ON TRANSVERSALITY IN FLAG MANIFOLDS AND LINEARITY OF AMALGAMS

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Abstract. We show that the fundamental group of the double of a complete negatively curved locally symmetric manifold along a closed geodesic is linear. More generally, we establish linearity of doubles of torsion-free *transverse* subgroups (also known in the literature as *regular antipodal* subgroups) of semisimple Lie groups along biproximal maximal cyclic subgroups.

Given a subgroup Δ of a group Γ, we will refer to the amalgam $\Gamma *_\Delta \Gamma$ induced by the inclusion into either factor as the *double of* Γ *along* Δ. If Γ is the fundamental group of a connected manifold M and Δ the fundamental group of a π_1-injective properly immersed connected submanifold N of M, then the double $\Gamma *_\Delta \Gamma$ is nothing but the fundamental group of the *double of* M *along* N, where the latter is analogously defined as a graph of spaces. In this note, we prove the following.

Theorem 1. *The fundamental group of the double of a negatively curved locally symmetric manifold*[1] M *along a primitive closed geodesic in* M *admits a faithful linear representation over* $\mathbb{R}$ *whose dimension depends only on that of* M.

The negative curvature assumption on M in Theorem 1 is indispensable. Indeed, as shown in [27], it follows from Margulis's superrigidity theorem [21] and work of Prasad–Rapinchuk [25] that if M is instead taken to be an irreducible finite-volume quotient of a *higher-rank* symmetric space of noncompact type, then the fundamental group of the double of M along a generic primitive closed geodesic will fail to admit a faithful finite-dimensional linear representation over any field. On the other hand, as already noted in [22, §3] in the arithmetic case, for any locally symmetric space M of noncompact type with finitely generated fundamental group, the fundamental group of the double of M along a closed geodesic, primitive or otherwise, is at the very least residually finite; see [28] and the references [2, 5, 14] therein.

In the case that M has *constant* negative curvature, i.e., the case that M is real hyperbolic, Theorem 1 follows from [9, Thm. 1.7].[2] In this setting, it moreover follows from work of Baker and Cooper [1] that if c is a primitive closed geodesic in M with trivial holonomy, then there is a finite cover $\hat{M}$ of M to which c lifts isometrically and such that the double of $\hat{M}$ along c is homotopy equivalent to a real hyperbolic manifold whose dimension depends only on that of M.

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[1] All locally symmetric spaces in this note are assumed complete, though not necessarily of finite volume.

[2] Theorem 1.7 in [9] ceases to apply when M is of variable curvature; indeed, any irreducible symmetric space containing a one-dimensional reflective submanifold must also contain a totally geodesic hypersurface and must thus be of constant curvature [17, 23].

Note however that, since the octonionic hyperbolic plane does not embed as a positive-codimension totally geodesic submanifold of any negatively curved symmetric space [29], it follows from the superrigidity theorem of Corlette [6] that the double of a finite-volume octonionic hyperbolic surface along a properly immersed totally geodesic submanifold (of positive codimension) will fail to be homotopy equivalent to any negatively curved locally symmetric space. It was shown in [27] that the latter also holds for doubles of compact quaternionic hyperbolic ($\geqslant 3$)-folds along closed geodesics. By a similar argument, the same will hold for the double of a finite-volume quaternionic hyperbolic ($\geqslant 3$)-fold along a closed geodesic c if c is taken within a properly immersed totally geodesic quaternionic curve.[3]

On the other hand, by the combination theorems of Bestvina–Feighn [3] and Dahmani [7], the fundamental group of any double D of a locally symmetric space M as in the previous paragraph along a primitive closed geodesic c will nevertheless be intrinsically negatively curved, in the sense that such $\pi_1(D)$ are hyperbolic relative to the fundamental groups of the ends of D. If c is moreover chosen so that the (pushforward of the) normalized length measure on c closely approximates the normalized volume measure on M, then $\pi_1(D)$ is thus perhaps a good candidate for a linear group that is hyperbolic relative to finitely generated nilpotent subgroups but admits no discrete embedding in any connected Lie group. Note however that, by work of the second-named author with Dey [8], in the case that M is compact, up to replacing M with a finite cover, one can always find a representation of $\pi_1(D)$ that is *Anosov* in the sense of Labourie [20] and Guichard–Wienhard [16], and is in particular discrete and faithful. As suggested to the second-named author by Hee Oh, more generally, it seems likely that if M is only assumed to be of finite volume then, up to replacing M with a finite cover, one can always find a representation of $\pi_1(D)$ that is, as defined in [18] and [31], *Anosov relative to* the fundamental groups of the ends of D.

Theorem 1 was previously established in work of the second-named author with Tholozan [27] in the case that M is convex cocompact (e.g., the case that M is compact). There, the statement followed from a more general linearity result for doubles of torsion-free Anosov subgroups of semisimple Lie groups along maximal cyclic subgroups. The approach here is similar, but Theorem 1 will instead follow from a linearity result (Theorem 2) for doubles of *transverse subgroups* in the language of Canary–Zhang–Zimmer [4].

To define the latter notion, we introduce some terminology/notation. For $n \geqslant 2$, let $\mathcal{F}_{1,n-1}$ be the space of all flags in $\mathbb{R}^n$ of the form (V, W), where V and W are linear subspaces of $\mathbb{R}^n$ of dimension 1 and $n-1$, respectively (for $n = 2$, one can understand $\mathcal{F}_{1,n-1}$ simply as $\mathbb{RP}^1$). Two flags $(V, W), (V', W') \in \mathcal{F}_{1,n-1}$ are *transverse* if both $V + W' = \mathbb{R}^n$ and $V' + W = \mathbb{R}^n$. A subset $\Lambda \subset \mathcal{F}_{1,n-1}$ is then *transverse* if any two distinct elements of Λ are transverse. A sequence $(\gamma_m)_{m\in\mathbb{N}}$ in Γ is ($P_{1,n-1}$-)*contracting* if there are flags $\phi^+, \phi^- \in \mathcal{F}_{1,n-1}$, the *attracting* and *repelling* flags of $(\gamma_m)_m$, respectively, such that γ_m converges to the constant function ϕ^+ uniformly on compact subsets of $\mathcal{F}_{1,n-1}$ consisting entirely of flags transverse to ϕ^-. A subgroup Γ of $\mathsf{SL}_n(\mathbb{R})$ is said to be (P_1-)*divergent* if each infinite sequence of distinct elements of Γ contains a contracting subsequence. The *limit set* Λ_Γ of such Γ in $\mathcal{F}_{1,n-1}$ is the closed Γ-invariant subset of $\mathcal{F}_{1,n-1}$ consisting of all attracting

[3]In light of Gromov–Schoen's arithmeticity theorem [13], such a quaternionic curve is always available; see [10, Prop. 4.1].

(equivalently, repelling) flags of contracting sequences in Γ. Note that a divergent subgroup of $\mathsf{SL}_n(\mathbb{R})$ is in particular discrete, but that, as soon as $n \geqslant 3$, discrete subgroups of $\mathsf{SL}_n(\mathbb{R})$ need not be divergent (for example, if $n \geqslant 3$, then no lattice in a Cartan subgroup of, and hence by [24] no lattice in, $\mathsf{SL}_n(\mathbb{R})$ is divergent). One says a divergent subgroup Γ of $\mathsf{SL}_n(\mathbb{R})$ is (P_1-)*transverse* if Λ_Γ is transverse. In the latter case, the action of Γ on Λ_Γ is a convergence action in the sense of Gehring and Martin [12].[4] An element $\gamma \in \mathsf{SL}_n(\mathbb{R})$ generating a transverse cyclic subgroup whose limit set consists of precisely two flags $\gamma^\pm \in \mathcal{F}_{1,n-1}$ is said to be *biproximal*. In the language of convergence actions, an infinite-order element γ of a transverse subgroup $\Gamma < \mathsf{SL}_n(\mathbb{R})$ is biproximal if and only if γ acts loxodromically on Λ_Γ.

The main result of this note is the following.

Theorem 2. *Suppose $\Gamma < \mathsf{SL}_n(\mathbb{R})$ is transverse and $\gamma \in \Gamma$ is biproximal. Let $\Delta = \mathrm{Stab}_\Gamma(\gamma^+)$. Then the double $\Gamma *_\Delta \Gamma$ embeds in $\mathsf{SL}_n(\mathbb{R})$.*

Remark 3. In the context of Theorem 2, we also have $\Delta = \mathrm{Stab}_\Gamma(\gamma^-)$, and Δ is virtually cyclic. In particular, if Γ is moreover torsion-free, then Δ is nothing but the maximal cyclic subgroup of Γ containing γ.

The above notions of transversality, contraction, divergence, limit set, and biproximality can be (and have been) extended to general reflexive flag manifolds G/P of semisimple Lie groups G; see Kapovich–Leeb–Porti [19], whose work precedes that of Canary–Zhang–Zimmer [4] (though we adopt here the terminology of the latter), and who instead refer to such subgroups as *regular antipodal*. Examples of P-transverse subgroups of such G include the relative P-Anosov subgroups [18, 31]. One can realize P-transverse subgroups of G for general pairs (G, P) as transverse subgroups of special linear groups via the following fact.

Proposition 4. ([4, Prop. B.1]) *Let G be a noncompact semisimple real algebraic group and $P < G$ a reflexive proper parabolic subgroup (considered up to conjugacy). Then there exist $n = n(G, P) \in \mathbb{N}$, a representation $\tau : G \to \mathsf{SL}_n(\mathbb{R})$, and a τ-equivariant smooth embedding $\xi : G/P \to \mathcal{F}_{1,n-1}$, such that a given subgroup $\Gamma < G$ is P-transverse if and only if $\tau(\Gamma)$ is transverse in $\mathsf{SL}_n(\mathbb{R})$, and in the latter case, one has $\Lambda_{\tau(\Gamma)} = \xi(\Lambda_\Gamma)$, where Λ_Γ denotes the limit set of Γ in G/P. In particular, an element $g \in G$ is P-biproximal if and only if $\tau(g)$ is biproximal.*

The representation τ in Proposition 4 is nothing but a so-called Plücker–Tits representation of G; see, for instance, [15, Prop. 3.3 and Lem. 3.7]. The following is then immediate from Theorem 2 and Proposition 4.

Corollary 5. *Let G be a noncompact semisimple real algebraic group and $P < G$ a reflexive proper parabolic subgroup (considered up to conjugacy). Then there exists $n = n(G, P) \in \mathbb{N}$ such that, for any torsion-free P-transverse subgroup $\Gamma < G$ and any maximal cyclic subgroup $\Delta < \Gamma$ generated by a P-biproximal element of G, the double $\Gamma *_\Delta \Gamma$ embeds in $\mathsf{SL}_n(\mathbb{R})$.*

Note that, since G contains only finitely many conjugacy classes of proper parabolic subgroups, it follows formally that the dimension n can in fact be made uniform in P.

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[4] In retrospect, the notion of a (nonelementary) convergence action was introduced earlier by Furstenberg [11]; see [30].

If G is a simple Lie group of real rank one and P is any proper parabolic subgroup of G, then G/P can be identified, as a homogeneous space for G, with the visual boundary of the symmetric space for G. In this setting, to say two elements of G/P are transverse is merely to say that they are distinct; a P-transverse subgroup Γ of G is nothing but a discrete subgroup of G; and the above notion of limit set in G/P coincides with the classical one. Theorem 1 then follows immediately from Corollary 5.

Theorem 2 will in turn be deduced from the following statement, which is inspired by techniques of Shalen [26].

Theorem 6. *Let $\Gamma_1, \Gamma_2 < \mathsf{SL}_n(K)$ and $\Delta < \Gamma_1 \cap \Gamma_2$, where K is any field. Suppose there is an integer $1 \leqslant m \leqslant \frac{n}{2}$ such that for each $h \in \Delta$, we have that h is of the form*

$$h = \begin{pmatrix} A_h & & \\ & B_h & \\ & & C_h \end{pmatrix}$$

for some $A_h, C_h \in \mathsf{GL}_m(K)$ and $B_h \in \mathsf{GL}_{n-2m}(K)$. Suppose moreover that for each $g_1 \in \Gamma_1 \smallsetminus \Delta$ the bottom-left $(m \times m)$-block of g_1 is invertible, and for each $g_2 \in \Gamma_2 \smallsetminus \Delta$, the top-right $(m \times m)$-block of g_2 is invertible. If L is a field extension of K and $t \in L$ is transcendental over K, then the subgroup $\langle \Gamma_1, a_t \Gamma_2 a_t^{-1} \rangle < \mathsf{SL}_n(L)$ decomposes as the amalgam $\Gamma_1 *_\Delta a_t \Gamma_2 a_t^{-1}$, where

$$a_t := \begin{pmatrix} tI_m & & \\ & I_{n-2m} & \\ & & t^{-1} I_m \end{pmatrix}.$$

Proof. We show that no element $g \in \mathsf{SL}_n(L)$ of the form

$$g = g_1^{(1)} g_2^{(1)} \dots g_1^{(r)} g_2^{(r)}$$

is unipotent, where $g_1^{(1)}, \dots, g_1^{(r)} \in \Gamma_1 \smallsetminus \Delta$ and $g_2^{(1)}, \dots, g_2^{(r)} \in a_t \Gamma_2 a_t^{-1} \smallsetminus \Delta$. This will suffice since any element of $\Gamma_1 *_\Delta \Gamma_2$, if not conjugate into one of the factors, is conjugate to an alternating product of even length.

Given a matrix $M \in \mathsf{M}_n(L)$, we write

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where the M_{ij} are matrices with entries in L, and $M_{22} \in \mathsf{M}_m(L)$. We say $M \in \mathsf{M}_n(L)$ is *sufficiently transcendental* if there is a nonnegative integer q such that

- $M_{11} = \ldots + t^{-1} W_{-1} + W_0 + t W_1 + \ldots + t^q W_q$ for some $W_i \in \mathsf{M}_{n-m}(K)$;
- $M_{12} = \ldots + t^{-1} X_{-1} + X_0 + t X_1 + \ldots + t^{q+1} X_{q+1}$ for some $X_i \in \mathsf{M}_{(n-m)\times m}(K)$;
- $M_{21} = \ldots + t^{-1} Y_{-1} + Y_0 + t Y_1 + \ldots + t^q Y_q$ for some $Y_i \in \mathsf{M}_{m \times (n-m)}(K)$;
- $M_{22} = \ldots + t^{-1} Z_{-1} + Z_0 + t Z_1 + \ldots + t^{q+1} Z_{q+1}$ for some $Z_i \in \mathsf{M}_m(K)$ with Z_{q+1} invertible.

Observe that since t is transcendental over K, the zero matrix is not sufficiently transcendental. Note also that the sum of a sufficiently transcendental matrix with an element of $\mathsf{M}_n(K)$ is sufficiently transcendental, and that a product of sufficiently transcendental matrices is sufficiently transcendental. In particular, by our first observation, a sufficiently transcendental matrix is not nilpotent. It follows that a sufficiently transcendental matrix M is not unipotent, since $M - \mathrm{Id}$ is sufficiently transcendental and therefore not nilpotent.

It is easy to verify that $g_1 g_2$ is sufficiently transcendental for $g_1 \in \Gamma_1 \smallsetminus \Delta$ and $g_2 \in a_t \Gamma_2 a_t^{-1} \smallsetminus \Delta$, so that g is sufficiently transcendental and therefore not unipotent by the previous discussion. □

Proof of Theorem 2. Up to conjugating Γ within $\mathsf{SL}_n(\mathbb{R})$, we have

$$\gamma^+ = (e_1, \operatorname{span}_{\mathbb{R}}\{e_1, \ldots, e_{n-1}\}),$$
$$\gamma^- = (e_n, \operatorname{span}_{\mathbb{R}}\{e_2, \ldots, e_n\}),$$

where $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. As $\Delta = \operatorname{Stab}_\Gamma(\gamma^-)$, we then have that any $h \in \Delta$ is of the form

$$h = \begin{pmatrix} a & & \\ & B & \\ & & c \end{pmatrix}$$

for some $a, c \in \mathbb{R}^*$ and $B \in \mathsf{GL}_{n-2}(\mathbb{R})$. Since Γ is discrete in $\mathsf{SL}_n(\mathbb{R})$ and hence countable, the entry field K of Γ is countable, so that $\mathbb{R}$ is a transcendental extension of K. By Theorem 6 (with $m = 1$), it thus suffices to show that, given $g \in \Gamma \smallsetminus \Delta$, both the bottom-left and top-right entries of g are nonzero. Indeed, since $g\gamma^+ \in \Lambda_\Gamma \smallsetminus \{\gamma^+\}$, and since Λ_Γ is transverse by assumption, we have that γ^+ and $g\gamma^+$ are transverse, and hence that $ge_1 \notin \langle e_1, \ldots, e_{n-1}\rangle$. In other words, the bottom-left entry of g is nonzero. Since we also have $\Delta = \operatorname{Stab}_\Gamma(\gamma^-)$, an identical argument shows that the top-right entry of g is likewise nonzero. □

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