}^{\vee}G = G(\mathbb{C})$$

$$\mathcal{K}\mathrm{Rep}G(k) \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_k(G), {}^{\vee}G) \qquad {}^{\vee}G \circlearrowright$$

$X_k(G)$



$$\mathcal{K}\mathrm{Rep}G(k') \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^{\vee}G)$$

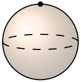
Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$X_k(G)$

$$\mathcal{K}\mathrm{Rep}G(k) \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \qquad {}^\vee G \circlearrowright$$

$$\mathcal{K}\mathrm{Rep}G(k') \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G) \qquad {}^\vee G \circlearrowright$$



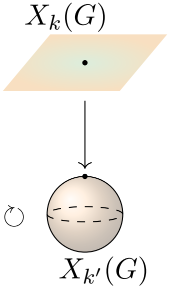
$X_{k'}(G)$

Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$$\mathcal{K}\mathrm{Rep}G(k) \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \qquad {}^\vee G \circlearrowright$$

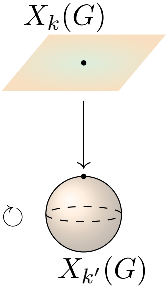
$$\mathcal{K}\mathrm{Rep}G(k') \xleftrightarrow{\text{LLC+}} \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G) \qquad {}^\vee G \circlearrowright$$



Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$$\begin{tikzcd}
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\mathrm{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] & {}^\vee G \circlearrowright \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[r, leftrightarrow, "\mathrm{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G) & {}^\vee G \circlearrowright
\end{tikzcd}$$



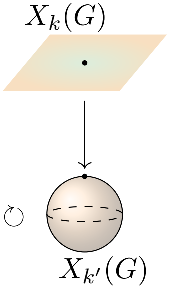
Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$$\begin{tikzcd}
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u] \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G)
\end{tikzcd}$$

${}^\vee G \circlearrowright$

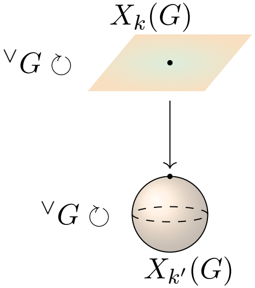
${}^\vee G \circlearrowright$



Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$$\begin{tikzcd}
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u] \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G)
\end{tikzcd}$$



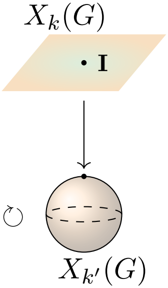
$$F : \mathrm{Rep}G(k') \to \mathrm{Rep}G(k)$$

Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$$\begin{tikzcd}
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u] \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G)
\end{tikzcd}$$

${}^\vee G \circlearrowright$



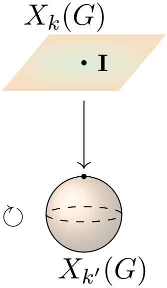
${}^\vee G \circlearrowright$

$$F : \mathrm{Rep}G(k') \to \mathrm{Rep}G(k)$$

Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

$$\begin{tikzcd}
 & \mathbf{II} & \\
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] & {}^\vee G \circlearrowright \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u] \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G) & {}^\vee G \circlearrowright
\end{tikzcd}$$



$$F : \mathrm{Rep}G(k') \to \mathrm{Rep}G(k)$$

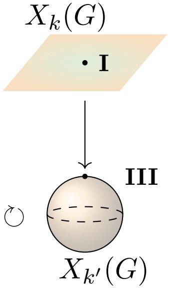
Outline

$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$

$$\begin{tikzcd}
 & \mathbf{II} \\
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u] \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G)
\end{tikzcd}$$

${}^\vee G \circlearrowright$

${}^\vee G \circlearrowright$

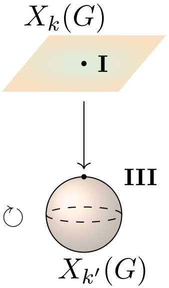


$F : \mathrm{Rep}G(k') \to \mathrm{Rep}G(k)$

Outline

$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$

$$\begin{tikzcd}
 & \textbf{II} & \\[-2em]
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] & {}^\vee G \circlearrowright \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u] \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G) & {}^\vee G \circlearrowright \\[-2em]
 & \textbf{IV} &
\end{tikzcd}$$



$F : \mathrm{Rep}G(k') \to \mathrm{Rep}G(k)$

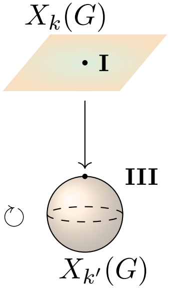
Outline

$G = GL_n \quad {}^\vee G = G(\mathbb{C})$

$$\begin{tikzcd}
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u, "\mathbf{V}"] \arrow[r, leftrightarrow, "\text{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G)
\end{tikzcd}$$

II   **IV**

${}^\vee G \circlearrowright$



${}^\vee G \circlearrowright$

$F : \mathrm{Rep}G(k') \to \mathrm{Rep}G(k)$

Outline

$$G = GL_n \qquad {}^\vee G = G(\mathbb{C})$$

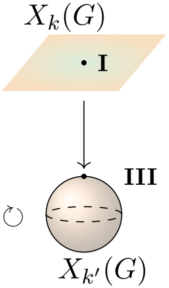
II

$$\begin{tikzcd}
\mathcal{K}\mathrm{Rep}G(k) \arrow[r, leftrightarrow, "\mathrm{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G) \arrow[d] \\
\mathcal{K}\mathrm{Rep}G(k') \arrow[u, "\mathbf{V}"] \arrow[r, leftrightarrow, "\mathrm{LLC+}"] & \mathcal{K}\,\mathrm{Con}(X_{k'}(G), {}^\vee G)
\end{tikzcd}$$

IV

${}^\vee G \circlearrowright$

${}^\vee G \circlearrowright$



$$F : \mathrm{Rep}G(k') \to \mathrm{Rep}G(k)$$

VI

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$

$$\sigma \in \mathfrak{h}$$

$$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$$

$$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$$

$$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$$

Example:

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T = \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$

$$\sigma \in \mathfrak{h}$$

$$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$$

$$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$$

$$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$$

Example:

$$\sigma = \rho^\vee$$
$$= \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T$$
$$= \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$

$\sigma \in \mathfrak{h}$

$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$

$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$

$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$

Example:

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T = \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$$

$$\sigma \in \mathfrak{h}$$

$$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$$

$$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$$

$$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$$

Example:

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T = \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$$

$$\sigma \in \mathfrak{h}$$

$$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$$

$$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$$

$$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$$

Example:

$$\sigma = \rho^\vee$$
$$= \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & & \\ & 0 & * & \ddots & & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T$$
$$= \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$

$\sigma \in \mathfrak{h}$

$$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$$

$$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$$

$$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$$

Example:

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T = \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$$

$$\sigma \in \mathfrak{h}$$

$$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$$

$$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$$

$$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$$

Example:

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T = \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$

$\sigma \in \mathfrak{h}$

$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$

$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$

$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$

Example:

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T = \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

I. Geometry of Langlands Parameters

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{h} = \{\text{diagonal matrices}\} \subset \mathfrak{g}$

$\sigma \in \mathfrak{h}$

$$\mathfrak{g}_1(\sigma) := \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)(x) = x\}$$

$$L(\sigma) := \mathrm{Stab}_{GL_n(\mathbb{C})}(\sigma)$$

$$\implies L(\sigma) \circlearrowright \mathfrak{g}_1(\sigma)$$

Example:

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3, \cdots, -n+3, -n+1)$$

$$\mathfrak{g}_1(\sigma) = \begin{bmatrix} 0 & * & 0 & & \\ & 0 & * & \ddots & \\ & & \ddots & * & 0 \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

$$L(\sigma) = T = \{M \in GL_n(\mathbb{C}) : M \text{ diagonal}\}$$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$$\mathrm{per}(\xi) \qquad\qquad \mathrm{con}(\xi)$$

$$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{per}(\xi)]\} \qquad \mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{con}(\xi)]\}$$

$$\begin{aligned} \chi : \mathcal{K}\,\mathrm{Per}(Y,H) &\xrightarrow{\ \sim\ } \mathcal{K}\,\mathrm{Con}(Y,H) \\ P &\longrightarrow \sum(-1)^i H^i P \end{aligned}$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$$\mathrm{per}(\xi) \qquad\qquad \mathrm{con}(\xi)$$

$$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span}\ \{[\mathrm{per}(\xi)]\} \qquad \mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span}\ \{[\mathrm{con}(\xi)]\}$$

$$\begin{aligned} \chi : \mathcal{K}\,\mathrm{Per}(Y,H) &\xrightarrow{\ \sim\ } \mathcal{K}\,\mathrm{Con}(Y,H) \\ P \quad &\longrightarrow \quad \textstyle\sum (-1)^i H^i P \end{aligned}$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$\mathrm{per}(\xi)$ $\qquad$ $\mathrm{con}(\xi)$

$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span}\ \{[\mathrm{per}(\xi)]\}$ $\qquad$ $\mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span}\ \{[\mathrm{con}(\xi)]\}$

$$\begin{array}{rcl} \chi : \mathcal{K}\,\mathrm{Per}(Y,H) & \xrightarrow{\ \sim\ } & \mathcal{K}\,\mathrm{Con}(Y,H) \\ P & \longrightarrow & \sum (-1)^i H^i P \end{array}$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$$\mathrm{per}(\xi) \qquad\qquad \mathrm{con}(\xi)$$

$$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{per}(\xi)]\} \qquad \mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{con}(\xi)]\}$$

$$\begin{aligned} \chi : \mathcal{K}\,\mathrm{Per}(Y,H) &\xrightarrow{\sim} \mathcal{K}\,\mathrm{Con}(Y,H) \\ P &\longrightarrow \sum(-1)^i H^i P \end{aligned}$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$\mathrm{per}(\xi)$ $\qquad\qquad$ $\mathrm{con}(\xi)$

$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span}\ \{[\mathrm{per}(\xi)]\}$ $\qquad$ $\mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span}\ \{[\mathrm{con}(\xi)]\}$

$$\chi : \mathcal{K}\,\mathrm{Per}(Y,H) \xrightarrow{\sim} \mathcal{K}\,\mathrm{Con}(Y,H)$$
$$P \longrightarrow \sum(-1)^i H^i P$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$$\mathrm{per}(\xi) \qquad\qquad \mathrm{con}(\xi)$$

$$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span}\,\{[\mathrm{per}(\xi)]\} \qquad \mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span}\,\{[\mathrm{con}(\xi)]\}$$

$$\chi : \mathcal{K}\,\mathrm{Per}(Y,H) \xrightarrow{\sim} \mathcal{K}\,\mathrm{Con}(Y,H)$$
$$P \longrightarrow \sum(-1)^i H^i P$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$\operatorname{per}(\xi)$ $\operatorname{con}(\xi)$

$\mathcal{K}\operatorname{Per}(Y,H) = \mathbb{Z}\text{-span }\{[\operatorname{per}(\xi)]\}$ $\mathcal{K}\operatorname{Con}(Y,H) = \mathbb{Z}\text{-span }\{[\operatorname{con}(\xi)]\}$

$$\chi : \mathcal{K}\operatorname{Per}(Y,H) \xrightarrow{\sim} \mathcal{K}\operatorname{Con}(Y,H)$$

$$P \longrightarrow \sum(-1)^i H^i P$$

but $\chi([\operatorname{per}(\xi)]) \neq [\operatorname{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$$\mathrm{per}(\xi) \qquad\qquad \mathrm{con}(\xi)$$

$$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span } \{[\mathrm{per}(\xi)]\} \qquad \mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span } \{[\mathrm{con}(\xi)]\}$$

$$\begin{aligned} \chi : \mathcal{K}\,\mathrm{Per}(Y,H) &\xrightarrow{\ \sim\ } \mathcal{K}\,\mathrm{Con}(Y,H) \\ P &\longrightarrow \sum (-1)^i H^i P \end{aligned}$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$\mathrm{per}(\xi)$ $\qquad$ $\mathrm{con}(\xi)$

$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{per}(\xi)]\}$ $\qquad$ $\mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{con}(\xi)]\}$

$$\begin{aligned} \chi : \mathcal{K}\,\mathrm{Per}(Y,H) &\xrightarrow{\sim} \mathcal{K}\,\mathrm{Con}(Y,H) \\ P &\longrightarrow \textstyle\sum (-1)^i H^i P \end{aligned}$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

Let H be a complex algebraic group

which acts on a complex algebraic variety Y with finitely many orbits.

$\xi = (\mathcal{O}, \mathcal{L})$: $\mathcal{O}$ an H-orbit on Y, $\mathcal{L}$ an irr. H-equiv. local system on $\mathcal{O}$

$\mathrm{per}(\xi)$ $\qquad\qquad$ $\mathrm{con}(\xi)$

$\mathcal{K}\,\mathrm{Per}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{per}(\xi)]\}$ $\qquad$ $\mathcal{K}\,\mathrm{Con}(Y,H) = \mathbb{Z}\text{-span }\{[\mathrm{con}(\xi)]\}$

$$\begin{aligned} \chi : \mathcal{K}\,\mathrm{Per}(Y,H) &\xrightarrow{\ \sim\ } \mathcal{K}\,\mathrm{Con}(Y,H) \\ P &\longrightarrow \sum (-1)^i H^i P \end{aligned}$$

but $\chi([\mathrm{per}(\xi)]) \neq [\mathrm{con}(\xi)]$

II. Variations on a geometric theme

$$\chi([\mathrm{per}(\xi)]) = \sum_{\zeta} c(\xi, \zeta)[\mathrm{con}(\zeta)]$$

II. Back to Langlands Parameters

Let $\mathcal{L}$ be an irreducible[1] $L(\sigma)$-equivariant local system on the $L(\sigma)$-orbit $\mathcal{O} \subset \mathfrak{g}_1(\sigma)$.

$$\{\xi = (\mathcal{O}, \mathcal{L})\} \longleftrightarrow \{\mathcal{O}\}$$

$$c(\mathcal{Q}, \mathcal{O}) = \sum (-1)^j \dim IH^j(\overline{\mathcal{O}})_x$$

for $x \in \mathcal{Q} \subset \overline{\mathcal{O}}$.

[1] For GL_n, turns out that $\mathrm{Stab}_{L(\sigma)}\, x$ is always connected $\Rightarrow$ each irr. local system is trivial

II. Back to Langlands Parameters

Let $\mathcal{L}$ be an irreducible[1] $L(\sigma)$-equivariant local system on the $L(\sigma)$-orbit $\mathcal{O} \subset \mathfrak{g}_1(\sigma)$.

$$\{\xi = (\mathcal{O}, \mathcal{L})\} \longleftrightarrow \{\mathcal{O}\}$$

$$c(\mathcal{Q}, \mathcal{O}) = \sum (-1)^j \dim IH^j(\overline{\mathcal{O}})_x$$

for $x \in \mathcal{Q} \subset \overline{\mathcal{O}}$.

[1] For GL_n, turns out that $\mathrm{Stab}_{L(\sigma)}\, x$ is always connected $\Rightarrow$ each irr. local system is trivial

II. Back to Langlands Parameters

Let $\mathcal{L}$ be an irreducible[1] $L(\sigma)$-equivariant local system on the $L(\sigma)$-orbit $\mathcal{O} \subset \mathfrak{g}_1(\sigma)$.

$$\{\xi = (\mathcal{O}, \mathcal{L})\} \longleftrightarrow \{\mathcal{O}\}$$

$$c(\mathcal{Q}, \mathcal{O}) = \sum (-1)^j \dim IH^j(\overline{\mathcal{O}})_x$$

for $x \in \mathcal{Q} \subset \overline{\mathcal{O}}$.

[1]For GL_n, turns out that $\mathrm{Stab}_{L(\sigma)}\, x$ is always connected $\Rightarrow$ each irr. local system is trivial

II. Problem

Calculate $c(\mathcal{Q}, \mathcal{O})$.

III. Kazhdan–Lusztig theory

$$L(\sigma) \cong GL(V_1) \times \cdots \times GL(V_k) = \begin{bmatrix} \boxed{*} & & 0 \\ & \boxed{\;*\;} & \\ 0 & & \boxed{\;\;*\;\;} \end{bmatrix}$$

III. Kazhdan–Lusztig theory

$$L(\sigma) \cong GL(V_1) \times \cdots \times GL(V_k) = \begin{bmatrix} \boxed{*} & & 0 \\ & \boxed{\;*\;} & \\ 0 & & \boxed{\;\;*\;\;} \end{bmatrix}$$

$$X(\sigma) := \{W_1 \subset \cdots \subset W_k = \mathbb{C}^n\}, \quad \text{where } \dim W_j = \textstyle\sum_{i \leq j} \dim V_i$$

III. Kazhdan–Lusztig theory

$$L(\sigma) \cong GL(V_1) \times \cdots \times GL(V_k) = \begin{bmatrix} \boxed{*} & & 0 \\ & \boxed{\;*\;} & \\ 0 & & \boxed{\;\;*\;\;} \end{bmatrix}$$

$$X(\sigma) := \{W_1 \subset \cdots \subset W_k = \mathbb{C}^n\}, \quad \text{where } \dim W_j = \textstyle\sum_{i \leq j} \dim V_i$$

$$P(\sigma) := \{g \in GL_n(\mathbb{C}) : g \cdot V_i \subseteq \oplus_{j \leq i} V_j\} = \begin{bmatrix} \boxed{*} & & * \\ & \boxed{\;*\;} & \\ 0 & & \boxed{\;\;*\;\;} \end{bmatrix}$$

III. Kazhdan–Lusztig theory

$$L(\sigma) \cong GL(V_1) \times \cdots \times GL(V_k) = \begin{bmatrix} \boxed{*} & & 0 \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{bmatrix}$$

$$X(\sigma) := \{W_1 \subset \cdots \subset W_k = \mathbb{C}^n\}, \quad \text{where } \dim W_j = \textstyle\sum_{i \leq j} \dim V_i$$

$$P(\sigma) := \{g \in GL_n(\mathbb{C}) : g \cdot V_i \subseteq \oplus_{j \leq i} V_j\} = \begin{bmatrix} \boxed{*} & & * \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{bmatrix}$$

$P(\sigma) \circlearrowright X(\sigma)$ with finitely many orbits

III. Example

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3\cdots, -n+3, -n+1)$$

$$L(\sigma) = GL(V_1) \times \cdots \times GL(V_n) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$$

$$X(\sigma) = \{W_1 \subset \cdots \subset W_n\} = F(\mathbb{C}^n)$$

$$P(\sigma) = B = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

$$P(\sigma) \text{ orbits on } X(\sigma) \longleftrightarrow W = S_n$$

III. Example

$$\sigma = \rho^\vee = \frac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$L(\sigma) = GL(V_1) \times \cdots \times GL(V_n) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$$

$$X(\sigma) = \{W_1 \subset \cdots \subset W_n\} = F(\mathbb{C}^n)$$

$$P(\sigma) = B = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

$$P(\sigma) \text{ orbits on } X(\sigma) \longleftrightarrow W = S_n$$

III. Example

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$L(\sigma) = GL(V_1) \times \cdots \times GL(V_n) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$$

$$X(\sigma) = \{W_1 \subset \cdots \subset W_n\} = F(\mathbb{C}^n)$$

$$P(\sigma) = B = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

$$P(\sigma) \text{ orbits on } X(\sigma) \longleftrightarrow W = S_n$$

III. Example

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$L(\sigma) = GL(V_1) \times \cdots \times GL(V_n) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$$

$$X(\sigma) = \{W_1 \subset \cdots \subset W_n\} = F(\mathbb{C}^n)$$

$$P(\sigma) = B = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

$P(\sigma)$ orbits on $X(\sigma) \longleftrightarrow W = S_n$

III. Example

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3\cdots, -n+3, -n+1)$$

$$L(\sigma) = GL(V_1) \times \cdots \times GL(V_n) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$$

$$X(\sigma) = \{W_1 \subset \cdots \subset W_n\} = F(\mathbb{C}^n)$$

$$P(\sigma) = B = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

$$P(\sigma) \text{ orbits on } X(\sigma) \longleftrightarrow W = S_n$$

IV. Kazhdan–Lusztig theory

Again, irreducible local systems are trivial

$$\{(\mathcal{O}, \mathcal{L})\} \longleftrightarrow \{\mathcal{O}\} \longleftrightarrow S_n$$

IV. Kazhdan–Lusztig theory

Again, irreducible local systems are trivial

$$\{(\mathcal{O},\mathcal{L})\} \longleftrightarrow \{\mathcal{O}\} \longleftrightarrow S_n$$

$$\chi\left([\mathrm{per}(w)]\right) = \sum c(y,w)[\mathrm{con}(y)]$$

IV. Kazhdan–Lusztig theory

Again, irreducible local systems are trivial

$$\{(\mathcal{O},\mathcal{L})\} \longleftrightarrow \{\mathcal{O}\} \longleftrightarrow S_n$$

$$\chi\left([\mathrm{per}(w)]\right) = \sum c(y,w)[\mathrm{con}(y)]$$

$$c(y,w) = \sum (-1)^j \dim IH^j(\overline{\mathcal{O}}(w))_x$$

$$\text{for } x \in \mathcal{O}(y) \subset \overline{\mathcal{O}}(w).$$

IV. Reduction to KL theory

$$\Psi : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow P(\sigma) \text{ orbits on } X(\sigma)$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

for $x \in \mathcal{Q} \subset \overline{\mathcal{O}}$ and $y \in \Psi(\mathcal{Q}) \subset \overline{\Psi(\mathcal{O})}$.

$$\Longrightarrow \quad c_{\mathfrak{g}_1}(\xi, \zeta) = c_X(\Psi(\xi), \Psi(\zeta))$$

IV. Reduction to KL theory

$$\Psi : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow P(\sigma) \text{ orbits on } X(\sigma)$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

for $x \in \mathcal{Q} \subset \overline{\mathcal{O}}$ and $y \in \Psi(\mathcal{Q}) \subset \overline{\Psi(\mathcal{O})}$.

$$\Longrightarrow \quad c_{\mathfrak{g}_1}(\xi, \zeta) = c_X(\Psi(\xi), \Psi(\zeta))$$

IV. Reduction to KL theory

$$\Psi : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow P(\sigma) \text{ orbits on } X(\sigma)$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

$$\text{for } x \in \mathcal{Q} \subset \overline{\mathcal{O}} \text{ and } y \in \Psi(\mathcal{Q}) \subset \overline{\Psi(\mathcal{O})}.$$

$$\Longrightarrow \quad c_{\mathfrak{g}_1}(\xi, \zeta) = c_X(\Psi(\xi), \Psi(\zeta))$$

IV. Example

$$\sigma = \rho^\vee = \frac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$T \circlearrowright \begin{bmatrix} 0 & * & 0 \\ & \ddots & * \\ & & 0 \end{bmatrix} \xrightarrow{\quad\Psi\quad} B \circlearrowright F(\mathbb{C}^n)$$

$$\Psi : \text{partitions of } n \longrightarrow S_n$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \begin{cases} 1 & \text{when } j = 2\dim\mathcal{O} \\ 0 & \text{else} \end{cases} \qquad (\overline{\mathcal{O}} \text{ is smooth})$$

$$= \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

IV. Example

$$\sigma = \rho^\vee = \frac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$T \circlearrowright \begin{bmatrix} 0 & * & 0 \\ & \ddots & * \\ & & 0 \end{bmatrix} \xrightarrow{\quad\Psi\quad} B \circlearrowright F(\mathbb{C}^n)$$

$$\Psi : \text{partitions of } n \longrightarrow S_n$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \begin{cases} 1 & \text{when } j = 2\dim \mathcal{O} \\ 0 & \text{else} \end{cases} \qquad (\overline{\mathcal{O}} \text{ is smooth})$$

$$= \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

IV. Example

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$T \circlearrowright \begin{bmatrix} 0 & * & 0 \\ & \ddots & * \\ & & 0 \end{bmatrix} \xrightarrow{\quad\Psi\quad} B \circlearrowright F(\mathbb{C}^n)$$

$$\Psi : \text{partitions of } n \longrightarrow S_n$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \begin{cases} 1 & \text{when } j = 2\dim \mathcal{O} \\ 0 & \text{else} \end{cases} \qquad (\overline{\mathcal{O}} \text{ is smooth})$$

$$= \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

IV. Example

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$T \circlearrowright \begin{bmatrix} 0 & * & 0 \\ & \ddots & * \\ & & 0 \end{bmatrix} \xrightarrow{\quad\Psi\quad} B \circlearrowright F(\mathbb{C}^n)$$

$$\Psi : \text{partitions of } n \longrightarrow S_n$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \begin{cases} 1 & \text{when } j = 2\dim\mathcal{O} \\ 0 & \text{else} \end{cases} \qquad (\overline{\mathcal{O}} \text{ is smooth})$$

$$= \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

IV. Example

$$\sigma = \rho^\vee = \tfrac{1}{2}(n-1, n-3 \cdots, -n+3, -n+1)$$

$$T \circlearrowright \begin{bmatrix} 0 & * & 0 \\ & \ddots & * \\ & & 0 \end{bmatrix} \xrightarrow{\quad\quad\Psi\quad\quad} B \circlearrowright F(\mathbb{C}^n)$$

$$\Psi : \text{partitions of } n \longrightarrow S_n$$

$$\dim IH^j(\overline{\mathcal{O}})_x = \begin{cases} 1 & \text{when } j = 2\dim\mathcal{O} \\ 0 & \text{else} \end{cases} \qquad (\overline{\mathcal{O}} \text{ is smooth})$$

$$= \dim IH^j(\overline{\Psi(\mathcal{O})})_y$$

V. Why is this calculation significant?

Local Langlands correspondence: k a local field of char 0

Let G be an $\overline{k}$ reductive algebraic group defined over k

$$^{\vee}G \circlearrowright P(^{\vee}G^{\Gamma}) := \{\varphi : W_k' \to {}^{\vee}G^{\Gamma} \mid \text{quasiadmissible}\}$$

$$\Pi(G(k)) = \coprod_{\gamma} \Pi_{\gamma}$$

L-packet

where γ is a $^{\vee}G$-orbit on $P(^{\vee}G^{\Gamma})$.

When $G = GL_n$, Π_{γ} is a single irr. adm. representation

V. Why is this calculation significant?

Local Langlands correspondence: k a local field of char 0

Let G be an $\overline{k}$ reductive algebraic group defined over k

$$^{\vee}G \circlearrowright P(^{\vee}G^{\Gamma}) := \{\varphi : W_k' \to {}^{\vee}G^{\Gamma} | \text{ quasiadmissible}\}$$

$$\Pi(G(k)) = \coprod_{\gamma} \Pi_{\gamma}$$

L-packet

where γ is a $^{\vee}G$-orbit on $P(^{\vee}G^{\Gamma})$.

When $G = GL_n$, Π_{γ} is a single irr. adm. representation

V. Why is this calculation significant?

Local Langlands correspondence: k a local field of char 0

Let G be an $\overline{k}$ reductive algebraic group defined over k

$$^{\vee}G \circlearrowright P(^{\vee}G^{\Gamma}) := \{\varphi : W_k' \to {}^{\vee}G^{\Gamma} |\ \text{quasiadmissible}\}$$

$$\Pi(G(k)) = \coprod_{\gamma} \Pi_{\gamma}$$

L-packet

where γ is a $^{\vee}G$-orbit on $P(^{\vee}G^{\Gamma})$.

When $G = GL_n$, Π_{γ} is a single irr. adm. representation

V. Why is this calculation significant?

Local Langlands correspondence: k a local field of char 0

Let G be an $\overline{k}$ reductive algebraic group defined over k

$${}^\vee G \circlearrowright P({}^\vee G^\Gamma) := \{\varphi : W_k' \to {}^\vee G^\Gamma \mid \text{quasiadmissible}\}$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

L-packet

where γ is a ${}^\vee G$-orbit on $P({}^\vee G^\Gamma)$.

When $G = GL_n$, Π_γ is a single irr. adm. representation

V. Why is this calculation significant?

Local Langlands correspondence: k a local field of char 0

Let G be an $\overline{k}$ reductive algebraic group defined over k

$${}^\vee G \circlearrowright P({}^\vee G^\Gamma) := \{\varphi : W_k' \to {}^\vee G^\Gamma | \text{ quasiadmissible}\}$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

L-packet

where γ is a ${}^\vee G$-orbit on $P({}^\vee G^\Gamma)$.

When $G = GL_n$, Π_γ is a single irr. adm. representation

V. Why is this calculation significant?

Local Langlands correspondence: k a local field of char 0

Let G be an $\overline{k}$ reductive algebraic group defined over k

$${}^\vee G \circlearrowright P({}^\vee G^\Gamma) := \{\varphi : W_k' \to {}^\vee G^\Gamma | \text{ quasiadmissible}\}$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

L-packet

where γ is a ${}^\vee G$-orbit on $P({}^\vee G^\Gamma)$.

When $G = GL_n$, Π_γ is a single irr. adm. representation

V. Why is this calculation significant?

Local Langlands correspondence: k a local field of char 0

Let G be an $\overline{k}$ reductive algebraic group defined over k

$${}^\vee G \circlearrowright P({}^\vee G^\Gamma) := \{\varphi : W_k' \to {}^\vee G^\Gamma \mid \text{quasiadmissible}\}$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

L-packet

where γ is a ${}^\vee G$-orbit on $P({}^\vee G^\Gamma)$.

When $G = GL_n$, Π_γ is a single irr. adm. representation

V. LLC for GL_n

$G = GL_n$:

$$\Pi(G(k)) \longleftrightarrow {}^\vee G\text{-orbits on } P({}^\vee G^\Gamma)$$

... not much hope for character formulas

á la variations on a geometric theme

Fix: Lusztig (non-archimedean),
Adams–Barbasch–Vogan (archimedean)

V. LLC for GL_n

$G = GL_n$:

$$\Pi(G(k)) \longleftrightarrow {}^\vee G\text{-orbits on } P({}^\vee G^\Gamma)$$

... not much hope for character formulas

á la variations on a geometric theme

Fix: Lusztig (non-archimedean),

Adams–Barbasch–Vogan (archimedean)

V. LLC for GL_n

$G = GL_n$:

$$\Pi(G(k)) \longleftrightarrow {}^\vee G\text{-orbits on } P({}^\vee G^\Gamma)$$

... not much hope for character formulas

á la variations on a geometric theme

Fix: Lusztig (non-archimedean),

Adams–Barbasch–Vogan (archimedean)

V. LLC wish list

$X_k(G)$ a complex algebraic variety

$${}^\vee G \circlearrowright X_k(G)$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

where γ is a ${}^\vee G$-orbit on $X_k(G)$.

$$\mathcal{K}\mathrm{Rep}G(k) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G), \mathbb{Z})$$

$\Longrightarrow$ geometric characters give us representation theoretic characters

V. LLC wish list

$X_k(G)$ a complex algebraic variety

$${}^\vee G \circlearrowright X_k(G)$$

$$\Pi(G(k)) = \coprod_{\gamma} \Pi_\gamma$$

where γ is a ${}^\vee G$-orbit on $X_k(G)$.

$$\mathcal{K}\mathrm{Rep}G(k) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G), \mathbb{Z})$$

$\Longrightarrow$ geometric characters give us representation theoretic characters

V. LLC wish list

$X_k(G)$ a complex algebraic variety

$${}^\vee G \circlearrowright X_k(G)$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

where γ is a ${}^\vee G$-orbit on $X_k(G)$.

$$\mathcal{K}\mathrm{Rep}G(k) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G), \mathbb{Z})$$

$\Longrightarrow$ geometric characters give us representation theoretic characters

V. LLC wish list

$X_k(G)$ a complex algebraic variety

$$^\vee G \circlearrowright X_k(G)$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

where γ is a $^\vee G$-orbit on $X_k(G)$.

$$\mathcal{K}\mathrm{Rep}G(k) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G), \mathbb{Z})$$

$\Longrightarrow$ geometric characters give us representation theoretic characters

V. LLC wish list

$X_k(G)$ a complex algebraic variety

$${}^\vee G \circlearrowright X_k(G)$$

$$\Pi(G(k)) = \coprod_\gamma \Pi_\gamma$$

where γ is a ${}^\vee G$-orbit on $X_k(G)$.

$$\mathcal{K}\mathrm{Rep}G(k) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_k(G), {}^\vee G), \mathbb{Z})$$

$\Longrightarrow$ geometric characters give us representation theoretic characters

V. Langlands parameters for graded affine Hecke algebras

If $\lambda \in \mathfrak{h}^* \leftrightarrow \sigma \in \mathfrak{h}$,

$$\mathcal{K}\mathrm{Rep}_\lambda(\mathbb{H}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z})$$

V. complex/real Langlands parameters

Fix an infinitesimal character $\lambda \in \mathfrak{h}^* \cong {}^\vee\mathfrak{h} \leftrightarrow \sigma \in \mathfrak{h}$

$$\mathcal{K}\mathrm{Rep}_\lambda(G(\mathbb{C})) \cong \mathrm{Hom}_\mathbb{Z}(\mathcal{K}\,\mathrm{Con}(X(\sigma), P(\sigma)), \mathbb{Z})$$

$$\mathcal{K}\mathrm{Rep}_\lambda(G(\mathbb{R})) \cong \mathrm{Hom}_\mathbb{Z}(\mathcal{K}\,\mathrm{Con}(X^\lambda_\mathbb{R}(G), {}^\vee G), \mathbb{Z})$$

V. complex/real Langlands parameters

Fix an infinitesimal character $\lambda \in \mathfrak{h}^* \cong {}^\vee\mathfrak{h} \leftrightarrow \sigma \in \mathfrak{h}$

$$\mathcal{K}\mathrm{Rep}_\lambda(G(\mathbb{C})) \cong \mathrm{Hom}_\mathbb{Z}(\mathcal{K}\,\mathrm{Con}(X(\sigma), P(\sigma)), \mathbb{Z})$$

$$\mathcal{K}\mathrm{Rep}_\lambda(G(\mathbb{R})) \cong \mathrm{Hom}_\mathbb{Z}(\mathcal{K}\,\mathrm{Con}(X^\lambda_\mathbb{R}(G), {}^\vee G), \mathbb{Z})$$

V. complex/real Langlands parameters

Fix an infinitesimal character $\lambda \in \mathfrak{h}^* \cong {}^\vee\mathfrak{h} \leftrightarrow \sigma \in \mathfrak{h}$

$$\mathcal{K}\mathrm{Rep}_\lambda(G(\mathbb{C})) \cong \mathrm{Hom}_\mathbb{Z}(\mathcal{K}\,\mathrm{Con}(X(\sigma), P(\sigma)), \mathbb{Z})$$

$$\mathcal{K}\mathrm{Rep}_\lambda(G(\mathbb{R})) \cong \mathrm{Hom}_\mathbb{Z}(\mathcal{K}\,\mathrm{Con}(X^\lambda_\mathbb{R}(G), {}^\vee G), \mathbb{Z})$$

V. Back to Ψ

$$\Psi_{\mathbb{C}} = \Psi : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow P(\sigma) \text{ orbits on } X(\sigma)$$

$$\Psi_{\mathbb{R}} : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow {}^{\vee}G \text{ orbits on } X_{\mathbb{R}}^{\lambda}(G)$$

$$\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)) \longrightarrow \mathcal{K}\,\mathrm{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G)$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G), \mathbb{Z})$$

V. Back to Ψ

$$\Psi_{\mathbb{C}} = \Psi : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow P(\sigma) \text{ orbits on } X(\sigma)$$

$$\Psi_{\mathbb{R}} : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow {}^{\vee}G \text{ orbits on } X_{\mathbb{R}}^{\lambda}(G)$$

$$\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)) \longrightarrow \mathcal{K}\,\mathrm{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G)$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G), \mathbb{Z})$$

V. Back to Ψ

$$\Psi_{\mathbb{C}} = \Psi : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow P(\sigma) \text{ orbits on } X(\sigma)$$

$$\Psi_{\mathbb{R}} : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow {}^\vee G \text{ orbits on } X^\lambda_{\mathbb{R}}(G)$$

$$\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)) \longrightarrow \mathcal{K}\,\mathrm{Con}(X^\lambda_{\mathbb{R}}(G), {}^\vee G)$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X^\lambda_{\mathbb{R}}(G), {}^\vee G), \mathbb{Z})$$

V. Back to Ψ

$$\Psi_{\mathbb{C}} = \Psi : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow P(\sigma) \text{ orbits on } X(\sigma)$$

$$\Psi_{\mathbb{R}} : L(\sigma) \text{ orbits on } \mathfrak{g}_1(\sigma) \longrightarrow {}^{\vee}G \text{ orbits on } X_{\mathbb{R}}^{\lambda}(G)$$

$$\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)) \xrightarrow{\qquad\qquad} \mathcal{K}\,\mathrm{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G)$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G), \mathbb{Z})$$

V. LLC + geometric digression

$$\operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}\operatorname{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) \longleftarrow \operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}\operatorname{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G), \mathbb{Z})$$

V. LLC + geometric digression

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) \arrow[d, phantom, "\wr\|"] & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X^{\lambda}_{\mathbb{R}}(G), {}^{\vee}G), \mathbb{Z}) \arrow[l] \arrow[d, phantom, "\wr\|"] \\
\mathcal{K}\mathrm{Rep}_{\lambda}(\mathbb{H}) & \mathcal{K}\mathrm{Rep}_{\lambda}(G(\mathbb{R})) \arrow[l, "\Psi"']
\end{tikzcd}$$

V. LLC + geometric digression

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) & \longleftarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X^{\lambda}_{\mathbb{R}}(G), {}^{\vee}G), \mathbb{Z}) \\
\wr\| & & \wr\| \\
\mathcal{K}\mathrm{Rep}_{\lambda}(\mathbb{H}) & \xleftarrow[\Psi]{\qquad\qquad} & \mathcal{K}\mathrm{Rep}_{\lambda}(G(\mathbb{R}))
\end{array}$$

Can we categorify?

V. LLC + geometric digression

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(\mathfrak{g}_1(\sigma), L(\sigma)), \mathbb{Z}) & \longleftarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}\,\mathrm{Con}(X_{\mathbb{R}}^{\lambda}(G), {}^{\vee}G), \mathbb{Z}) \\ \wr\| & & \wr\| \\ \mathcal{K}\mathrm{Rep}_{\lambda}(\mathbb{H}) & \xleftarrow[\Psi]{\qquad\qquad} & \mathcal{K}\mathrm{Rep}_{\lambda}(G(\mathbb{R})) \end{array}$$

Can we categorify?

Arakawa–Suzuki: $G(\mathbb{C})$

Ciubotaru–Trapa: $G(\mathbb{R})$

VI. Arakawa–Suzuki functors for $GL_n(\mathbb{R})$

$$F : (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathbb{H}\text{-modules}$$

$$F(X) := \mathrm{Hom}_{O(n)}(\det, X \otimes V^{\otimes n})$$

(relatively) easy calculation $\Rightarrow$ F maps[2] standard reps to standard reps

$+$ LLC

$\Longrightarrow$ F maps simple $(\mathfrak{g}, K)$-modules to simple $\mathbb{H}$-modules

[2] under certain assumptions

VI. Arakawa–Suzuki functors for $GL_n(\mathbb{R})$

$$F : (\mathfrak{g}, K)\text{-mod} \quad \longrightarrow \quad \mathbb{H}\text{-modules}$$

$$F(X) := \mathrm{Hom}_{O(n)}(\det, X \otimes V^{\otimes n})$$

(relatively) easy calculation $\Rightarrow$ F maps[2] standard reps to standard reps

$+$ LLC

$\Longrightarrow$ F maps simple $(\mathfrak{g}, K)$-modules to simple $\mathbb{H}$-modules

[2] under certain assumptions

VI. Arakawa–Suzuki functors for $GL_n(\mathbb{R})$

$$F : (\mathfrak{g}, K)\text{-mod} \quad \longrightarrow \quad \mathbb{H}\text{-modules}$$

$$F(X) := \mathrm{Hom}_{O(n)}(\det, X \otimes V^{\otimes n})$$

(relatively) easy calculation $\Rightarrow$ F maps[2] standard reps to standard reps

+ LLC

$\Longrightarrow$ F maps simple $(\mathfrak{g}, K)$-modules to simple $\mathbb{H}$-modules

[2] under certain assumptions

VI. Arakawa–Suzuki functors for $GL_n(\mathbb{R})$

$$F : (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathbb{H}\text{-modules}$$

$$F(X) := \mathrm{Hom}_{O(n)}(\det, X \otimes V^{\otimes n})$$

(relatively) easy calculation $\Rightarrow$ F maps[2] standard reps to standard reps

$+$ LLC

$\Longrightarrow$ F maps simple $(\mathfrak{g}, K)$-modules to simple $\mathbb{H}$-modules

[2] under certain assumptions

VI. Arakawa–Suzuki functors for $GL_n(\mathbb{R})$

$$F : (\mathfrak{g}, K)\text{-mod} \quad \longrightarrow \quad \mathbb{H}\text{-modules}$$

$$F(X) := \mathrm{Hom}_{O(n)}(\det, X \otimes V^{\otimes n})$$

(relatively) easy calculation $\Rightarrow$ F maps[2] standard reps to standard reps

+ LLC

$\Longrightarrow$ F maps simple $(\mathfrak{g}, K)$-modules to simple $\mathbb{H}$-modules

[2] under certain assumptions

VII. Signatures of Hermitian forms & Unitary Representations, joint with Peter Trapa

$\sigma \circlearrowright IH^j(\overline{\mathcal{O}}, \mathcal{L})_x$,

$$\operatorname{tr} \sigma_{\mathbb{H}} \circlearrowright IH^j(\overline{\mathcal{O}})_x = \operatorname{tr} \sigma_{\mathfrak{g}} \circlearrowright IH^j(\overline{\Psi(\mathcal{O})})_y$$

$\Rightarrow$ signature characters for $\mathbb{H}$ are a subset of signature characters of $(\mathfrak{g}, K)$.

Thank you for listening.

References

Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston, Inc., Boston, MA, 1992. MR 1162533

Tomoyuki Arakawa and Takeshi Suzuki, *Duality between* $\mathfrak{sl}_n(\mathbf{C})$ *and the degenerate affine Hecke algebra*, J. Algebra **209** (1998), no. 1, 288–304. MR 1652134

Dan Ciubotaru and Peter E. Trapa, *Duality between* $GL(n,\mathbb{R})$, $GL(n,\mathbb{Q}_p)$, *and the degenerate affine Hecke algebra for* $\mathfrak{gl}(n)$, Amer. J. Math. **134** (2012), no. 1, 141–170. MR 2876142

David Kazhdan and George Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no. 1, 153–215. MR 862716

David A. Vogan, Jr., *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 305–379. MR 1216197

A. V. Zelevinskiĭ, *Two remarks on graded nilpotent classes*, Uspekhi Mat. Nauk **40** (1985), no. 1(241), 199–200. MR 783619