From $GL_2(\mathbb{Q}_p)$ to $SL_2(\mathbb{Q}_p)$

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The 2020 Paul J. Sally, Jr. Midwest Representation Theory Conference

Local Langlands correspondence for $GL_n(F)$

Let F be a finite extension of $\mathbb{Q}_p$, and let WD_F denote the Weil-Deligne group of F. The local Langlands correspondence for $GL_n(F)$ is a bijection

$$\phi \longleftrightarrow \pi(\phi)$$

between the equivalence classes of n-dimensional complex representations of the Weil-Deligne group WD_F on which Frobenius acts semisimply, and the equivalence classes of irreducible admissible smooth complex representations of $GL_n(F)$.

- Proved by Kutzko for $GL_2(F)$.
- Proved by Harris-Taylor and Henniart for $GL_n(F)$ [H-T], [Hen].

The idea of p-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$ was first formulated and investigated by Breuil. It was established by the work of many people, and proved by Colmez [Col2010].

p-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$

Let E be a finite extension of $\mathbb{Q}_p$. Let

$$\psi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(E)$$

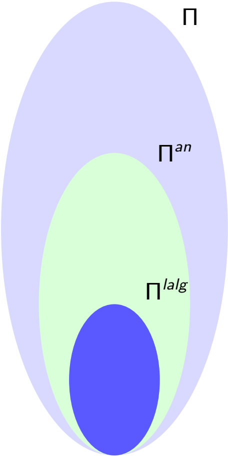
be an absolutely irreducible Galois representation. Attached to ψ by the p-adic Langlands correspondence is the representation

$$\Pi = \Pi(\psi)$$

such that

- Π is a continuous E-Banach space representation of $G = GL_2(\mathbb{Q}_p)$;
- Π is unitary, that is, $||g.v|| = ||v||$, for all $g \in G$, $v \in \Pi$;
- Π is admissible in the sense of [SchT] (this means that the continuous dual Π' is finitely generated as a module over the Iwasawa algebra $E[[K]]$ for some (hence any) compact open subgroup K of G); and
- Π is non-ordinary, that is, Π is not a subquotient of a unitary parabolic induction from a unitary character.

The structure of $\Pi(\psi)$



$\Pi = \Pi(\psi)$ the irreducible admissible Banach space representation attached to ψ

Π^{an} = locally analytic vectors in Π

Π^{lalg} = locally algebraic vectors in Π

$$\Pi^{lalg} = \pi(\psi) \otimes \Pi^{alg}$$

$\pi(\psi)$ smooth

Π^{alg} algebraic

$\Pi^{lalg} \neq 0$ if and only if ψ is de Rham with distinct Hodge-Tate weights

Suppose that ψ is de Rham with distinct Hodge-Tate weights. Then

- If ψ is trianguline, then $\pi(\psi)$ is a principal series or the Steinberg representation.
- If ψ is not trianguline, then $\pi(\psi)$ is supercuspidal.

Filtered modules

We work with $\Pi = \Pi(\psi)$ such that $\Pi^{lalg} \neq 0$ because

- we know smooth representations of G, and
- ψ corresponds to a 2-dimensional filtered module D which can be described explicitly [Fon1994a,b].

For example, if $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ is an absolutely irreducible crystabelline representation (i.e., becomes crystalline over an abelian extension of $\mathbb{Q}_p$), we associate to it an admissible filtered $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-module $D_{\mathrm{cris}}(\psi)$. We will describe it explicitly later.

The jumps in the filtration on D are the negatives of the Hodge-Tate weights of ψ. That is, if h is a Hodge-Tate weight, one has $Fil^h(D) \neq Fil^{h+1}(D)$.

The smooth part of $\Pi(\psi)$

From now on, we assume that ψ is de Rham with distinct H-T weights $a < b$. Then

$$\begin{aligned}\Pi^{lalg} &= \pi(\psi) \otimes \Pi^{alg} \\ &= \pi(\psi) \otimes \det{}^a \otimes \mathrm{Sym}^{b-a-1}(E^2).\end{aligned}$$

Let

$$\phi = WD(\psi)$$

be the Weil-Deligne representation associated to ψ. Let $\pi_{LL}(\phi)$ be the irreducible smooth representation of G over $\overline{\mathbb{Q}}_p$ attached to ϕ by the classical local Langlands correspondence. A twist of $\pi_{LL}(\phi)$ has a canonical model over E [BrSch]. We denote this smooth E-representation of G by

$$\pi(\phi).$$

If $\pi(\psi)$ is irreducible, then

$$\pi(\psi) = \pi(\phi).$$

If $\pi(\psi)$ is reducible, then we have the following exact sequence

$$0 \to \pi(\psi)^{\mathrm{irr}} \to \pi(\psi) \to \pi(\phi) \to 0,$$

where $\pi(\psi)^{\mathrm{irr}}$ is the unique irreducible subrepresentation of $\pi(\psi)$. ($\pi(\psi)$ is a standard module, and $\pi(\phi)$ is the corresponding Langlands quotient).

The classical local Langlands correspondence for $H = SL_2(\mathbb{Q}_p)$

Let $\mathrm{pr} : GL_2(E) \to PGL_2(E)$ be the canonical projection. Given a Weil-Deligne representation ϕ, we set

$$\overline{\phi} = \mathrm{pr} \circ \phi.$$

We denote by $S_{\overline{\phi}}$ the centralizer of the image of $\overline{\phi}$ in $PGL_2(\overline{E})$ (considered as an algebraic group) and $S^{\circ}_{\overline{\phi}}$ its identity component, and we define

$$\mathcal{S}_{\overline{\phi}} \cong S_{\overline{\phi}}/S^{\circ}_{\overline{\phi}}.$$

Denote by

$$\{\pi(\overline{\phi})\}$$

the L-packet associated to $\overline{\phi}$ by the classical Langlands corrspondence. It consists of irreducible components of $\pi(\phi)|_H$ and it has size 1, 2, or 4. The elements of the packet $\{\pi(\overline{\phi})\}$ are parameterized by the characters of $\mathcal{S}_{\overline{\phi}}$.

We would like to understand the corresponding picture for p-adic Langlands correspondence. More specifically, for $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$, we study the restriction of $\Pi(\psi)$ to H and

$$\overline{\psi} = \mathrm{pr} \circ \psi.$$

The parameterization of trianguline representations

We denote by $\widehat{\mathfrak{T}}(E)$ the set of continuous characters $\delta : \mathbb{Q}_p^\times \to E^\times$.
For $\delta \in \widehat{\mathfrak{T}}(E)$, let $w(\delta)$ be its weight ($w(\delta) = \frac{\log \delta(u)}{\log u}$, where $u \in \mathbb{Z}_p^\times$ is not a root of unity).
Set $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, and let $\mathcal{S}$ be the set of triples

$$s = (\delta_1, \delta_2, \mathcal{L}),$$

where $\delta_1, \delta_2 \in \widehat{\mathfrak{T}}(E)$ and $\mathcal{L} \in \mathbb{P}^0(E) = \{\infty\}$ if $\delta_1\delta_2^{-1}$ is not of the form x^{-i} with $i \geq 0$, nor of the form $x|x^i|$ with $i \geq 1$, and $\mathcal{L} \in \mathbb{P}^1(E)$ otherwise. We denote by $\Delta(s)$ the (φ, Γ)-module associated to s, and by

$$\psi(s)$$

the corresponding Galois representation. See [Col2014, Introduction] for details.

Set $u(s) = v_p(\delta_1(p))$ and $w(s) = w(\delta_1) - w(\delta_2)$, and define

$$\begin{array}{lcll} \mathcal{S}_* & = & \{s \in \mathcal{S} & \mid \quad v_p(\delta_1(p)) + v_p(\delta_2(p)) = 0 \text{ and } u(s) > 0\}\ , \\ \mathcal{S}_*^{\mathrm{ng}} & = & \{s \in \mathcal{S}_* & \mid \quad w(s) \notin \mathbb{Z}_{\geq 1}\}\ , \\ \mathcal{S}_*^{\mathrm{cris}} & = & \{s \in \mathcal{S}_* & \mid \quad w(s) \in \mathbb{Z}_{\geq 1}\, ,\ u(s) < w(s)\, ,\ \mathcal{L} = \infty\}\ , \\ \mathcal{S}_*^{\mathrm{st}} & = & \{s \in \mathcal{S}_* & \mid \quad w(s) \in \mathbb{Z}_{\geq 1}\, ,\ u(s) < w(s)\, ,\ \mathcal{L} \neq \infty\}\ , \\ \mathcal{S}_*^{\mathrm{ord}} & = & \{s \in \mathcal{S}_* & \mid \quad w(s) \in \mathbb{Z}_{\geq 1}\, ,\ u(s) = w(s)\}\ . \end{array}$$

If $\psi(s)$ is an irreducible Hodge-Tate representation, we have the following:

- $\psi(s)$ is de Rham if and only if $s \in \mathcal{S}_*^{\mathrm{cris}} \sqcup \mathcal{S}_*^{\mathrm{st}}$,
- $\psi(s)$ is crystabelline (i.e., becomes crystalline over an abelian extension of $\mathbb{Q}_p$) if and only if $s \in \mathcal{S}_*^{\mathrm{cris}}$,
- $\psi(s)$ is the twist of a semistable non-crystalline representation by a character of finite order if and only if $s \in \mathcal{S}_*^{\mathrm{st}}$.

Every 2-dimensional absolutely irreducible trianguline representation of $\mathcal{G}_{\mathbb{Q}_p}$ is of the form $\psi(s)$ for some $s \in \mathcal{S}_*^{\mathrm{ng}} \sqcup \mathcal{S}_*^{\mathrm{cris}} \sqcup \mathcal{S}_*^{\mathrm{st}}$.

The structure of Π^{an}

Suppose

$$s = (\delta_1, \delta_2, \infty) \in \mathcal{S}_*^{\mathrm{cris}}.$$

Set

$$\delta_1' = x^{w(s)}\delta_2, \quad \delta_2' = x^{-w(s)}\delta_1, \quad \text{and} \quad s' = (\delta_1', \delta_2', \infty).$$

Assume $s \neq s'$ (s is not exceptional).

We denote by $x \in \widehat{\mathcal{T}}(E)$ the character $x \mapsto x$ induced by the inclusion $\mathbb{Q}_p \subseteq E$. Set $\chi_{\mathrm{cyc}} = x|x|$. For $\delta_1, \delta_2 \in \widehat{\mathcal{T}}(E)$ define

$$B^{\mathrm{an}}(\delta_1, \delta_2) = \mathrm{Ind}_P^G(\delta_2 \otimes \delta_1\chi_{\mathrm{cyc}}^{-1})^{\mathrm{an}},$$

the locally analytic principal series representation. Then

$$\Pi^{\mathrm{lalg}} \cong B^{\mathrm{lalg}}(\delta_1, \delta_2) \cong B^{\mathrm{lalg}}(\delta_1', \delta_2').$$

The structure of Π^{an} (continued)

By [Col2014, Proposition 8.97], we have the following exact sequences

$$0 \longrightarrow \Pi^{\mathrm{lalg}} \longrightarrow B^{\mathrm{an}}(\delta_1, \delta_2) \oplus B^{\mathrm{an}}(\delta_1', \delta_2') \longrightarrow \Pi^{\mathrm{an}} \longrightarrow 0,$$

$$0 \longrightarrow \Pi^{\mathrm{lalg}} \longrightarrow \Pi^{\mathrm{an}} \longrightarrow B^{\mathrm{an}}(\delta_2, \delta_1) \oplus B^{\mathrm{an}}(\delta_2', \delta_1') \longrightarrow 0.$$

Since $B^{\mathrm{an}}(\delta_1, \delta_2)$ and $B^{\mathrm{an}}(\delta_1', \delta_2')$ both embed into Π^{an}, it follows

$$\Pi^{\mathrm{an}} = B^{\mathrm{an}}(\delta_1, \delta_2) \oplus_{\Pi^{\mathrm{lalg}}} B^{\mathrm{an}}(\delta_1', \delta_2').$$

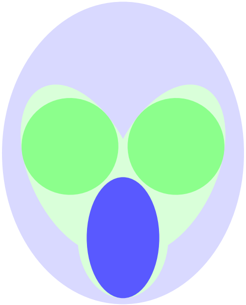
The structure of Π^{an} (continued)

By [Col2014, Proposition 8.97], we have the following exact sequences

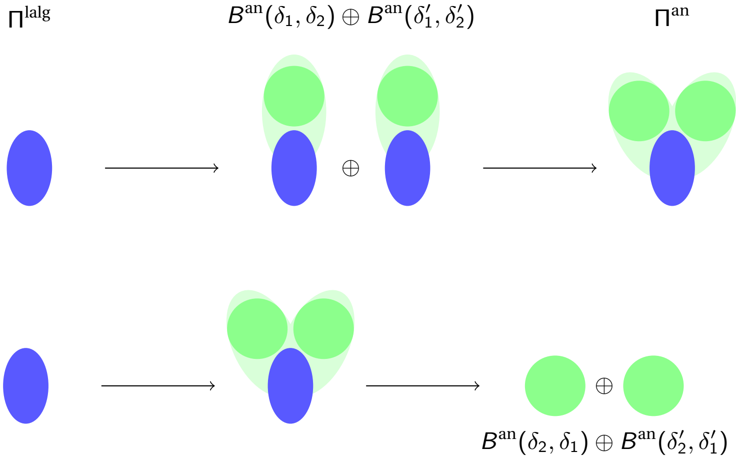
$$0 \longrightarrow \Pi^{\mathrm{lalg}} \longrightarrow B^{\mathrm{an}}(\delta_1, \delta_2) \oplus B^{\mathrm{an}}(\delta_1', \delta_2') \longrightarrow \Pi^{\mathrm{an}} \longrightarrow 0,$$
$$0 \longrightarrow \Pi^{\mathrm{lalg}} \longrightarrow \Pi^{\mathrm{an}} \longrightarrow B^{\mathrm{an}}(\delta_2, \delta_1) \oplus B^{\mathrm{an}}(\delta_2', \delta_1') \longrightarrow 0.$$

Since $B^{\mathrm{an}}(\delta_1, \delta_2)$ and $B^{\mathrm{an}}(\delta_1', \delta_2')$ both embed into Π^{an}, it follows

$$\Pi^{\mathrm{an}} = B^{\mathrm{an}}(\delta_1, \delta_2) \oplus_{\Pi^{\mathrm{lalg}}} B^{\mathrm{an}}(\delta_1', \delta_2').$$



The structure of Π^{an} in pictures



The structure of Π^{lalg}

Set $\psi = \psi(s)$ and $\Pi = \Pi(\psi)$. Let $\delta_s = \chi_{\mathrm{cyc}}^{-1}\delta_1\delta_2^{-1}$. The space Π^{alg} is described in [Col2014]:

1. if $s \in \mathcal{S}_*^{\mathrm{cris}}$, $\delta_s \neq x^{w(s)-1}, |x|^{-2}x^{w(s)-1}$, then

$$\Pi^{\mathrm{lalg}} = B^{\mathrm{lalg}}(\delta_1, \delta_2) = \mathrm{ind}(|x|\delta_s x^{1-w(s)} \otimes |x|^{-1}) \otimes \mathrm{Sym}^{w(s)-1} \otimes (\delta_2 \circ \det);$$

2. if $s \in \mathcal{S}_*^{\mathrm{cris}}$, $\delta_s = |x|^{-2}x^{w(s)-1}$, then $\Pi^{\mathrm{lalg}} = B^{\mathrm{lalg}}(\delta_1, \delta_2)$;
3. if $s \in \mathcal{S}_*^{\mathrm{cris}}$, $\delta_s = x^{w(s)-1}$, then

$$\Pi^{\mathrm{lalg}} = B^{\mathrm{lalg}}(\delta_1', \delta_2') = \mathrm{ind}(|x| \otimes |x|^{-1}) \otimes \mathrm{Sym}^{w(s)-1} \otimes (\delta_2 \circ \det);$$

4. if $s \in \mathcal{S}_*^{\mathrm{st}}$, then $\Pi^{\mathrm{lalg}} = \mathrm{St} \otimes \mathrm{Sym}^{w(s)-1} \otimes (\delta_2 \circ \det)$.

Cases (2) and (3) correspond to each other by the involution $s \mapsto s'$, so it is enough to consider one of them, because for $s \in \mathcal{S}_*^{\mathrm{cris}}$ we have $\Delta(s') \cong \Delta(s)$ [Proposition 8.3 (ii) Col2014].

Restriction of representations of p-adic Lie groups

Proposition 1

Let G be a p-adic Lie group and H an open normal subgroup of G of finite index. Let Π be an irreducible admissible E-Banach space representation of G. Then

$$\Pi|_H = \Pi_1 \oplus \cdots \oplus \Pi_r \tag{1}$$

where Π_i are irreducible E-Banach space representations of H.

For the proof, we have to show that $\Pi|_H$ contains an irreducible (closed) subrepresentation Π_1. For this, we use the admissibility of Π. The rest of the proof is standard; in particular

$$\Pi|_H = \Pi_1 \oplus g_2.\Pi_1 \cdots \oplus g_r.\Pi_1, \tag{2}$$

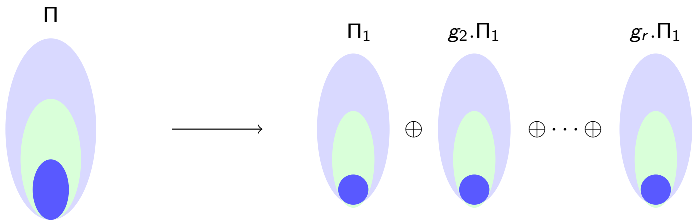
where $\{1, g_2, \ldots, g_r\}$ is a subset of the set of representatives of G/H.

Locally algebraic vectors in $\Pi|_H$

Proposition 2

Let G be a p-adic Lie group and H an open normal subgroup of G of finite index. Let Π be an irreducible admissible E-Banach space representation of G. Suppose that the subspace of locally algebraic vectors Π^{lalg} is dense in Π. Write $\Pi = \Pi_1 \oplus \ldots \oplus \Pi_r$ as in (1). Then for each i, the set $(\Pi_i)^{\mathrm{lalg}}$ is dense in Π_i, hence non-zero, and

$$(\Pi^{\mathrm{lalg}})|_H = (\Pi_1)^{\mathrm{lalg}} \oplus \ldots \oplus (\Pi_r)^{\mathrm{lalg}} .$$



The structure of $\Pi|_H$ for ψ trianguline

From now on, $G = GL_2(\mathbb{Q}_p)$ and $H = SL_2(\mathbb{Q}_p)$.

Theorem 3

Let $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ be an absolutely irreducible trianguline representation which is de Rham with distinct Hodge-Tate weights. Denote by $\Pi = \Pi(\psi)$ the corresponding absolutely irreducible unitary admissible Banach space representation of $G = GL_2(\mathbb{Q}_p)$, and let Π^{lalg} be the subspace of locally algebraic vectors of Π. Then the following assertions are equivalent:

1. *$\Pi|_H$ is reducible.*
2. *$\Pi|_H$ is decomposable.*
3. *$(\Pi^{\mathrm{lalg}})|_H$ is decomposable.*

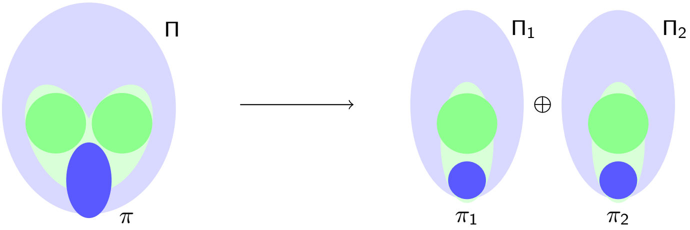
If one (equivalently all) of the above cases occurs, then both $\Pi|_H$ and $(\Pi^{\mathrm{lalg}})|_H$ have two absolutely irreducible inequivalent constituents.

The proof in pictures

If $(\Pi^{\text{lalg}})|_H$ is indecomposable, then Theorem 3 follows from Proposition 2. Also, the results for the smooth principal series representations of G imply

$$\Pi^{\text{lalg}} \text{ reducible and indecomposable} \implies (\Pi^{\text{lalg}})|_H \text{ indecomposable.}$$

It remains to consider the case when Π^{lalg} is irreducible, but $(\Pi^{\text{lalg}})|_H$ is reducible. Assume for simplicity that the Hodge-Tate weights are 0 and 1. It turns out that Π decomposes as follows:



$\pi|_H = \pi_1 \oplus \pi_2$ and $\{\pi_1, \pi_2\}$ form an L-packet of smooth representations of H. It seems natural to consider $\{\Pi_1, \Pi_2\}$ as an L-packet of admissible p-adic Banach space representations of H.

Centralizers

Proposition 4

Let $\psi : G_{\mathbb{Q}_p} \to GL_2(E)$ *be an absolutely irreducible trianguline de Rham representation with distinct Hodge-Tate weights.*

(i) *The centralizer* $S_{\overline{\psi}}$ *in* $PGL_2(\overline{E})$ *of the image of* $\overline{\psi}$ *has one or two elements. The latter case occurs if and only if* ψ *is equivalent to* $\vartheta\psi$ *for some quadratic character* $\vartheta \neq 1$.

(ii) *Denote by* ϕ *the Weil group representation on* $\mathrm{WD}(\psi)$ *associated to* ψ, *and set* $\overline{\phi} = \mathrm{pr} \circ \phi$. *Then*

$$S_{\overline{\psi}} \cong S_{\overline{\phi}} / S_{\overline{\phi}}^{\circ} ,$$

where $S_{\overline{\phi}}$ *denotes the centralizer of the image of* $\overline{\phi}$ *in* $PGL_2(\overline{E})$ *(considered as an algebraic group) and* $S_{\overline{\phi}}^{\circ}$ *its identity component.*

The strategy is to determine the centralizer using the filtered modules attached to ψ. Note that we can twist ψ by a power of the cyclotomic character so that its Hodge-Tate weights are 0 and $k-1$, where $k \geq 2$.

Filtered modules in the crystabelline case.

Here we assume that ψ is crystabelline, and we consider the filtered $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-module $D_{\mathrm{cris}}(\psi)$ as defined in [BeBr]. Let $\alpha, \beta : \mathbb{Q}_p^\times \to E^\times$ be locally constant characters such that

$$-(k-1) < \mathrm{val}(\alpha(p)) \leq \mathrm{val}(\beta(p)) < 0 \quad \text{and} \quad \mathrm{val}(\alpha(p)) + \mathrm{val}(\beta(p)) = -(k-1)$$

and which are trivial on $1 + p^n\mathbb{Z}_p$ for some $n \geq 1$. We define on $D(\alpha, \beta) = E \cdot e_\alpha \oplus E \cdot e_\beta$ the structure of a filtered $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-module:
If $\alpha \neq \beta$, then:

$$\begin{cases} \varphi(e_\alpha) = \alpha(p) e_\alpha \\ \varphi(e_\beta) = \beta(p) e_\beta \end{cases} \qquad \text{and if } g \in \Gamma, \text{ then:} \qquad \begin{cases} g(e_\alpha) = \alpha(\varepsilon(g)) e_\alpha \\ g(e_\beta) = \beta(\varepsilon(g)) e_\beta \end{cases}$$

and

$$\mathrm{Fil}^i(E_n \otimes_E D(\alpha, \beta)) = \begin{cases} E_n \otimes_E D(\alpha, \beta) & \text{if } i \leq -(k-1) \\ E_n \cdot (e_\alpha + G(\beta\alpha^{-1}) \cdot e_\beta) & \text{if } -(k-2) \leq i \leq 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Here, $E_n = E \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n})$, $\varepsilon : \mathcal{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the cyclotomic character, and $G(\beta\alpha^{-1})$ is the Gauss sum.

Filtered modules in the crystabelline case, continued

If $\alpha = \beta$, then:

$$\begin{cases} \varphi(e_\alpha) = \alpha(p)e_\alpha \\ \varphi(e_\beta) = \alpha(p)(e_\beta - e_\alpha) \end{cases} \quad \text{and if } g \in \Gamma, \text{ then:} \quad \begin{cases} g(e_\alpha) = \alpha(\varepsilon(g))e_\alpha \\ g(e_\beta) = \alpha(\varepsilon(g))e_\beta \end{cases}$$

and

$$\mathrm{Fil}^i(E_n \otimes_E D(\alpha,\beta)) = \begin{cases} E_n \otimes_E D(\alpha,\beta) & \text{if } i \le -(k-1) \\ E_n \cdot e_\beta & \text{if } -(k-2) \le i \le 0 \\ 0 & \text{if } i \ge 1 \end{cases}$$

Then:

- If $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ is an absolutely irreducible crystabelline representation with Hodge-Tate weights 0 and $k-1$, where $k \ge 2$, then there exist characters α and β as above such that $D_{\mathrm{cris}}(\psi) = D(\alpha,\beta)$.
- Conversely, if α and β are such characters, then there exists an absolutely irreducible crystabelline representation $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ such that $D_{\mathrm{cris}}(\psi) = D(\alpha,\beta)$.

Some linear algebra

Lemma 5

Suppose $D(\alpha,\beta)$ is equivalent to $\vartheta \otimes D(\alpha,\beta)$, for some character ϑ of $\mathbb{Q}_p^\times$. Then $\beta = \vartheta\alpha$ and $\vartheta^2 = 1$. Conversely, if $\beta = \vartheta\alpha$ with a non-trivial quadratic character ϑ, then $D(\alpha,\beta)$ is equivalent to $\vartheta \otimes D(\alpha,\beta)$.

Proof. Suppose $D(\alpha,\beta) \cong \vartheta \otimes D(\alpha,\beta)$, and the equivalence is given with respect to the basis (e_α, e_β) by $y \in GL_2(E)$. First, we consider the case $\alpha \neq \beta$. Then, for all $t \in \mathbb{Q}_p^\times$,

$$y \begin{pmatrix} \alpha(t) & 0 \\ 0 & \beta(t) \end{pmatrix} y^{-1} = \begin{pmatrix} \vartheta(t)\alpha(t) & 0 \\ 0 & \vartheta(t)\beta(t) \end{pmatrix} \tag{3}$$

and, since y respects filtration,

$$y \begin{pmatrix} 1 \\ G(\beta\alpha^{-1}) \end{pmatrix} = c \begin{pmatrix} 1 \\ G(\beta\alpha^{-1}) \end{pmatrix} \tag{4}$$

for some $c \in E^\times$. Because $\alpha \neq \beta$, (3) implies that y must be either a diagonal matrix or an anti-diagonal matrix (i.e., the entries on the diagonal vanish). If y is a diagonal matrix, then equation (3) gives $\vartheta = 1$ and equation (4) implies that y is a scalar matrix.

More linear algebra

Now suppose that y is an anti-diagonal matrix. Equation (3) becomes

$$y\begin{pmatrix}\alpha(t) & 0\\ 0 & \beta(t)\end{pmatrix}y^{-1}=\begin{pmatrix}\beta(t) & 0\\ 0 & \alpha(t)\end{pmatrix}=\begin{pmatrix}\vartheta(t)\alpha(t) & 0\\ 0 & \vartheta(t)\beta(t)\end{pmatrix}$$

It follows $\beta=\vartheta\alpha$, $\alpha=\vartheta\beta$, and hence $\vartheta^2=1$. Finally, equation (4) implies that y is a scalar multiple of the matrix $y_0=\begin{pmatrix}0 & G(\beta\alpha^{-1})^{-1}\\ G(\beta\alpha^{-1}) & 0\end{pmatrix}$.

Conversely, if $\beta=\vartheta\alpha$ with a non-trivial quadratic character ϑ, then the matrix y_0 defines an equivalence $D(\alpha,\beta)\cong\vartheta\otimes D(\alpha,\beta)$.

If $\alpha=\beta$, then

$$y\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}y^{-1}=\vartheta(p)\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\quad\text{and}\quad y\begin{pmatrix}0\\ 1\end{pmatrix}=c\begin{pmatrix}0\\ 1\end{pmatrix}$$

imply that y is a scalar matrix, and hence $\vartheta=1$. □

Perfect matching

Theorem 6

Let $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(E)$ *be an absolutely irreducible crystabelline or semi-stable representation with distinct Hodge-Tate weights. Let* $\Pi = \Pi(\psi)$ *and let* $\pi = \pi(\psi)^{\mathrm{irr}}$ *be the unique irreducible subrepresentation of* $\pi(\psi)$*.*

(i) *Both* $\Pi|_H$ *and* $\pi|_H$ *decompose as direct sums of inequivalent irreducible components.*

(ii) *There is a canonical bijection between the set of components of* $\Pi|_H$ *and the set of components of* $\pi|_H$*. The number of components is equal to either 1 or 2.*

(iii) *Denote by* ϕ *the Weil-Deligne representation associated to* ψ*, and set* $\overline{\phi} = \mathrm{pr} \circ \phi$*. Then*

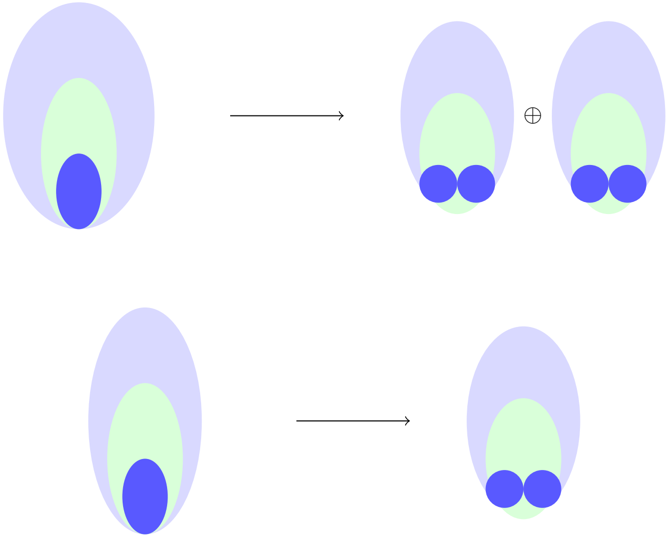
$$\mathcal{S}_{\overline{\psi}} \cong \mathcal{S}_{\overline{\phi}},$$

and this group is isomorphic to either 1 *or* $\mathbb{Z}/2\mathbb{Z}$*.*

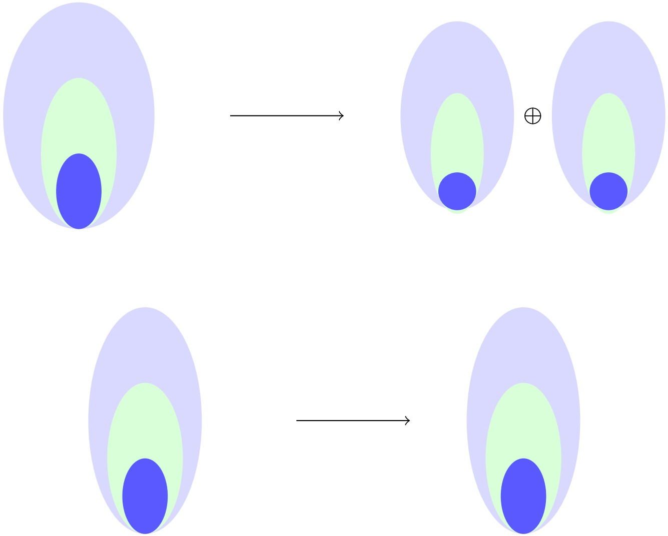
(iv) $\Pi|_H$ *is reducible if and only if* $\mathcal{S}_{\overline{\psi}} \cong \mathbb{Z}/2\mathbb{Z}$*.*

Triangulline case is too good

The matching from Theorem 6 between $\Pi|_H$ and $\pi|_H$ does not hold for non-triangulline case. We have the following situations



More cases



At most two irreducible constituents

Proposition 7

Let Π be an absolutely irreducible admissible unitary p-adic Banach space representation of G. Then $\Pi|_{SL_2(\mathbb{Q}_p)}$ decomposes into at most two irreducible components.

Proof. Put $\overline{\Pi} = \Pi_{\leq 1} \otimes_{\mathcal{O}_E} k_E$, where $\Pi_{\leq 1} = \{v \in \Pi \mid \|v\| \leq 1\}$ and k_E is the residue field of E. This is a smooth G-representation. By [ColDoPa], after possibly replacing E by an unramified quadratic extension, there are two possibilities for $\overline{\Pi}$, namely

(i) $\overline{\Pi}$ is an absolutely irreducible supersingular representation.

(ii) The semisimplification $\overline{\Pi}^{\mathrm{ss}}$ of $\overline{\Pi}$ embeds into

$$\pi\{\chi_1, \chi_2\} := \left(\mathrm{Ind}_P^G(\chi_1 \otimes \chi_2\omega^{-1})\right)^{\mathrm{ss}} \oplus \left(\mathrm{Ind}_P^G(\chi_2 \otimes \chi_1\omega^{-1})\right)^{\mathrm{ss}},$$

where χ_1 and χ_2 are smooth characters $\mathbb{Q}_p^\times \to k_E^\times$, and $\omega : \mathbb{Q}_p^\times \to k_E^\times$ is the reduction of the cyclotomic character.

It is a result of Ramla Abdellatif that in case (i) $\overline{\Pi}|_H$ decomposes into two irreducible representations [Abd]. In particular, $\Pi|_H$ cannot have more than two irreducible components.

Now suppose we are in case (ii). We consider the list given in [ColDoPa] which provides an explicit description of the decomposition of $\pi\{\chi_1, \chi_2\}$ into irreducible constituents. $\pi\{\chi_1, \chi_2\}$ is isomorphic to one (and only one) of the following:

1. $\mathrm{ind}_P^G(\chi_1 \otimes \chi_2\omega^{-1}) \oplus \mathrm{ind}_P^G(\chi_2 \otimes \chi_1\omega^{-1})$, if $\chi_1\chi_2^{-1} \neq 1, \ \omega^{\pm 1}$;
2. $\mathrm{ind}_P^G(\chi \otimes \chi\omega^{-1})^{\oplus 2}$, if $\chi_1 = \chi_2 = \chi$ and $p \geq 3$;
3. $\left(1 \oplus \mathrm{St} \oplus \mathrm{ind}_P^G(\omega \otimes \omega^{-1})\right) \otimes \chi \circ \det$, if $\chi_1\chi_2^{-1} = \omega^{\pm 1}$ and $p \geq 5$;
4. $\left(1 \oplus \mathrm{St} \oplus \omega \circ \det \oplus \mathrm{St} \otimes \omega \circ \det\right) \otimes \chi \circ \det$, if $\chi_1\chi_2^{-1} = \omega^{\pm 1}$ and $p = 3$;
5. $\left(1 \oplus \mathrm{St}\right)^{\oplus 2} \otimes \chi \circ \det$, if $\chi_1 = \chi_2$ and $p = 2$.

Write $\Pi|_H = \Pi_1 \oplus \ldots \oplus \Pi_r$, with irreducible H-representations Π_i. By Prop. 1, the irreducible representations Π_i are permuted by the action of G, and they must hence be all infinite-dimensional. Therefore, the representation $(\overline{\Pi})^{\mathrm{ss}}|_H$ must have at least r infinite-dimensional irreducible constituents. Then we analyze reducibility cases, and conclude that $r \leq 2$.

Theorem 8

Let $E/\mathbb{Q}_p$ be a finite extension, and let $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ be an absolutely irreducible de Rham representation with distinct Hodge-Tate weights, which we assume to be 0 and 1 if ψ is not trianguline. Let $\Pi = \Pi(\psi)$ be the absolutely irreducible p-adic Banach representation of G associated to ψ by the p-adic Langlands correspondence, and let $\Pi|_{SL_2(\mathbb{Q}_p)} = \Pi_1 \oplus \ldots \oplus \Pi_r$ be the decomposition into (topologically) irreducible representations of H. Denote by $S_{\overline{\psi}}$ the centralizer in $PGL_2(\overline{E})$ of the image of the associated projective Galois $\overline{\psi}$.

(i) $r \leq 2$ *and* $|S_{\overline{\psi}}| \leq 2$.

(ii) *If ψ is trianguline, then $r = |S_{\overline{\psi}}|$.*

(iii) *If ψ is not trianguline, then $r \geq |S_{\overline{\psi}}|$.*

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