ON REFLECTIONS OF CONGRUENCE HYPERBOLIC MANIFOLDS

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Abstract. We show that the standard method for constructing closed hyperbolic manifolds of arbitrary dimension possessing reflective symmetries typically produces reflections whose fixed point sets are nonseparating.

Given a complete oriented Riemannian manifold M, we will call an isometric involution $\tau : M \to M$ a *reflection* if the fixed point set $\mathrm{Fix}(\tau)$ of τ is a (possibly disconnected, but nonempty) 2-sided hypersurface of M. Note that $\mathrm{Fix}(\tau)$ is then necessarily totally geodesic. For a closed nonpositively curved manifold M, denote by $\mathrm{sys}(M)$ the systole of M, that is, the length of a shortest closed geodesic in M, and by

$$h(M) := \inf_{\Omega} \frac{\mathrm{area}(\partial\Omega)}{\mathrm{vol}(\Omega)}$$

the Cheeger constant of M, where the infimum is taken over all domains $\Omega \subset M$ with rectifiable boundary and of volume at most $\frac{1}{2}\mathrm{vol}(M)$. One says a sequence $(M_i)_{i\in\mathbb{N}}$ of such manifolds M_i with $\mathrm{vol}(M_i) \to \infty$ is *expander* if there is some $\epsilon > 0$ such that $h(M_i) > \epsilon$ for all $i \in \mathbb{N}$. The purpose of this note is to observe the following.

Theorem 1. *Suppose $(M_i)_{i\in\mathbb{N}}$ is an expander sequence of closed hyperbolic manifolds M_i each admitting a reflection $\tau_i : M_i \to M_i$, and with $\mathrm{sys}(M_i) \to \infty$. Then the (possibly disconnected) hypersurface $\mathrm{Fix}(\tau_i)$ of M_i does not separate M_i for all sufficiently large $i \in \mathbb{N}$.*

While it may seem difficult to arrange for all the conditions of Theorem 1 to hold simultaneously, the standard arithmetic method for constructing closed hyperbolic manifolds admitting reflections indeed results in sequences of manifolds as in Theorem 1; see Remark 5. We felt compelled to share Theorem 1, despite its rather straightforward proof, as there appear to be misconceptions in the literature (for instance, in certain subsequent accounts of the Gromov–Thurston construction [8] of exotic negatively curved manifolds) regarding the content of Remark 5. This confusion is perhaps due to the misleading cartoons often used to depict reflections.

We also remark that it follows from superstrong approximation [6, 4] that, given any closed hyperbolic orbifold $M = \Gamma\backslash\mathbb{H}^n$, arithmetic or otherwise, there is an expander sequence of regular oriented finite manifold covers $M_i \to M$ with $\mathrm{sys}(M_i) \to \infty$. Thus, for instance, if Γ contains a reflection of $\mathbb{H}^n$, then the M_i, and the reflections τ_i induced on the M_i, are as in Theorem 1.

The proof of Theorem 1 uses the following (known) elementary estimate.

Lemma 2. *Let M be a closed hyperbolic manifold and $\Sigma \subset M$ be a (possibly disconnected) separating closed embedded totally geodesic hypersurface. If Σ has an open embedded collar of width $w > 0$ in M, then $h(M) \le \mathrm{csch}(w)$.*

Proof. The open collar $C_w := \Sigma \times (-w, w) \subset M$ of width w around Σ has metric

$$\cosh^2(t) g_\Sigma(x) + dt^2, \quad x \in \Sigma, -w < t < w,$$

where g_Σ denotes the metric on Σ and t denotes the distance to Σ. By the coarea formula, each component of $C_w \setminus \Sigma$ has volume

$$\int_0^w \cosh(t)\mathrm{area}(\Sigma) dt = \sinh(w)\mathrm{area}(\Sigma). \tag{1}$$

Hence, for each component Ω of $M \setminus \Sigma$ we have that $\frac{\mathrm{area}(\partial\Omega)}{\mathrm{vol}(\Omega)} \leq \mathrm{csch}(w)$, and the lemma follows. ☐

Proof of Theorem 1. A shortest orthogeodesic segment ω_i to $\Sigma_i := \mathrm{Fix}(\tau_i)$ in M_i has length at least $\frac{\mathrm{sys}(M_i)}{2}$, as $\omega_i \cup \tau_i(\omega_i)$ is a closed geodesic in M_i. It follows that Σ_i has an open embedded collar of width $\frac{\mathrm{sys}(M_i)}{4}$ in M_i. Now suppose (up to extraction) that Σ_i separates M_i for all $i \in \mathbb{N}$. We then obtain from Lemma 2 that $h(M_i) \to 0$, a contradiction. ☐

Remark 3. Note that the above proof in fact shows that if $h(M_i) > \epsilon$, then the conclusion of Theorem 1 holds as soon as $\mathrm{sys}(M_i) \geq 4\,\mathrm{csch}^{-1}(\epsilon)$.

Remark 4. Observe that, for its application in the proof of Theorem 1, we need only that Equation (1) in the proof of Lemma 2 provide a lower bound for the volume of a collar in terms of the area of Σ and a multiplicative factor that depends only on the width w. The latter holds more generally in the nonpositively curved setting by the Rauch–Berger comparison theorems, so that if M is a closed nonpositively curved Riemannian manifold containing a separating closed embedded totally geodesic hypersurface Σ with an open embedded collar of width $w > 0$, then $h(M) \leq w$. Similarly, Theorem 1 persists if "hyperbolic" is replaced with "nonpositively curved." In analogy with Remark 3, we have in the latter case that if $h(M_i) > \epsilon$, then the conclusion of Theorem 1 holds as soon as $\mathrm{sys}(M_i) \geq 4\epsilon^{-1}$.

Remark 5. Let $n \geq 2$, let $k \subset \mathbb{R}$ be a totally real number field with ring of integers $\mathcal{O}_k$, and let f be a quadratic form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$ with coefficients in k such that f^σ is positive definite for each nonidentity embedding $\sigma : k \to \mathbb{R}$. Suppose moreover that f is anisotropic over k (note that this is automatic if $k \neq \mathbb{Q}$). We identify the level set $\{x \in \mathbb{R}^{n+1} | f(x) = -1, x_{n+1} > 0\}$ with n-dimensional hyperbolic space $\mathbb{H}^n$ and $\mathrm{O}'_f(\mathbb{R})$ with $\mathrm{Isom}(\mathbb{H}^n)$, where $\mathrm{O}'_f(\mathbb{R})$ denotes the index-2 subgroup of $\mathrm{O}_f(\mathbb{R})$ preserving $\mathbb{H}^n$. By the Borel–Harish-Chandra theorem [3], we have that $\Gamma := \mathrm{O}'_f(\mathcal{O}_k)$ is a uniform lattice in $\mathrm{Isom}(\mathbb{H}^n)$. We suppose henceforth that Γ contains a reflection τ of $\mathbb{H}^n$ (this is true, for instance, if the form f is diagonal).

Now let $I_1 \supset I_2 \supset \ldots$ be a sequence of nonzero ideals in $\mathcal{O}_k$ with trivial intersection, and for $i \geq 1$, denote by $\Gamma(I_i)$ the principal congruence subgroup of Γ of level I_i. There is some $N > 0$ such that for all $i \geq N$, the quotient $M_i := \Gamma(I_i)\backslash\mathbb{H}^n$ is an oriented manifold, and the fixed point set $\Sigma_i := \mathrm{Fix}(\tau_i)$ of the reflection τ_i of M_i induced by τ is a (possibly disconnected) 2-sided closed embedded totally geodesic hypersurface of M_i. Since $\bigcap_i I_i = \{0\}$, we have $\mathrm{sys}(M_i) \to \infty$. Moreover, it is well known that there is some $\epsilon > 0$ such that $h(M_i) > \epsilon$ for $i \geq N$; see Burger and Sarnak [5] and the references therein. It then follows from Remark 3 that if $N_0 \geq N$ is chosen such that $\mathrm{sys}(M_{N_0}) \geq 4\,\mathrm{csch}^{-1}(\epsilon)$, then Σ_i does not separate M_i for all $i \geq N_0$.

In the notation of Remark 5, for $i \geq N_0$, let p_i be the projection $M_i \to M_{N_0}$. Given an embedded collar C around Σ_{N_0} in M_{N_0}, note that $p_i^{-1}(C)$ is an embedded collar of the same width around $p_i^{-1}(\Sigma_{N_0})$ in M_i. As in the proof of Theorem 1, it then follows again from Lemma 2 that in fact $p_i^{-1}(\Sigma_{N_0})$ does not separate M_i for $i \geq N_0$. Note that the latter is stronger than saying that Σ_i does not separate M_i for $i \geq N_0$, since $\Sigma_i \subset p_i^{-1}(\Sigma_{N_0})$. On the other hand, it is known (see the final page in the proof of [1, Thm. 4.1]) that the number of components of $p_i^{-1}(\Sigma_{N_0})$ approaches infinity as $i \to \infty$. Hence, given $r \in \mathbb{N}$, for i sufficiently large, one can express $\pi_1(M_i)$ as the fundamental group of a graph of groups whose underlying graph is a rose with $\geq r$ petals. One thus recovers the following result of Lubotzky [9].

Corollary 6. *For any $r \in \mathbb{N}$, there is some $i \geq N_0$ such that $\Gamma(I_i)$ admits an epimorphism onto a free group of rank r.*

While there are by now several known arguments for the existence of a finite-index subgroup of Γ admitting an epimorphism onto a nonabelian free group, Corollary 6 guarantees the existence of a *congruence* such subgroup, as does Lubotzky's original argument. We highlight also that Remark 5 and Corollary 6 concern the nontriviality in homology of *possibly disconnected* totally geodesic hypersurfaces of congruence hyperbolic manifolds. This distinguishes the discussion above from classical work of Millson [10] (see also Millson–Raghunathan [11]), where this question was considered for connected hypersurfaces.

We conclude with the following proposition, which shows that reflections with nonseparating fixed point sets occur quite generally.

Proposition 7. *Let M be a closed negatively curved manifold admitting a reflection τ, and suppose that $\pi_1(M)$ is residually finite. Then M admits a finite cover to which τ lifts as a reflection whose fixed point set is nonseparating.*

Proof. We may assume that $\mathrm{Fix}(\tau)$ separates M. Let M_{cut} be one of the two isometric components of the manifold with totally geodesic boundary obtained by cutting M along $\mathrm{Fix}(\tau)$, and let $\Sigma \subset M_{\mathrm{cut}}$ be a component of $\partial M_{\mathrm{cut}}$. Then $\pi_1(\Sigma)$ is separable in $\pi_1(M)$, hence in $\pi_1(M_{\mathrm{cut}})$, since $\pi_1(M)$ is residually finite and since $\pi_1(\Sigma)$ is the fixed point set of an automorphism of $\pi_1(M)$ (corresponding to a lift of τ to $\widetilde{M}$). The index of $\pi_1(\Sigma)$ in $\pi_1(M_{\mathrm{cut}})$ is moreover infinite since M is negatively curved (this can be seen, for instance, by considering the limit sets of $\pi_1(\Sigma)$ and $\pi_1(M_{\mathrm{cut}})$ on $\partial_\infty \widetilde{M}$). There is thus a finite cover $\hat{M}_{\mathrm{cut}}$ of M_{cut} to which Σ has at least 3 lifts, so that $\partial \hat{M}_{\mathrm{cut}}$ has $r \geq 3$ components. Let $\hat{M}$ be the finite cover of M obtained by doubling $\hat{M}_{\mathrm{cut}}$ across $\partial \hat{M}_{\mathrm{cut}}$. Then τ lifts to a reflection $\hat{\tau}$ of $\hat{M}$ whose fixed point set is $\partial \hat{M}_{\mathrm{cut}}$.

Given an integer $s \geq 3$, let Θ_s be the graph consisting of two vertices and s edges joining those vertices, and let ι_s be the involution of Θ_s that interchanges the vertices, and inverts each edge of, Θ_s. Then Θ_r is the dual graph to the decomposition of $\hat{M}$ along $\partial \hat{M}_{\mathrm{cut}}$, and ι_r the involution of Θ_r induced by $\hat{\tau}$. Now Θ_3 admits a degree-3 cover that is a complete bipartite graph $K_{3,3}$ and to which ι_3 lifts as an involution $\iota_{3,3}$ whose fixed point set is nonseparating. We can obtain from $K_{3,3}$ a degree-3 cover of Θ_r to which ι_r lifts as an involution whose fixed point set is nonseparating by adding $r-3$ edges joining each pair of vertices of $K_{3,3}$ interchanged by $\iota_{3,3}$. The corresponding cover of $\hat{M}$ is then as desired. $\square$

As is apparent from the proof, Proposition 7 holds under milder assumptions than negative curvature, but we chose to phrase the statement in that form given the longstanding open question of Gromov [7] as to whether indeed every Gromov-hyperbolic group is residually finite. In light of the ideas that went into the proof of the virtual Haken conjecture (see [2] and the references therein), it seems more plausible that the fundamental group of any closed negatively curved manifold is residually finite under the assumption that the manifold contains a closed embedded totally geodesic hypersurface.

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