SYSTOLES OF HYPERBOLIC HYBRIDS

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ABSTRACT. We exhibit closed hyperbolic manifolds with arbitrarily small systole in each dimension that are not quasi-arithmetic in the sense of Vinberg, and are thus not commensurable to those constructed by Agol, Belolipetsky–Thomson, and Bergeron–Haglund–Wise. This is done by taking hybrids of the manifolds constructed by the latter authors.

Given a lattice Γ in $\mathrm{Isom}(\mathbb{H}^n)$, the *systole* of the hyperbolic orbifold $M = \Gamma\backslash\mathbb{H}^n$ is the minimal length of a closed geodesic in M, i.e., the minimal translation length of a loxodromic element in Γ. Using the geometry of hyperbolic right-angled polygons, one can easily construct for each sufficiently small $\epsilon > 0$ a closed oriented hyperbolic surface of genus 2 and systole precisely ϵ. By Mostow rigidity, the topology of a closed hyperbolic n-manifold determines the systole as soon as $n \geq 3$; in particular, the systoles of such manifolds take on only countably many values. It is nevertheless true that for every $n \geq 3$ and every $\epsilon > 0$, there is a closed hyperbolic n-manifold of systole $< \epsilon$. For $n = 3$, this follows from Thurston's theory of hyperbolic Dehn filling.[1] No such tool is available in dimension 4, and it was in this dimension that Agol [1] provided a strategy for producing closed hyperbolic manifolds of arbitrarily small systole by "inbreeding" arithmetic manifolds. The separability arguments required to implement Agol's strategy in every dimension were later supplied independently by Belolipetsky–Thomson [2] and Bergeron–Haglund–Wise [3].

An interesting feature of the manifolds arising from this "inbreeding" construction is that they are all quasi-arithmetic in the sense of Vinberg [14], as observed by Thomson [13]. If $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is a lattice, then Γ is said to be *quasi-arithmetic* if there is a totally real number field $k \subset \mathbb{R}$ and a k-group $\mathbf{G}$ such that $\mathbf{G}(\mathbb{R})$ is isogenous to $\mathrm{Isom}(\mathbb{H}^n)$ with Γ virtually contained in $\mathbf{G}(k)$ via this isogeny, but $\mathbf{G}^\sigma(\mathbb{R})$ is compact for every embedding $\sigma : k \to \mathbb{R}$ distinct from the inclusion $k \subset \mathbb{R}$. In this case, the adjoint trace field of Γ coincides with k, and Γ is arithmetic if and only if $\mathrm{tr}\,\mathrm{Ad}(\gamma)$ is an algebraic integer for each $\gamma \in \Gamma$.

The purpose of this note is to demonstrate the following.

Theorem 1. *For each $n \geq 2$ and $\epsilon > 0$, there is a closed hyperbolic n-manifold of systole $< \epsilon$ that is not quasi-arithmetic.*

It follows that the manifolds M with small systole that we exhibit below are not commensurable to those constructed by Agol, Belolipetsky–Thomson, and Bergeron–Haglund–Wise. These M are nevertheless pseudo-arithmetic in the sense of Emery and Mila [6], and hence we do not address the question of the latter authors as to whether every lattice in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 4$ is pseudo-arithmetic.

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[1]In retrospect, there is a more elementary construction again involving hyperbolic right-angled polyhedra [4], though this also can ultimately be conceptualized as resulting from "Dehn filling" of Coxeter 3-polyhedra.

Proof of Theorem 1. Set $k = \mathbb{Q}(\sqrt{2})$, and let $a \in \mathbb{Q}_{>0}$ be a non-square in k. For $n \geq 2$, let f_1 and f_2 be the quadratic forms in $n+1$ variables with coefficients in k given by

$$x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2,$$
$$ax_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2,$$

respectively. For $i = 1, 2$, denote by $\mathbb{H}_{f_i}$ the f_i-hyperboloid model of $\mathbb{H}^n$, and by $\mathrm{O}'(f_i; \mathbb{R}) = \mathrm{Isom}(\mathbb{H}_{f_i})$ the index-2 subgroup of $\mathrm{O}(f_i; \mathbb{R})$ preserving $\mathbb{H}_{f_i}$. Let A_i be the 1-dimensional subgroup of $\mathrm{O}'(f_i; \mathbb{R})$ consisting of all matrices of the form

$$\begin{pmatrix} * & & * \\ & I_{n-1} & \\ * & & * \end{pmatrix}.$$

Since the k-points of A_i are dense in the identity component A_i° of A_i, we can find $g_i \in A_i^\circ$ with entries in k such that the leading eigenvalue λ_i of g_i satisfies $\log(\lambda_i) < \epsilon/4$. Denote by $H_i \subset \mathbb{H}_{f_i}$ the geodesic hyperplane $\{x_1 = 0\}$. Belolipetsky and Thomson [2] show that there is an ideal $I \subset \mathbb{Z}[\sqrt{2}]$ such that for $i = 1, 2$, the hyperplanes H_i and $g_i H_i$ project to disjoint embedded 2-sided totally geodesic hypersurfaces Σ_i and Σ_i', respectively, in the orbifold $M_i := \Gamma_i(I)\backslash\mathbb{H}_{f_i}$, where $\Gamma_i(I)$ denotes the principal congruence subgroup of level I in $\Gamma_i := \mathrm{O}'(f_i; \mathbb{Z}[\sqrt{2}])$. Up to diminishing I, we can assume moreover that the $\Gamma_i(I)$ are torsion-free, so that the M_i are manifolds, and that the projection ω_i to M_i of the orthogeodesic $\tilde{\omega}_i \subset \mathbb{H}_{f_i}$ joining the hyperplanes H_i and $g_i H_i$ does not cross Σ_i or Σ_i'. Let N_i be the component containing ω_i of the hyperbolic manifold with totally geodesic boundary obtained by cutting M_i along Σ_i and Σ_i'. Identify Σ_i and Σ_i' with the two boundary components of N_i joined by ω_i. Finally, let M be the manifold obtained by first gluing the N_i along the Σ_i via a canonical isometry in the language of Mila [10], so that the ω_i glue to an orthogeodesic ω joining the Σ_i' of length $\log(\lambda_1) + \log(\lambda_2)$, and then doubling the resulting manifold along its boundary; see Figure 1. Then M contains a closed geodesic of length $2(\log(\lambda_1) + \log(\lambda_2)) < \epsilon$, namely, the double of ω. To see that M is not quasi-arithmetic, one can for instance use the fact that a Zariski-dense subgroup of a quasi-arithmetic lattice has the same adjoint trace field as the lattice itself; see [5, Cor. 2]. In particular, since the $\pi_1(N_i)$ are Zariski-dense (see [8, Lem. 1.7.A]), they have adjoint trace field k. On the other hand, as explained in [10], the adjoint trace field of $\pi_1(M)$ is $k(\sqrt{a}) \neq k$. Since $\pi_1(N_1) \subset \pi_1(M)$, we conclude that $\pi_1(M)$ is not quasi-arithmetic. $\qquad\square$

Remark 2. We define a *classical hybrid* to be any closed hyperbolic orbifold V obtained by gluing two (possibly disconnected) compact hyperbolic orbifolds O_1 and O_2 with totally geodesic boundary along their boundaries via isometries, such that each component of the double $2O_i$ of O_i along ∂O_i is arithmetic for $i = 1, 2$, and such that $2O_1$ is not commensurable to $2O_2$. Note that to say $2O_i$ is arithmetic is to say that there is an arithmetic lattice containing $\pi_1(O_i)$ and the reflections in the boundary components of the universal cover $\widetilde{O_i}$, so that the particular examples of nonarithmetic hyperbolic manifolds constructed by Gromov and Piatetski-Shapiro in [8, 2.8.C and 2.9] are classical hybrids in the above sense.

For fixed D and μ, there are only finitely many monic integer polynomials of degree $\leq D$ and Mahler measure $\leq \mu$. Thus, for fixed d and n, there is some $\epsilon_{d,n}$
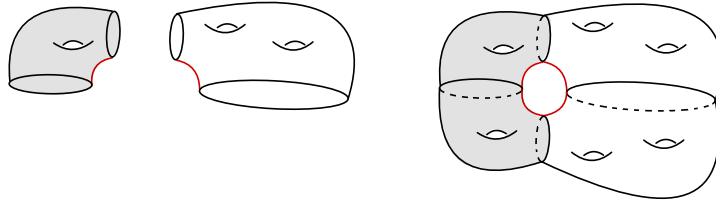
FIGURE 1. A schematic of the construction of the manifold M in the proof of Theorem 1.

such that any arithmetic hyperbolic n-orbifold with adjoint trace field of degree $\leq d$ has systole $\geq \epsilon_{d,n}$. It then follows that any classical hybrid of dimension n with adjoint trace field of degree $\leq d$ has systole $\geq \epsilon_{d,n}$. Indeed, let γ be a closed geodesic in a classical hybrid V as above. If γ is contained entirely in O_i for some $i = 1, 2$, then γ has length $\geq \text{sys}(2O_i) \geq \epsilon_{d,n}$. Otherwise,

$$\text{length}(\gamma) = \text{length}(\gamma \cap O_1) + \text{length}(\gamma \cap O_2) \geq \frac{1}{2}\text{sys}(2O_1) + \frac{1}{2}\text{sys}(2O_2) \geq \epsilon_{d,n}.$$

By [12, Lem. 3.3], for $n \geq 3$, any n-orbifold commensurable to a classical hybrid is again a classical hybrid. Since all the manifolds M as in the proof of Theorem 1 share the same adjoint trace field $k(\sqrt{a})$, we obtain from the above discussion that for $n \geq 3$ and $\epsilon < \epsilon_{4,n}$, the output n-manifold M is not commensurable to any classical hybrid in the above sense.

Remark 3. The approach in the proof of Theorem 1 can also be used to construct a lattice in $O'(n,1)$ for each $n \geq 2$ that is neither quasi-arithmetic nor has integral traces.[2] For instance, if in the proof of Theorem 1 we pick the g_i so that $\lambda_1 \lambda_2$ is not an algebraic integer (regardless of how small the $\log(\lambda_i)$ are), then $\Gamma = \pi_1(M)$ has the desired property. Indeed, by an argument similar to [9, Lem. 1.1], up to conjugation within $O'(f_1; \mathbb{R})$, we have $\Gamma \subset O'(f_1; K)$, where $K = k(\sqrt{a})$. Thus, if $\text{trAd}(g) \in \mathcal{O}_K$ for each $g \in \Gamma$, then the same is true of $\text{tr}(g)$ for each $g \in \Gamma$ by [15, Theorems 1 and 2 and Lem. 2]. On the other hand, if $\gamma \in \Gamma$ is an element representing the double of ω in M, then since γ is a pure hyperbolic element, we have that $\text{tr}(\gamma)$ is a monic palindromic Laurent polynomial in $\lambda_1 \lambda_2$ with integer coefficients, so that if $\text{tr}(\gamma)$ is an algebraic integer, then the same is true of $\lambda_1 \lambda_2$.

For instance, one can take $a = 3$ and

$$g_1 = \begin{pmatrix} 3 + 2\sqrt{2} & & 4 + 2\sqrt{2} \\ & I_{n-1} & \\ 2 + 2\sqrt{2} & & 3 + 2\sqrt{2} \end{pmatrix},$$

$$g_2 = \begin{pmatrix} \frac{11 + 6\sqrt{2}}{7} & & \frac{4 + 6\sqrt{2}}{7} \\ & I_{n-1} & \\ \frac{18 + 6\sqrt{2}}{7} & & \frac{11 + 6\sqrt{2}}{7} \end{pmatrix}.$$

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[2] Note that, even having fixed the adjoint trace field, small systoles are not enough to guarantee nonintegral traces, since the ambient groups of the relevant lattices are not admissible.

One can arrange in the proof of Theorem 1 for the double of ω to be the shortest closed geodesic in M. Indeed, a similar approach yields the following (compare, for instance, [11, Thm. 1.5]).

Theorem 4. *For every $n \geq 3$ and $m \in \mathbb{Z}_{>0}$, one can find $\epsilon_m < \frac{1}{m}$ and m pairwise incommensurable closed hyperbolic n-manifolds of the same volume each possessing a unique shortest closed geodesic of length precisely ϵ_m.*

Proof. Fix n and m. In the proof of Theorem 1, take $\epsilon \leq 2^{-m} \min\{\frac{1}{m}, \epsilon_{2,n}\}$, where $\epsilon_{2,n}$ is as in Remark 2. For $i = 1, 2$, let $p_i \in H_i$ be the point at which the axis of g_i meets H_i, and let q_i be the midpoint of the geodesic segment $\tilde{\omega}_i \subset \mathbb{H}_{f_i}$. For a point $y_i \in \mathbb{H}_{f_i}$ and a subgroup $\Delta_i < \Gamma_i$, let $D_i(\Delta_i, y_i)$ be the Dirichlet domain for Δ_i in $\mathbb{H}_{f_i}$ centered at y_i. Given an ideal $J \subset \mathbb{Z}[\sqrt{2}]$, denote by $\Delta_i(J)$ (resp., $\Delta_i'(J)$) the stabilizer in $\Gamma_i(J)$ of H_i (resp., gH_i), and let $\Phi_i(J) = \langle \Delta_i(J), \Delta_i'(J) \rangle < \Gamma_i(J)$. We pass to an ideal $J \subset I$ such that

    (i) the walls of $D_i(\Delta_i(J), p_i)$ and the walls of $D_i(\Delta_i'(J), gp_i)$ together comprise the walls of $D_i(\Phi_i(J), q_i)$;

    (ii) no wall of $D_i(\Delta_i(J), p_i)$ enters the $\epsilon_{2,n}$-neighborhood of gH_i in $\mathbb{H}_{f_i}$, and no wall of $D_i(\Delta_i'(J), gp_i)$ enters the $\epsilon_{2,n}$-neighborhood of H_i.

From (i), we have that $\Phi_i(J)$ is geometrically finite, and hence separable in $\Gamma_i(J)$ by aforementioned work of Bergeron–Haglund–Wise [3]. It follows that there is a finite-index subgroup $\Lambda_i < \Gamma_i(J)$ containing $\Phi_i(J)$ such that no wall of $D_i(\Lambda_i, q_i)$ that is not a wall of $D_i(\Phi_i(J), q_i)$ enters the $\epsilon_{2,n}$-neighborhood of $H_i \cup gH_i$. We replace M_i with the manifold $\Lambda_i \backslash \mathbb{H}_{f_i}$. Then any orthogeodesic segment in M_i with endpoints on $\Sigma_i \cup \Sigma_i'$ apart from ω_i now has length $\geq \epsilon_{2,n}$.

Let N_i' be the double of N_i across all boundary components of N_i apart from Σ_i, and $\omega_i' \subset N_i'$ the double of ω_i. Then any connected closed manifold C built out of a total of 2^m copies of the N_i' by gluing boundary components via isometries so that the copies of ω_i' match up has a single shortest closed geodesic, namely, the closed geodesic obtained by gluing up all the copies of ω_i', and this geodesic has length $< 2^m \frac{\epsilon}{2} < \frac{1}{m}$. Moreover, the systole and volume of such C depend only on the number of times each of N_1' and N_2' features in the construction of C.

Now we claim that if we set $a = 17$ in the proof of Theorem 1, then the manifolds C as above represent at least m distinct commensurability classes, even under the stipulation that N_1' and N_2' feature equally in the construction of C. Indeed, for our choice of a, Gelander and Levit [7] show that the forms f_1 and f_2 are not similar over k. Since $n \geq 3$, it then follows from work of Raimbault[3] [12] that if C_1 and C_2 are commensurable manifolds of the form C above, and α_j is the cyclic sequence of 1's and 2's dictating the decomposition of C_j into copies of N_1' and N_2' for $j = 1, 2$, then α_1 and α_2 are related by a dihedral symmetry. Now it suffices to observe that one can easily produce m cyclic sequences of 1's and 2's of length 2^m no two of which are related by such a symmetry and each of which features an equal number of 1's and 2's. $\qquad\square$

Remark 5. In the proof of Theorem 1, if there is no other orthogeodesic in $M_i = \Gamma_i(I) \backslash \mathbb{H}_{f_i}$ joining Σ_i to Σ_i' of precisely the same length as ω_i, then the argument

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[3]Raimbault's work [12] deals with manifolds constructed cyclically out of pieces of *arithmetic* manifolds the ambient groups of whose associated lattices are distinct. However, his arguments persist if "arithmetic" is weakened to "quasi-arithmetic" in the previous sentence, as observed in [5], and as is the case in our context.

of Belolipetsky and Thomson [2] (more precisely, the proof of their generalized Margulis–Vinberg lemma [2, Lem. 3.1]) in fact shows that, for any $\delta > 0$, it is possible to pass to an ideal $I_\delta \subset I \subset \mathbb{Z}[\sqrt{2}]$ such that, replacing M_i with $\Gamma_i(I_\delta)\backslash\mathbb{H}_{f_i}$, every orthogeodesic in M_i with endpoints on $\Sigma_i \cup \Sigma_i'$ apart from ω_i now has length $\geq \delta$. Taking $\delta = \epsilon_{2,n}$, one could then instead use the above congruence covers M_i in the proof of Theorem 4. It has been suggested to us by Misha Belolipetsky that this can potentially be arranged for an appropriate (indeed, generic) choice of g_i, but we have not pursued this here.

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