ON LOCAL MINIMALITY OF HAUSDORFF DIMENSION FOR ROUND LIMIT SETS

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Abstract. We address a question of Canary–Taylor [CT94, Remark 4] and Canary–Minsky–Taylor [CMT99, Conjecture 3] for sufficiently symmetric compact hyperbolic 3-orbifolds with totally geodesic boundary. We give sufficient conditions on the symmetries of such an orbifold to show that, among the complete convex cocompact structures on the interior, the structure with Fuchsian ends is a critical point (respectively, a strict local minimum) for the critical exponent. We also provide an infinite family of non-rigid examples fulfilling these conditions.

1. Introduction

Denote by $\mathrm{Isom}(\mathbb{H}^3)$ the isometry group of hyperbolic 3-space $\mathbb{H}^3$, and by $\partial_\infty\mathbb{H}^3$ the conformal sphere at infinity. The *limit set* Λ_Γ of a Kleinian group Γ, i.e., of a discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$, is the set of all accumulation points on $\partial_\infty\mathbb{H}^3$ of some (any) Γ-orbit in $\mathbb{H}^3$. Given a representation $\varphi : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$, the *limit set* Λ_φ of φ is simply $\Lambda_{\varphi(\Gamma)}$. We will say that Λ_Γ is *round* if Λ_Γ is the complement of a union of disjoint round open disks. One says Γ is *convex cocompact* if Γ acts cocompactly on the intersection of all half-spaces in $\mathbb{H}^3$ whose closures in $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ contain Λ_Γ. A representation $\varphi : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ is then said to be *convex cocompact* if φ is faithful and has convex cocompact image.

If Γ was a convex cocompact Kleinian group, then for any convex cocompact representation $\varphi : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$, the limit sets Λ_Γ and Λ_φ are φ-equivariantly homeomorphic. In particular, if the limit set of Γ was moreover round, then Λ_φ remains the complement of a union of disjoint open topological disks, but these disks may cease to be round. The following is a conjecture of Canary–Taylor [CT94] and Canary–Minsky–Taylor [CMT99].

Conjecture 1.1. *Let Γ be a convex cocompact Kleinian group with round limit set. Then, among limit sets of convex cocompact representations $\varphi : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$, those with round limit set are precisely the minimizers for the Hausdorff dimension δ_φ of Λ_φ.*

Our notation δ_φ for the Hausdorff dimension of Λ_φ alludes to the classical fact that this quantity coincides with the critical exponent of the group $\varphi(\Gamma)$ [Sul79, Theorem 7]. In the setting of Conjecture 1.1, note that if Λ_Γ is a round circle, then $\delta_\varphi \geq 1 = \delta_\Gamma$, where δ_Γ denotes the Hausdorff dimension of Λ_Γ, since the Hausdorff dimension is bounded below by the topological dimension. In this case, Conjecture 1.1 is known and is due to Bowen [Bow79].

In the remaining cases, the complement of Λ_Γ is a Schottky set, more precisely, is a union of a countably infinite collection of round open disks whose closures remain pairwise disjoint, so that Λ_Γ is topologically a Sierpiński curve. It then follows from Mostow's rigidity theorem that any convex cocompact representation

$\Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ with round limit set is conjugate[1] within $\mathrm{Isom}(\mathbb{H}^3)$ to the inclusion $\eta : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$. Thus, if we denote by $CC(\Gamma)$ the space of convex cocompact representations $\Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ considered up to conjugation, and by $H : CC(\Gamma) \to \mathbb{R}$ the function $[\varphi] \mapsto \delta_\varphi$, then Conjecture 1.1 predicts that $[\eta]$ is the unique point of $CC(\Gamma)$ at which H is minimized.

We discuss further the context of Conjecture 1.1. The authors of [CT94] and [CMT99] propose the invariant

$$D(M) := \inf_{\Gamma \cong \pi_1(M)} \mathrm{Hdim}(\Lambda_\Gamma),$$

as a measure of the topological complexity of M, where M denotes here a compact 3-orbifold with boundary whose interior admits a complete hyperbolic metric, and where the infimum is taken over all Kleinian groups $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ isomorphic to the (orbifold) fundamental group $\pi_1(M)$ of M. By a result of Bishop–Jones [BJ97, Theorem 1.2], this infimum remains unchanged if it is only taken over all geometrically finite such Γ. In the case that a convex cocompact such Γ exists, the infimum in fact need only be taken over all convex cocompact such Γ. Indeed, given a geometrically finite Γ, one can take a sequence Γ_n of Klein–Maskit combinations on each rank-1 cusp of Γ such that Γ_n strongly converges to Γ, and such that $\lim_{n\to\infty} \mathrm{Hdim}(\Lambda_{\Gamma_n}) = \mathrm{Hdim}(\Lambda_\Gamma)$ (using, for instance, [LVP23, Lemma 3.2]). By Dehn filling, one can approximate under strong convergence each Γ_n by convex cocompact groups isomorphic to Γ. Then by a diagonal argument we have that $\mathrm{Hdim}(\Lambda_\Gamma)$ belongs to the closure of $\mathrm{Hdim}(\Lambda_{(.)})$ evaluated at convex cocompact representations of Γ.

The pursuit of this infimum $D(M)$ is motivated by the fact that, for geometrically finite Kleinian groups $\Gamma \cong \pi_1(M)$, the quantity $\mathrm{Hdim}(\Lambda_\Gamma)$ is coarsely comparable (by constants depending only on the topology of M) to the volume of the 1-neighbourhood of the convex core of the quotient $\Gamma\backslash\mathbb{H}^3$, and by the fact that volume serves as a measure of topological complexity in the finite-volume case.

While we have already mentioned that $D(M) = 1$ when $\pi_1(M)$ is a surface group, the equality $D(M) = 1$ also holds for M a generalized book of I-bundles; see [CMT99]. On the other hand, as also explained in [CMT99], it follows from work of Sullivan [Sul79], Braam [Bra88], Canary–Taylor [CT94], and Bishop–Jones [BJ97] that $D(M) = 0$ in the case that M is a handlebody or thickened torus, and $D(M) > 1$ in the remaining cases, which are those for which at least one component of the incompressible core is acylindrical.

Thus, a natural next step is to describe $D(M)$ when M is acylindrical. In the deformation space of convex cocompact hyperbolic metrics on a tame acylindrical hyperbolizable 3-manifold M, there is a unique structure (up to isotopy) with Fuchsian ends, which we will denote by M_{geod}. This is precisely the convex cocompact structure for which the convex core has totally geodesic boundary, or, put another way, the limit set of whose holonomy representation is round. Moreover, as conjectured by Bonahon [Bon98] and established by Storm [Sto07], the structure M_{geod} is the global minimum for the volume of the convex core. Then as in [CMT99] one can conjecture that $D(M)$ is realized by $\mathrm{Hdim}(\Lambda_{M_{\mathrm{geod}}})$, where $\Lambda_{M_{\mathrm{geod}}}$ denotes the limit set of the holonomy representation $\pi_1(M_{\mathrm{geod}}) \to \mathrm{Isom}(\mathbb{H}^3)$.

We provide here the first non-rigid examples where Conjecture 1.1 holds locally.

[1]On the other hand, there may be *geometrically finite* representations of Γ with round limit set that are not conjugate to the inclusion.

Theorem A. *There are infinitely many pairwise incommensurable non-rigid convex cocompact acylindrical hyperbolic* 3*-orbifolds* M_{geod} *with Fuchsian ends such that* M_{geod} *is a strict local minimum of* $\mathrm{Hdim}(\Lambda_{(.)}) : CC(M_{\mathrm{geod}}) \to \mathbb{R}$*, where* $CC(M_{\mathrm{geod}})$ *denotes the space of (marked) convex cocompact hyperbolic metrics on* M_{geod}*.*

The orbifolds we exhibit are finite covers of reflection orbifolds corresponding to Coxeter polyhedra with hyperideal vertices; see §4. The strategy is to show that M_{geod} is *symmetric* (i.e., its automorphism group has no non-zero fixed vectors in the tangent space $T_{M_{\mathrm{geod}}}CC(M_{\mathrm{geod}})$) and *highly symmetric* (i.e., every $\mathbb{C}$-irreducible component of $T_{M_{\mathrm{geod}}}CC(M_{\mathrm{geod}})$ for the above action is also $\mathbb{R}$-irreducible), and then use that these properties imply that M_{geod} is a critical point and that the Hessian at M_{geod} is positive-definite, respectively.

Theorem B. *Let* M *be a compact hyperbolizable acylindrical* 3*-orbifold, and let* $M_{\mathrm{geod}} \in CC(M)$ *be the unique structure with Fuchsian ends, where* $CC(M)$ *denotes the space of (marked) convex cocompact hyperbolic metrics on* M*.*

(a) If M_{geod} *is symmetric then it is a critical point for* $\mathrm{Hdim}(\Lambda_{(.)})$*.*
(b) If M_{geod} *is highly symmetric then the Hessian of* $\mathrm{Hdim}(\Lambda_{(.)})$ *at* M_{geod} *is positive-definite. In particular* M_{geod} *is a strict local minimum of* $\mathrm{Hdim}(\Lambda_{(.)})$*.*

For more details on the notions of "symmetric" and "highly symmetric" we refer the reader to §3.

The proof of Theorem B uses the simple fact that, given a group action on a manifold and a smooth function invariant by the action, a fixed point of the action is a critical point of the function, assuming that the derivative of the action at that point has no fixed vectors. For the positivity of the Hessian of $\mathrm{Hdim}(\Lambda_{(.)})$ at M_{geod} we use the positivity of the pressure metric at Zariski-dense representations. This idea was used in [BPSW22] for particular directions. It moreover follows from the techniques of the latter authors that, in the setting of Theorem B(a), the second derivative of $\mathrm{Hdim}(\Lambda_{(.)})$ at M_{geod} is positive-definite in a subspace with at least half the real dimension of $CC(M)$.

Another observation to make the case for $D(M) = \mathrm{Hdim}(\Lambda_{M_{\mathrm{geod}}})$ comes from the fact (by applying [Bow79]) that M_{geod} minimizes the Hausdorff dimension associated to the peripheral subgroups obtained via the inclusion $\partial M \hookrightarrow M$. While this alone does not imply Conjecture 1.1 (as the limit set of each peripheral circle is meager within the entire Sierpiński curve) it is at the very least suggestive. For a recent work that can be interpreted as an attempt to minimize $\mathrm{Hdim}(\Lambda_{(.)})$, this time via lengths of given elements and in the setting of complex hyperbolic triangle groups, see Schwartz [Sch25].

The paper is organized as follows. In Section 2, we explain the basic principle that sufficiently symmetric functions are forced to have critical points and that the computation of the Hessian moreover simplifies in this case. In Section 3, we explain the notions of symmetric and highly-symmetric 3-orbifolds and show that these conditions imply that M_{geod} is a critical point for $\mathrm{Hdim}(\Lambda_{(.)})$ and has positive-definite Hessian, respectively (Theorem B). In Section 4, we present an infinite family of highly symmetric compact hyperbolizable acylindrical 3-orbifolds, completing the proof of Theorem A. Finally in Section 5, we answer in the negative a recent question of Dey–Frączyk–Hurtado ([DFH26, Question 5.7]), and provide a conjecture as to the minimizer of $D(M)$ when M is allowed to vary among all compact hyperbolizable acylindrical 3-manifolds.

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2. Criticality under symmetry

The following elementary proposition states roughly that if a smooth function Φ on a manifold X is sufficiently symmetric about a point $x_0 \in X$, then x_0 is a critical point of Φ, and the computation of the second derivative of Φ at x_0 simplifies.

Proposition 2.1. *Let X be a smooth manifold and G a finite group acting by diffeomorphisms on X. Let $x_0 \in X$ be a fixed point of G and let $\rho : G \to \mathrm{GL}_{\mathbb{R}}(T_{x_0}X)$ be the canonical linear action given by the derivative at x_0. Let $\Phi : X \to \mathbb{R}$ be a smooth G-invariant function.*

(a) If ρ has no non-zero fixed vectors, then x_0 is a critical point of Φ.
(b) If moreover V is a real irreducible component of ρ, then V is contained in an eigenspace of the Hessian $\mathrm{Hess}_{x_0}\Phi$ of Φ at x_0 with respect to any ρ-invariant Riemannian metric on X.

Proof. As G is a finite group, we can fix a metric on X such that G acts by isometries. By the chain rule, we have that

$$D_x\Phi(v) = D_{\gamma x}\Phi \circ D_x\gamma(v)$$

for $\gamma \in G$, $x \in X$, and $v \in T_xX$. In particular at $x = x_0$ we get

$$D_{x_0}\Phi(v) = D_{x_0}\Phi \circ \rho(\gamma)(v),$$

and as ρ acts by isometries, this implies that the gradient vector of Φ at x_0 is fixed by ρ. Hence under condition (a) it follows that x_0 is a critical point of Φ.

Assuming condition (b) and using that x_0 is a critical point of Φ, we can apply the chain rule once again to get that for any $\gamma \in G$ and $v, w \in T_{x_0}X$,

$$\mathrm{Hess}_{x_0}\Phi(v, w) = \mathrm{Hess}_{x_0}\Phi(\rho(\gamma)v, \rho(\gamma)w)$$

where the Hessian is calculated with respect to the chosen metric on X. As this implies that the eigenspaces of $\mathrm{Hess}_{x_0}\Phi$ are preserved by the representation ρ, we have that each real irreducible component of ρ is contained in an eigenspace of $\mathrm{Hess}_{x_0}\Phi$. □

We now present the particular space and group action relevant for our purposes. Let (M, S) be a pair consisting of an orientable hyperbolizable 3-orbifold M with non-empty boundary, together with a subset $S \subseteq \partial M$ that is a union of connected

components of ∂M. We will assume that ∂M is incompressible. One says the pair (M, S) is *relatively acylindrical* if there are no non-trivial cylinders in M with boundary in S.

Given a paired hyperbolizable orbifold (M, S) and Y a fixed (marked) conformal structure on $\partial M \setminus S$, we denote by $CC(M, S, Y)$ the space of marked convex cocompact hyperbolic metrics on M, up to isotopy, inducing the structure Y on $\partial M \setminus S$. Then the conformal boundary map

$$\Psi : CC(M, S, Y) \to \mathcal{T}(S)$$

is a homeomorphism by classical work of Ahlfors, Bers, Marden, and Sullivan, where $\mathcal{T}(S)$ denotes the Teichmüller space of S.

If the pair (M, S) is relatively acylindrical, for any choice of Y, there is a unique element of $CC(M, S, Y)$ such that the ends of M corresponding to the components of S are all Fuchsian. We denote this element by M_{geod}; here, we are suppressing the dependence on S and Y to simplify the notation.

Finally, denote by $\widehat{MCG}(M, S, Y)$ the group of isotopy classes of homeomorphisms of M preserving S and fixing the marked conformal structure Y on $\partial M \setminus S$, and denote by $MCG(M, S, Y)$ the orientation-preserving subgroup, which is of index ≤ 2. Then $\widehat{MCG}(M, S, Y)$ acts smoothly on the complex manifold $CC(M, S, Y)$ (by pulling back metrics) with global fixed point $M_{\text{geod}} \in CC(M, S, Y)$. We obtain in particular an action of $\widehat{MCG}(M, S, Y)$ on the hyperbolic manifold M_{geod} by isometries.

For acylindrical (M, S, Y) as above, we will apply Proposition 2.1 with $X = CC(M, S, Y)$, $x_0 = M_{\text{geod}}$, and $G = MCG(M, S, Y)$ (or $G = \widehat{MCG}(M, S, Y)$). The following lemma is a criterion for Proposition 2.1(a) in this setting.

Lemma 2.1. *If (M, S, Y) is acylindrical and the quotient of M_{geod} by $MCG(M, S, Y)$ is rigid, then the canonical linear representation*

$$\rho : MCG(M, S, Y) \to \mathrm{GL}_{\mathbb{C}}(T_{M_{\text{geod}}} CC(M, S, Y))$$

has no non-zero fixed vectors.

Proof. Assume that $v \in T_{M_{\text{geod}}} CC(M, S, Y)$ is a non-zero vector fixed by ρ. As any $\gamma \in MCG(M, S, Y)$ acts by isometries with respect to the Teichmüller metric of $CC(M, S, Y) \cong \mathcal{T}(S)$, this implies that the Teichmüller geodesic ray at M_{geod} tangent to v is also fixed by any $\gamma \in MCG(M, S)$. Hence the quotient of M_{geod} by $MCG(M, S)$ has a deformation space that is at least 1-dimensional, contradicting our assumption that this quotient is rigid. □

3. Hausdorff dimension of limit sets

Let (M, S) be a paired orientable hyperbolizable acylindrical 3-orbifold, and let $M_{\text{geod}} \in CC(M, S)$ be the structure with Fuchsian ends. Let $\rho : MCG(M, S) \to \mathrm{GL}_{\mathbb{C}}(T_{M_{\text{geod}}} CC(M, S))$ denote the canonical linear representation given by taking the derivative at M_{geod}. Observe that, as $MCG(M, S)$ is finite, the representation ρ splits as a direct sum of irreducible components.

- We say that M_{geod} is *symmetric* if ρ has no non-zero fixed vectors.
- Let $V \subseteq T_{M_{\text{geod}}} CC(M, S)$ be a complex irreducible component of ρ. We say that M_{geod} is *highly symmetric with respect to V* if the action of ρ on V is also irreducible when viewed as a real representation.

- We say that M_{geod} is *highly symmetric* if it is highly symmetric with respect to any complex irreducible component of ρ.

Observe that M_{geod} is symmetric if and only if ρ does not contain the trivial representation as an irreducible component.

Denote by $H : CC(M,S) \to \mathbb{R}$ the function that assigns to each $M_0 \in CC(M,S)$ the Hausdorff dimension δ_φ of the limit set Λ_φ, where $\varphi : \pi_1(M_0) \to \mathrm{PSL}_2(\mathbb{C})$ is the holonomy representation of $\pi_1(M_0)$. Note that H is smooth [BCLS15] and is invariant under the action of $\widehat{MCG}(M,S,Y)$ on $CC(M,S)$.

Theorem 3.1. *Let (M,S) be a paired orientable hyperbolizable acylindrical 3-orbifold, and let $M_{\text{geod}} \in CC(M,S)$ be the structure with Fuchsian ends.*

(a) If M_{geod} is symmetric then it is a critical point for H. Moreover, the second derivative of H at M_{geod} is positive-definite on a subspace with at least half the real dimension of $CC(M,S)$.

(b) If M_{geod} is moreover highly symmetric with respect to a complex irreducible component V of ρ, then the second derivative of H at M_{geod} is positive-definite on V.

In particular, if M_{geod} is highly symmetric then it is a strict local minimum of H.

Proof. We fix a ρ-invariant metric on $T_{M_{\text{geod}}}CC(M,S)$ that is also compatible with the complex structure. Assume that ρ has no non-zero fixed vectors. As H is a smooth function, it follows from Proposition 2.1(a) that M_{geod} is a critical point of H. Let $\eta : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ be the holonomy representation of M_{geod}, so that $M_{\text{geod}} = \eta(\pi_1(M))\backslash\mathbb{H}^3$, and let $v \in T_{M_{\text{geod}}}CC(M,S)$ be a non-zero vector. Then as η is Zariski-dense, by the thermodynamic formalism [BCLS15] we have

$$\frac{\ddot{H}_\eta(v)}{H(\eta)} + \ddot{\mathrm{I}}_\eta(v) = \mathbb{P}_\eta(v) > 0, \tag{1}$$

where $\mathbb{P}_\eta$ denotes the pressure metric at η, where $\mathrm{I}(\cdot,\cdot)$ denotes the intersection form for representations, and where $\mathrm{I}_\eta(\cdot) := \mathrm{I}(\eta,\cdot)$. As I_η is pluriharmonic, we have $\ddot{\mathrm{I}}_\eta(\iota v) = -\ddot{\mathrm{I}}_\eta(v)$. Following [BPSW22], from the latter and from (1) it follows that for any non-zero $v \in T_{M_{\text{geod}}}CC(M,S)$ we have

$$\frac{\ddot{H}_\eta(v)}{H(\eta)} + \frac{\ddot{H}_\eta(\iota v)}{H(\eta)} = \mathbb{P}_\eta(v) + \mathbb{P}_\eta(\iota v) > 0, \tag{2}$$

so at least one of $\ddot{H}_\eta(v), \ddot{H}_\eta(\iota v)$ is positive. Hence, the second derivative of H at M_{geod} is positive-definite on a subspace with at least half the real dimension of $CC(M,S)$. This follows simply because if $V \subset T_{M_{\text{geod}}}CC(M,S)$ is the maximal linear subspace where $\mathrm{Hess}_{M_{\text{geod}}} H$ is negative-semidefinite, then we must have $V \cap \iota V = \{0\}$.

Assume now that $V \subseteq T_{M_{\text{geod}}}CC(M,S)$ is a complex irreducible component of ρ that is also irreducible over $\mathbb{R}$. Then by Proposition 2.1(b) for any nonzero $v \in V$ we have that $\ddot{H}_\eta(\iota v) = \ddot{H}_\eta(v)$. Subsequently (2) implies that $\ddot{H}_\eta(v) > 0$, so that $\mathrm{Hess}_{M_{\text{geod}}} H$ is positive-definite on V. Finally, it is then clear that if each complex irreducible component of ρ is irreducible over $\mathbb{R}$, then $\mathrm{Hess}_{M_{\text{geod}}} H$ is positive-definite on all of $T_{M_{\text{geod}}}CC(M,S)$, from which it follows that M_{geod} is a strict local minimum of H. □

Remark 3.1. *Observe that in the previous argument, both $\mathbb{P}$ and $I(\cdot,\cdot)$ are invariant under $MCG(M,S)$. Hence, assuming that M is symmetric, we have that $\ddot{\mathrm{I}}_\eta(\cdot)$ vanishes on any subspace $V \subset T_{M_{\mathrm{geod}}}CC(M,S)$ with respect to which M_{geod} is highly symmetric. In particular, if M_{geod} is highly symmetric then $\ddot{\mathrm{I}}_\eta \equiv 0$ and the pressure metric $\mathbb{P}(v)$ is given by $\frac{\ddot{H}_\eta(v)}{H(\eta)}$. This mirrors what happens for quasi-Fuchsian deformations of Fuchsian subgroups, where the Weil–Petersson metric is recovered as the second derivative of the Hausdorff dimension of the limit set (see* [McM08]*).*

4. Examples of highly symmetric orbifolds

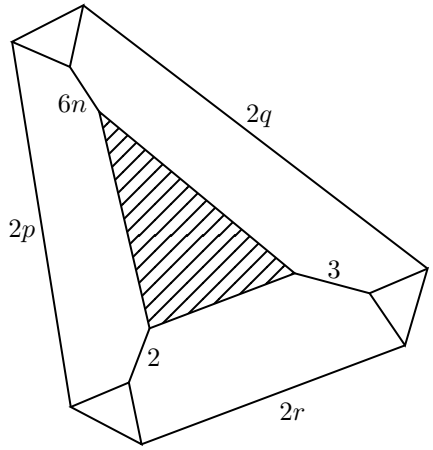

FIGURE 1. Example of a Coxeter polyhedron Π satisfying conditions (i)-(iii), stereographically projected onto the page, where the shaded face corresponds to F and edges with no labels are right-angled. Note that even for fixed $p,q,r>1$, varying n gives rise to infinitely many pairwise incommensurable orbifolds $\mathcal{O}$.

In this section, we provide examples of deformable orientable acylindrical hyperbolic 3-orbifolds with Fuchsian ends that are highly symmetric in our language, establishing Theorem A. Let $n \geq 2$ be an integer, and Π be a compact Coxeter polyhedron in $\mathbb{H}^3$ with a triangular facet F such that

(i) the three facets F_1, F_2, F_3 of Π adjacent to F are orthogonal to F;
(ii) the facet F is a $(2,3,6n)$ triangle;
(iii) the dihedral angle formed with F_i by any facet of Π adjacent to F_i apart from the F_j is an even submultiple of π for $i=1,2,3$.

The existence of (infinitely many) such polyhedra is guaranteed by Andreev's theorem [And71]; see Figure 1 for some examples. We think of the infinite-volume polyhedron in $\mathbb{H}^3$ bounded by all walls of Π except that containing the facet F as a convex cocompact hyperbolic 3-orbifold $\mathcal{O}$ with a single (Fuchsian) end corresponding to F, and whose orbifold fundamental group is the subgroup W of $\mathrm{Isom}(\mathbb{H}^3)$ generated by the reflections in the walls of this infinite-volume polyhedron. It is easy to see that both $\mathcal{O}$ and its orientation cover $\mathcal{O}^+$ are rigid. Let W_F be the subgroup of W generated by the reflections in the F_i. Then condition (iii) above

ensures that the map $r : W \to W_F$ that is the identity on W_F, and under which the reflection in each facet of Π that is not among the F_i has trivial image, is a well-defined retraction.

The $(2, 3, 6n)$-triangle 2-orbifold possesses a degree-12 regular cover T that is conformally equivalent to the modular torus but is instead compact with a single cone point of order n. Let B be the corresponding normal subgroup of W_F and let $\Gamma = r^{-1}(B) \triangleleft W$. Then $N_{\text{geod}} := \Gamma\backslash\mathbb{H}^3$ is a convex cocompact nonorientable hyperbolic 3-orbifold with a single (Fuchsian) end, and the boundary of the convex core $\text{core}(N_{\text{geod}})$ of N_{geod} is isometric to T. Let now M_{geod} be the orientation cover of N_{geod}. Then the boundary of $\text{core}(M_{\text{geod}})$ is isometric to $T \cup \overline{T}$, where $\overline{T}$ denotes T with the opposite orientation. Moreover, each orientation-preserving isometry (i.e., conformal automorphism) of $\partial\,\text{core}(M_{\text{geod}})$ induced by an element of the orientation subgroup of W_F extends to an orientation-preserving isometry of M_{geod}.

Then the quotient of M_{geod} by $MCG(M_{\text{geod}}, T \cup \overline{T})$ is finitely covered by the rigid orbifold $\mathcal{O}^+$, so by Lemma 2.1 it follows that M_{geod} is symmetric. Moreover we have the isomorphism

$$T_{M_{\text{geod}}} CC(M_{\text{geod}}, T \cup \overline{T}) \cong \mathbb{C} \oplus \mathbb{C},$$

where each $\mathbb{C}$ factor corresponds to the conformal deformations of T and $\overline{T}$, respectively. As the order-3 orientation-preserving conformal symmetries of T and $\overline{T}$ act as a $2\pi/3$ rotations on their respective deformation spaces and these symmetries extend to orientation-preserving isometries of M_{geod} by the previous paragraph, we have that M_{geod} is highly symmetric as well. Hence this family of examples fulfills all the conditions in Theorem 3.1. We summarize the above discussion in the following proposition.

Proposition 4.1. *The manifold M_{geod} obtained as above as a degree-24 cover of the reflection orbifold $\mathcal{O}$ arising from a Coxeter polyhedron in $\mathbb{H}^3$ satisfying (i), (ii), and (iii) is symmetric and highly symmetric.*

5. On the Apollonian gasket

Given a complete hyperbolic 3-manifold M, set $\Lambda_M := \Lambda_{\pi_1(M)}$, where $\pi_1(M)$ is viewed as a Kleinian group via the holonomy representation. As discussed in §1, we have that 1 is the minimal value of $\text{Hdim}(\Lambda_M)$ as M varies across all convex cocompact hyperbolic 3-manifolds with incompressible boundary, and that this minimum is achieved precisely when M_0 is Fuchsian. It is then natural to inquire about the analogous infimum when M is instead taken to vary across all convex cocompact *acylindrical* hyperbolic 3-manifolds.

In this section, we discuss the motivation for the following conjecture.

Conjecture 5.1. *We have*

$$\inf_{\substack{M \text{convex cocompact} \\ M \text{acylindrical}}} \text{Hdim}(\Lambda_M) = \text{Hdim}(\mathcal{A}),$$

where $\text{Hdim}(\mathcal{A})$ *is the Hausdorff dimension of the Apollonian gasket $\mathcal{A}$. Moreover, equality does not occur for any convex cocompact manifold M; indeed, if M is instead taken to vary across all geometrically finite acylindrical hyperbolic 3-manifolds, then the infimum is achieved precisely at the Apollonian manifolds, that*

is, at those geometrically finite manifolds whose fundamental groups are commensurable to the stabilizer of $\mathcal{A}$.

One motivation for this conjecture is as follows. Work of Canary [Can92] and Burger–Canary [BC94]implies that there exists constant $K > 0$ such that any geometrically finite hyperbolic 3-manifold N satisfies

$$\frac{K}{\mathrm{vol}(\mathrm{core}_1(N))^2} \leq 2 - \mathrm{Hdim}(\Lambda_N) \leq \frac{4\pi|\chi(\partial(C(N)))|}{\mathrm{vol}(\mathrm{core}(N))}, \tag{3}$$

where $\mathrm{core}(N)$ and $\mathrm{core}_1(N)$ denote the convex core of N and the 1-neighbourhood of the convex core of N, respectively. Thus, to minimize $\mathrm{Hdim}(\Lambda_{(\cdot)})$, one is led naturally to try to minimize $\mathrm{vol}(\mathrm{core}(N))$. The hyperbolic 3-manifolds with totally geodesic boundary of least volume were found by Miyamoto [Miy94, Theorem 5.2], where they were shown to be Apollonian. Moreover, by combining results of Miyamoto [Miy94, Theorem 4.2] and Storm [Sto07], the Apollonian manifolds are precisely the maximizers of the ratio $\frac{4\pi|\chi(\partial(C(N)))|}{\mathrm{vol}(\mathrm{core}(N))}$ among all geometrically finite acylindrical hyperbolic 3-manifolds N.

By considering the construction in Figure 2 one can show that the infimum in Conjecture 5.1 is no larger then $\mathrm{Hdim}(\mathcal{A})$, answering negatively a question of Dey–Frączyk–Hurtado [DFH26, Question 5.7].

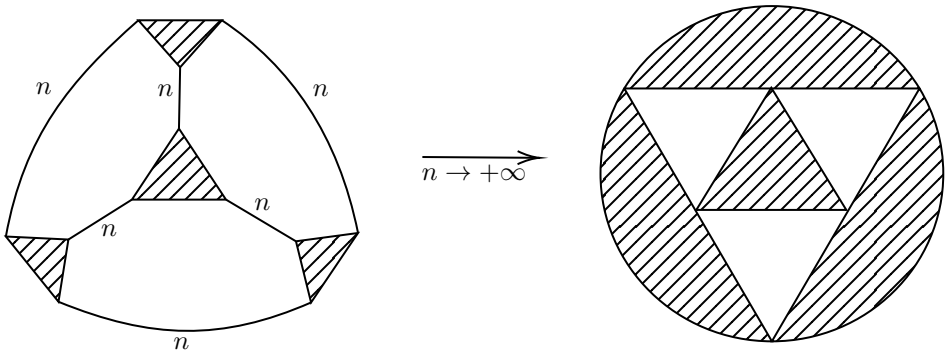

FIGURE 2. Sequences of reflection groups converging to a group with Apollonian gasket limit set.

Proposition 5.1. *We have*

$$\inf_{\substack{M\text{convex cocompact} \\ M\text{acylindrical}}} \mathrm{Hdim}(\Lambda_M) \leq \mathrm{Hdim}(\mathcal{A}).$$

Proof. Figure 2 describes a sequence of reflection groups Γ_n associated to regular truncated tetrahedra, where the label n on an edge indicates a dihedral angle of π/n, while edges without any label are right-angled. Each group Γ_n is generated by the reflections along each of the four hexagonal facets. As $n \to +\infty$ it is not hard to see that this sequence strongly converges to the group Γ generated by the reflections in alternating facets of a regular right-angled ideal octahedron. The limit set of the latter group is the Apollonian gasket $\mathcal{A}$.

As $\mathrm{Hdim}(\mathcal{A}) > 1$, we have by [McM99, Theorem 1.2] that $\lim_{n\to+\infty} \mathrm{Hdim}(\Lambda_{\Gamma_n}) = \mathrm{Hdim}(\mathcal{A})$. Note here that we have assumed the results of [McM99] apply for Kleinian groups with torsion. What follows is an explanation (following [LVP23],

for instance) as to why this is so. Our sequence consists of orbifolds with finite-degree manifold covers, and we can assume that these covers are all regular for convenience. For such orbifolds we still have the correspondence under Patterson–Sullivan theory between the first L^2 eigenvalue of the Laplace–Beltrami operator and the critical exponent of the limit set, given by

$$\lambda_0(\Gamma_n\backslash\mathbb{H}^3) = \mathrm{Hdim}(\Lambda_{\Gamma_n})(2 - \mathrm{Hdim}(\Lambda_{\Gamma_n})), \tag{4}$$

where $\lambda_0(n) = \lambda_0(\Gamma_n\backslash\mathbb{H}^3)$ is such that $-\Delta f_{0,n} = \lambda_0(n) f_{0,n}$ for $f_{0,n}$ a positive function in $\Gamma_n\backslash\mathbb{H}^3$ with $\int_{\Gamma_n\backslash\mathbb{H}^3} f_{0,n}^2 = 1$. Here, the Laplacian Δ is taken in a finite-degree regular manifold cover, where we use that the lift of $f_{0,n}$ is invariant by deck transformations, as it is the unique (up to scaling) positive L^2 integrable eigenvalue of the Laplace–Beltrami operator.

Having established this, one would like to apply [LVP23, Proposition 3.1] to show that there are constants $\epsilon, r, \eta > 0$ such that

$$\int_{B_r(C(\Gamma_n\backslash\mathbb{H}^3)^\epsilon)} f_{0,n}^2 \geq \eta,$$

where $C(\Gamma_n\backslash\mathbb{H}^3)^\epsilon$ are the points in the convex core of $\Gamma_n\backslash\mathbb{H}^3$ with injectivity radius greater than ϵ, and $B_r(.)$ denotes the r-neighborhood. The proof follows from a Rayleigh quotient estimate for a decomposition of $f_{0,n}$ by cut-off functions as in [Ham04], which uses the estimates from [Ham04, Lemma 2.1] and [Ham04, Lemma 2.3] for λ_0 along quotients of cyclic loxodromic subgroups and geometrically finite ends, respectively. But these bounds then follow for finite-degree orbifold quotients, which is our setting. For [Ham04, Lemma 2.1] one can take any loxodromic γ preserving the axis of an elliptic element to produce a positive $\left(\Delta + \frac{(n-1)^2}{4}\right)$-superharmonic function on $\langle\gamma\rangle\backslash\mathbb{H}^3$. This function lifts to a $\left(\Delta + \frac{(n-1)^2}{4}\right)$-superharmonic function that is radially symmetric with respect to the axis preserved by γ, and hence descends to a positive $\left(\Delta + \frac{(n-1)^2}{4}\right)$-superharmonic function in any elliptic quotient. Then as in the proof of [Ham04, Lemma 2.1] we can use that the bottom of the L^2-spectrum for $-\Delta$ is the top of the positive spectrum of $-\Delta$ (see [Sul87]). To establish the analogue of [Ham04, Lemma 2.3] regardless of the presence of elliptic elements one can take an equidistant foliation of the end and argue identically. Then, as in [LVP23, Lemma 3.2], from the sequence $f_{0,n}$ of L^2 integrable positive eigenfunctions of the Laplace–Beltrami operator, we can obtain as a limit a non-negative function $f_0 \in L^2(\Gamma\backslash\mathbb{H}^3)$ such that $\int_{\Gamma\backslash\mathbb{H}^3)} f_0^2 \geq \eta$ and $-\Delta f_0 = \lambda_0 f_0$ with $\lambda_0 = \lim_{n\to+\infty} \lambda_{0,n}$. Hence f_0 must be the unique (up to scaling) positive L^2 integrable eigenfunction of the Laplace–Beltrami operator for $\Gamma\backslash\mathbb{H}^3$. Then the convergence of Hausdorff dimensions follows from (4) applied to Γ_n, Γ, and the limit $\lambda_0 = \lim_{n\to+\infty} \lambda_{0,n}$.

Finally, since passing to a finite-index subgroup does not alter the limit set, any choice of torsion-free finite-index subgroup $\Gamma_n' < \Gamma$ gives rise to a convex cocompact acylindrical hyperbolic 3-manifold M_n such that the Hausdorff dimensions $\mathrm{Hdim}(\Lambda_{M_n})$ converge to $\mathrm{Hdim}(\mathcal{A})$. □

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