CONVEX COCOMPACT GROUPS WITH THREE-DIMENSIONAL LIMIT SETS

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ABSTRACT. We provide a general construction of convex cocompact hyperbolic reflection groups with three-dimensional limit sets. More precisely, our construction takes as input an arbitrary simplicial complex of dimension 3 on n vertices, and outputs a convex cocompact right-angled reflection group acting on real hyperbolic n-space whose nerve is precisely the Przytycki–Świątkowski subdivision of L. Moreover, the output reflection group is a thin subgroup of an n-dimensional cocompact arithmetic hyperbolic lattice. This answers affirmatively a question of M. Kapovich concerning the existence of a convex cocompact group acting on some real hyperbolic space with limit set a Čech cohomology sphere other than the standard sphere.

1. INTRODUCTION

The Gromov boundary of a Gromov-hyperbolic group Γ is a well-defined topological invariant of Γ that is compact and metrizable [Gro87]. It is natural to ask which compact metrizable spaces arise as the boundary of some Gromov-hyperbolic group. While the possible boundaries of topological dimension one are well understood [KK00], this problem remains largely open in higher dimensions. Indeed, beyond spheres, relatively few examples of dimension > 1 are known; notable exceptions include certain Menger and Sierpiński compacta, Pontryagin surfaces, and many trees of manifolds [Dra99; Fis03; DO07; PS09; Ś20a].

A closely related question arises in the study of discrete subgroups of the isometry group $\mathrm{Isom}(\mathbb{H}^n)$ of real hyperbolic n-space $\mathbb{H}^n$. The action of such a subgroup Γ on $\mathbb{H}^n$ extends to the visual boundary $\partial_\infty \mathbb{H}^n$ of $\mathbb{H}^n$, namely, the conformal sphere $\mathbb{S}^{n-1}$, and in the case that Γ is not virtually abelian, there is a unique closed nonempty minimal Γ-invariant subset $\Lambda_\Gamma \subset \partial_\infty \mathbb{H}^n$, called the *limit set* of Γ. The limit set is a fundamental object of study, as it encodes key geometric and dynamical information about the group Γ. In particular, if Γ is convex cocompact, then Γ is Gromov-hyperbolic, and the Gromov boundary of Γ is equivariantly homeomorphic to Λ_Γ [Kap09; Sul79]. Despite considerable interest, the possible topological types of limit sets remain poorly understood; for work in this direction, see, for instance, [Bou97; Dou+26; Ma26; MZ25].

In the present work, we address the following question of M. Kapovich.

Question 1.1. [Kap08, Question 9.4] *Is there a convex cocompact subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$ *for some dimension n whose limit set is a Čech cohomology sphere but is not homeomorphic to a sphere?*

We remark that, if $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is convex cocompact, then Λ_Γ is a Čech cohomology sphere if and only if Γ is a virtual Poincaré duality group [BM91]. In this paper, we resolve Question 1.1 affirmatively by providing a general construction of convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ with 2- and 3-dimensional limit sets.

It is known that an abstract right-angled Coxeter group is Gromov-hyperbolic if and only if its nerve is so-called *flag-no-square*; see §2.2 for definitions. Dranishnikov [Dra97] introduced a procedure that takes as input an arbitrary simplicial 2-complex L and outputs a flag-no-square subdivision $L^\#$ of L. This procedure was generalized to dimension $\leqslant 3$ by Przytycki and Świątkowski [PS09]; we will continue to denote the Przytycki–Świątkowski subdivision of a simplicial 3-complex L by $L^\#$. Our main result is as follows.

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Theorem 1.2. *For each $n \geqslant 4$, there is a cocompact arithmetic lattice $\Delta_n < \mathrm{Isom}(\mathbb{H}^n)$ such that, for any simplicial complex L on n vertices and of dimension $d \leqslant 3$, the lattice Δ_n contains a right-angled reflection subgroup Γ_L whose nerve is $L^\#$. Moreover, if $d \geqslant 1$, then Γ_L can be chosen to be Zariski-dense*[1] *in* $\mathrm{Isom}(\mathbb{H}^n)$.

We remark that, for $n \leqslant 3$, we have that L is a subcomplex of the 2-simplex, so that $L^\#$ is a full subcomplex of the icosahedron. Thus, in this case, there is a reflection subgroup with nerve $L^\#$ inside the reflection group associated to the right-angled dodecahedron in $\mathbb{H}^3$.

Note that a finitely generated reflection group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is geometrically finite, so that if Γ is moreover contained in a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n)$, then Γ is automatically convex cocompact. Nevertheless, it will be clear from our argument that the reflection groups we construct are convex cocompact, using the fact that a group generated by the reflections in the walls of a finite-sided right-angled hyperbolic polyhedron P is convex cocompact if and only if no two walls of P are asymptotic (see [DH13, Theorem 4.7]). In particular, we recover Przytycki–Świątkowski's result [PS09, Proposition 2.13] that their subdivision $L^\#$ is flag-no-square.

Remark 1.3. Any subgroup $\Gamma_L < \mathrm{Isom}(\mathbb{H}^n)$ as in Theorem 1.2 is necessarily of infinite covolume, so that the subgroups $\Gamma_L < \Delta_n$ we construct are *thin* in the sense of Sarnak (see [BO14]). Indeed, if such a subgroup Γ_L were to be cocompact in $\mathrm{Isom}(\mathbb{H}^n)$, then, since the boundary of a compact convex polyhedron in $\mathbb{H}^n$ is topologically an $(n-1)$-sphere S^{n-1}, the complex L would have to be an n-vertex triangulation of S^{n-1}, and no such triangulation exists.

The following are applications of Theorem 1.2.

Corollary 1.4. *Let $d \leqslant 3$ and let N be a closed connected d-manifold admitting a triangulation with n vertices. Then there is a convex cocompact reflection group in* $\mathrm{Isom}(\mathbb{H}^n)$ *whose limit set is homeomorphic to the tree of manifolds $\mathscr{X}(N)$ (if N is nonorientable) or $\mathscr{X}(N\#\overline{N})$ (if N is orientable).*

See §2.3.1 for details on trees of manifolds (also known as *Jakobsche spaces*). In particular, using triangulations of the sphere S^d with n vertices, one obtains thin convex cocompact reflection groups acting on $\mathbb{H}^n$ with limit set S^d. On the other hand, using triangulations of the real projective plane, the torus, and the Poincaré homology sphere, one obtains the following limit sets. For Item (**3.**) in Corollary 1.5, we use a triangulation of the Poincaré homology 3-sphere with 16 vertices due to Björner and Lutz [BL00], which is the triangulation with the smallest number of vertices known to the authors.

Corollary 1.5. *There are convex cocompact reflection groups*

1. *in* $\mathrm{Isom}(\mathbb{H}^6)$ *whose limit set is the Pontryagin surface Π_2;*
2. *in* $\mathrm{Isom}(\mathbb{H}^7)$ *whose limit set is the orientable Pontryagin sphere;*
3. *in* $\mathrm{Isom}(\mathbb{H}^{16})$ *whose limit set is a Čech cohomology 3-sphere not homeomorphic to S^3.*

Ma–Zheng [MZ25] previously constructed convex cocompact reflection groups in $\mathrm{Isom}(\mathbb{H}^5)$ whose limit sets are the Pontryagin surfaces Π_2 and Π_3. In [Dou+26], the authors exhibited a convex cocompact reflection group in $\mathrm{Isom}(\mathbb{H}^4)$ whose limit set is the orientable Pontryagin sphere.

Remark 1.6. From Corollary 1.4, one in fact obtains infinitely many pairwise non-homeomorphic Čech cohomology 3-spheres as limit sets of convex cocompact groups. This can for instance be achieved by letting N vary among the Brieskorn homology 3-spheres; see [Jak91, §11] and [Mil75].

For the next result, we use triangulations of the p-fold dunce hat; see §2.3.2 and Figure 3.

Corollary 1.7. *For each prime $p \geqslant 2$, there is a convex cocompact reflection group in* $\mathrm{Isom}(\mathbb{H}^n)$ *with $n = \lceil \frac{3p}{2} \rceil + 3$ whose limit set is a Pontryagin surface Π_p.*

[1] It is not difficult to see that the condition $d \geqslant 1$ is also necessary here.

We also show that many of the groups obtained in Corollary 1.4 admit convex cocompact reflection subgroups with limit set a Menger curve; see [Dou+26, Proposition 5.1] and Proposition 2.4.

We remark that the dimension n in the construction of Theorem 1.2 has no reason to be optimal. For example, for $d \leqslant 3$, the minimal number of vertices in a triangulation of the d-sphere is $d+2$, and so Theorem 1.2 produces a subgroup of $\mathrm{Isom}(\mathbb{H}^{d+2})$ whose limit set is a d-sphere within $\partial_\infty \mathbb{H}^{d+2}$. However, any cocompact lattice in $\mathrm{Isom}(\mathbb{H}^{d+1})$, which in these dimensions may be taken to be a right-angled reflection group, has limit set S^d.

Similarly, we expect that there is a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ for some integer $d < 16$ whose limit set is a Čech cohomology sphere not homeomorphic to the standard sphere. Indeed, each homology 3-sphere admits a locally flat embedding in S^4 by a result of Freedman [Fre82, Theorem 1.4]. Hence, using [Jak91, Example 9.1], one obtains that the Čech cohomology 3-sphere given by Corollary 1.5 embeds in S^4. A natural question for future research is the following.

Question 1.8. *Let $X \subseteq \mathbb{S}^d = \partial_\infty \mathbb{H}^{d+1}$ and suppose there is a convex cocompact subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$ *for some integer $n \geqslant 4$ with limit set homeomorphic to X. Is there a convex cocompact subgroup of* $\mathrm{Isom}(\mathbb{H}^{d+1})$ *with limit set homeomorphic to X?*

Note that if Γ is a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ that does not split over virtually cyclic subgroups, then the topological type of Λ_Γ is well understood. Namely, the limit set Λ_Γ must either be a circle, a Sierpiński carpet, or S^2 itself (see [KK00]). For a survey on existing knowledge regarding which Gromov-hyperbolic (Coxeter) groups can be realized as convex cocompact (reflection) subgroups of $\mathrm{Isom}(\mathbb{H}^n)$, we refer the reader to [Dav24, §4.2.8].

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2. Preliminaries

2.1. **Simplicial complexes and the subdivisions of Dranishnikov and Przytycki–Świątkowski.**

Let L be a finite simplicial complex. We will use the following terminology and notation:

- The *d-skeleton* of L is denoted by $L^{(d)}$.
- If $K \subseteq L$ is a subcomplex, we say that K is *full* (or *induced*) if whenever $d+1$ vertices of K span a d-simplex in L, they also span a d-simplex in K.
- L is *flag* if any collection of $d+1$ pairwise adjacent vertices spans a d-simplex in L. The complex L is *flag-no-square* if L is flag and has no induced squares (i.e., every square has a chord). A full subcomplex of a flag (respectively, flag-no-square) complex is flag (respectively, flag-no-square).

Dranishnikov [Dra99] introduced a subdivision procedure to turn every 2-dimensional simplicial complex into a 2-dimensional flag-no-square simplicial complex. This procedure consists of adding a midpoint to every edge and then subdividing each triangle as shown in Figure 1. Note that the complex on the right-hand side of the picture can be obtained by fixing a triangle τ in the standard icosahedron I and then taking the full subcomplex of I spanned by all the vertices not contained in τ. If L is 2-dimensional, then we denote by $L^\#$ the flag-no-square complex obtained by applying this subdivision procedure.

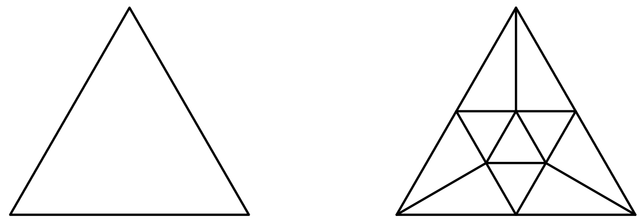

FIGURE 1. Dranishnikov's subdivision procedure for 2-dimensional simplicial complexes.

Dranishnikov's subdivision was generalized by Przytycki–Świątkowski [PS09] for simplicial complexes of dimension $\leqslant 3$. In the latter procedure, one applies Dranishnikov's subdivision to the 2-skeleton and then subdivides each tetrahedron into a copy of the 3-dimensional complex obtained as follows: Fix a tetrahedron τ in the 600-cell C_{600} and then take the full subcomplex of C_{600} spanned by all the vertices not contained in τ. If L is 3-dimensional, then we continue to denote the Przytycki–Świątkowski subdivision of L by $L^\#$.

Note that both the icosahedron I and the 600-cell C_{600} are dual to compact right-angled polyhedra in real hyperbolic space, namely, the dodecahedron and the 120-cell respectively (note such polyhedra cease to exist in dimensions higher than 4; see [PV05]). The Gromov hyperbolicity of the associated reflection groups is manifested in the fact that I and C_{600} are flag-no-square; see §2.2. In the language of right-angled Coxeter groups (RACGs) introduced below, one can think of these subdivision procedures as a way to "hyperbolize" given RACGs using classical real hyperbolic right-angled reflection groups to produce (abstract) Gromov-hyperbolic RACGs. Our goal will be to realize the output groups geometrically as concrete reflection groups acting on $\mathbb{H}^n$.

2.2. **Right-angled Coxeter groups.** Given a flag simplicial complex L, the *right-angled Coxeter group* (RACG) defined by L is the group W_L generated by one involution for each vertex of L, with two generators commuting precisely when the corresponding vertices are adjacent. In other words, the group W_L is given by the presentation

$$W_L = \langle s \in L^{(0)} \mid s^2 = 1,\, [s,t] = 1 \text{ for all edges } (s,t) \text{ of } L \rangle.$$

The reader is referred to [Dav08] or [Dav24] for background. The flag complex L is called the *nerve* of the group W_L (and depends only on the isomorphism type of W_L; see [Rad03]). Note that W_L is completely determined by the 1-skeleton of L. It is well-known that W_L is Gromov-hyperbolic if and only if L is flag-no-square; see [Mou88; DGK18]. Moreover, for any full subcomplex $K \subseteq L$, the subgroups generated by $\langle K^{(0)} \rangle$ is naturally isomorphic to W_K and is quasiconvex (with respect to the generating set $L^{(0)}$ for W_L). We call W_K the *standard subgroup* defined by the full subcomplex K.

The following argument is well known to experts; see [Dav02, §9] or [Dav24, Example 4.35]. We sketch it for the reader's convenience.

Lemma 2.1. *Let L be a flag-no-square triangulation of an (integral) homology 3-sphere N. Then the Gromov boundary of W_L is a Čech cohomology 3-sphere S. Moreover, we have that $S \cong S^3$ if and only if $N \cong S^3$.*

Proof. The Davis complex X_L of W_L is a 4-dimensional CAT(0) cube complex. The link of each vertex of X_L is a copy of N. The commutator subgroup W_L' of W_L acts freely and cocompactly on the Davis complex, and the quotient P is a closed 4-pseudomanifold. Each vertex of P has a neighborhood homeomorphic to the cone over N. Since N is a 3-manifold, the complement in P of the vertex set of P is a 4-manifold.

By Freedman's work, the homology 3-sphere N bounds a compact contractible 4-manifold B; see [Fre82, Theorem 1.4']. If we replace a small neighborhood of every vertex of P with a copy of B, we obtain a closed 4-manifold that is homotopy equivalent to P, so that W_L' is a 4-dimensional Poincaré duality group. It then follows from work of Bestvina–Mess [BM91, Corollary 1.3(c)] that the Gromov boundary S of W'_L is a Čech cohomology 3-sphere. Since W'_L has finite index in W_L, they share the same Gromov boundary S. Moreover, if $N \not\cong S^3$, then $S \not\cong S^3$, as proved by Davis [Dav02, Proposition 9.4]. □

The RACGs considered in this paper arise concretely as *right-angled hyperbolic reflection groups*, that is, as discrete groups generated by reflections in the codimension-1 faces of certain right-angled polyhedra in real hyperbolic n-space $\mathbb{H}^n$. The *limit set* Λ_Γ of a discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is the closed subset of $\mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$ consisting of all accumulation points of the Γ-orbit of some (any) point of $\mathbb{H}^n$. We say that Γ is *convex cocompact* if Γ acts cocompactly on some closed convex Γ-invariant subset of $\mathbb{H}^n$. If Γ is convex cocompact, then Γ is Gromov-hyperbolic as an abstract group, and its limit set Λ_Γ is equivariantly homeomorphic to the Gromov boundary of Γ. In particular, the topological type of the limit set of a convex cocompact right-angled hyperbolic reflection group Γ is entirely determined by the nerve of Γ as an abstract RACG.

2.3. **Some low-dimensional continua.** The purpose of this section is to define the continua that appear as Gromov boundaries of certain Gromov-hyperbolic RACGs whose nerve is not a sphere. (That the Coxeter groups in this discussion be Gromov-hyperbolic or right-angled is not relevant here, and one has identical statements for arbitrary finitely generated Coxeter groups and the visual boundaries of their Davis complexes. In this paper, we are however only interested in the hyperbolic and right-angled case.)

2.3.1. *Trees of manifolds.* Trees of manifolds were introduced by Jakobsche [Jak91]; see also [Ś20a; Ś20b], whose notation we follow. Roughly speaking, given a closed connected manifold N, the tree of manifolds $\mathscr{X}(N)$ is obtained as an inverse limit of connected sums of copies of N. When $N = S^d$ is a sphere, one has $\mathscr{X}(N) \cong S^d$, but when $N \not\cong S^d$, the tree of manifolds $\mathscr{X}(N)$ fails to be a manifold. For instance, if N is an orientable surface of positive genus, then $\mathscr{X}(N\#\overline{N})$ is the so-called Pontryagin sphere, and if N is a nonorientable surface, then $\mathscr{X}(N)$ is the nonorientable Pontryagin sphere.

Trees of manifolds arise as boundaries of Coxeter groups with manifold nerve as follows (see [Fis03; Ś20a]). If L is a flag-no-square triangulation of a closed connected manifold N, then the Davis complex X_L of W_L is a pseudomanifold with isolated vertex singularities, where the link of each vertex is homeomorphic to L. The idea is then to realize the Gromov boundary of W_L as the visual boundary of X_L, which can be computed as the inverse limit of an exhausting sequence of metric spheres centered at a point $o \in X_L$. Informally, as the metric sphere centered at o expands, each encounter with a vertex of X_L contributes a connected sum with the link (a copy of N) of that vertex.

Lemma 2.2 (Theorem 3.7 in [Fis03], Theorem 2 in [Ś20a])**.** *Suppose L is a flag-no-square triangulation of a closed connected manifold N. Then the Gromov boundary of W_L is homeomorphic to the tree of manifolds $\mathscr{X}(N)$ (if N is non-orientable) or $\mathscr{X}(N\#\overline{N})$ (if N is orientable).*

2.3.2. *Pontryagin surfaces.* Fix an integer $p \geqslant 2$. Let M_p be the mapping cylinder of the degree p cover $f_p : S^1 \to S^1, f(z) = z^p$, i.e.,

$$M_p = \big(([0,1] \times S^1) \sqcup S^1\big)/\sim,$$

where $\sim$ is the equivalence relation generated by $(1,x) \sim f(x)$ for all $x \in S^1$. The *boundary* of M_p is the circle given by the image of $\{0\} \times S^1$ in M_p. Note that M_2 is a Möbius band, while for $p \geqslant 3$ the space M_p is neither a surface nor a planar set. The *core* of M_p is the circle given by the image of $\{1\} \times S^1$ in M_p. Note that collapsing the core of M_p to a point results in a 2-disk.

We now define a sequence Σ_k of 2-dimensional spaces as follows: we set $\Sigma_1 = S^2$, and Σ_{k+1} is obtained by triangulating Σ_k and then replacing each 2-simplex of Σ_k with a copy of M_p, gluing along their boundaries. Collapsing the core of each copy of M_p to a point provides a natural collapsing map $q_k^{k+1} : \Sigma_{k+1} \to \Sigma_k$. This defines an inverse system, and the *Pontryagin surface* Π_p is defined as the associated inverse limit. Since any two triangulations of Σ_k admit a common refinement (see [Ran+96] and references therein), inverse limit does not depend on the triangulations chosen at each step. We note that Π_p is not a manifold, but Π_2 is the nonorientable Pontryagin sphere, i.e., the tree of manifolds $\mathscr{X}(\mathbb{RP}^2)$. According to [Dra05, Example 1.9, Theorem 1.10], the Pontryagin surfaces Π_p do not embed in $\mathbb{R}^3$.

Pontryagin surfaces appear as boundaries of Coxeter groups as follows. Let $\widehat{M}_p$ be the p-fold dunce hat, i.e., the 2-complex obtained by gluing a disk to the boundary of M_p. The space $\widehat{M}_p$ is a Moore space $M(\mathbb{Z}/p\mathbb{Z},1)$.

Lemma 2.3 (Proof of [Dra97, Corollary 3])**.** *Let $p \geqslant 2$ be prime, and let L be a flag-no-square triangulation of $\widehat{M}_p$. Then the Gromov boundary of W_L is homeomorphic to the Pontryagin surface Π_p.*

2.3.3. *Quasiconvex subgroups with limit set a Menger curve.* In [Dou+26], we showed that if L is a flag-no-square triangulation of a surface of positive genus, then W_L contains a quasiconvex subgroup with limit set homeomorphic to a Menger curve. We now prove a similar criterion for triangulations of $\widehat{M}_p$, which applies to the triangulations in Figure 3 below. The argument is very similar to that in [Dou+26, §5], to which we refer for the relevant terminology and notation.

Proposition 2.4. *Let L be a triangulation of $\widehat{M}_p$ with at least one vertex not in the core. Then $W_{L^\#}$ admits a convex cocompact subgroup with limit set a Menger curve.*

Proof. Let v be a vertex of L not in the core of $\widehat{M}_p$. Then v gives rise to a vertex of $L^\#$, which we still denote by v, whose link is a circle and consists of vertices not in the core of $\widehat{M}_p$. Consider the full subcomplex K on $(L^\#)^{(0)} \smallsetminus \{v\}$. The standard subgroup W_K is quasiconvex, and we claim the limit set of W_K is a Menger curve. By [DKŚ24, Theorem 0.1.(2)], it is enough to check that K is non-planar and inseparable, is not a join, and has puncture-respecting cohomological dimension $\mathrm{pcd}(K) = 1$.

Note that K deformation retracts to the core of $\widehat{M}_p$, hence $K \simeq S^1$. In particular, we have that K is connected. Since K is a full subcomplex of $L^\#$, we have that K is automatically flag-no-square. The complex K is also non-planar as it contains a neighborhood of the core of $\widehat{M}_p$. It follows that K cannot split as a join of two subcomplexes. Moreover, for any (possibly empty) simplex $\sigma \subseteq K$, we have that $H^k(L \smallsetminus \sigma) = 0$ for all $k \geqslant 2$. But $K \simeq S^1$, so $\mathrm{pcd}(K) = 1$. Finally, the complex K cannot be disconnected by removing a simplex or the suspension of a simplex, because K contains a neighborhood of the core of $\widehat{M}_p$. □

3. Proof of Theorem 1.2

3.1. **A particularly nice Lorentzian bilinear form.** Denote by φ the golden ratio $\frac{1+\sqrt{5}}{2}$. For $n \geqslant 2$, let B_n be the $(n+1)\times(n+1)$ symmetric matrix with entries given by

$$(B_n)_{i,j} = \begin{cases} 1 & \text{for } i = j \leqslant n \\ -\varphi & \text{otherwise.} \end{cases} \tag{3.1}$$

We first claim that B_n has signature $(n,1)$. Indeed, we can explicitly compute the eigenvalues of B_n; denoting by $e_1,\dots,e_{n+1}$ the standard basis vectors for $\mathbb{R}^{n+1}$, we have that $e_i - e_{i+1}$ is an eigenvector of B_n with eigenvalue $1+\varphi$ for $i = 1,\dots,n-1$. Moreover, the vector $\sum_{i=1}^n e_i + s_\pm e_{n+1}$ is an eigenvector of B_n with eigenvalue $1-(n-1+s_\pm)\varphi$, where

$$s_\pm = \frac{1-\varphi(n-2) \pm \sqrt{(\varphi n - 1)^2 + 4\varphi(1+\varphi)}}{2\varphi}.$$

Then $1-(n-1+s_-)\varphi$ is positive, whereas $1-(n-1+s_+)\varphi$ is negative, since

$$1-(n-1+s_\pm)\varphi = -\frac{1}{2}\left(\varphi n - 1 \pm \sqrt{(\varphi n-1)^2 + 4\varphi(1+\varphi)}\right).$$

It follows that B_n has exactly one negative eigenvalue, and hence signature $(n,1)$.

Since B_n has signature $(n,1)$, we may identify one of the two components of the level set $\{x \in \mathbb{R}^{n+1} \mid x^T B_n x = -1\}$ with n-dimensional real hyperbolic space $\mathbb{H}^n$, and $\mathrm{O}'(B_n,\mathbb{R})$ with $\mathrm{Isom}(\mathbb{H}^n)$, where $\mathrm{O}'(B_n,\mathbb{R})$ denotes the index-2 subgroup of the orthogonal group $\mathrm{O}(B_n,\mathbb{R})$ preserving $\mathbb{H}^n$. We will fix this model of $\mathbb{H}^n$ for the rest of the argument.

Note that for each spacelike vector v, the linear hyperplane of $\mathbb{R}^{n,1}$ that is orthogonal to v with respect to B_n defines an oriented geodesic hyperplane of $\mathbb{H}^n$, and therefore a halfspace of $\mathbb{H}^n$ (namely, that consisting of $x \in \mathbb{H}^n \subseteq \mathbb{R}^{n,1}$ with $x^T B_n v > 0$). Moreover, in this model if H and H' are geodesic hyperplanes of $\mathbb{H}^n$ determined by spacelike unit vectors $v, v' \in \mathbb{R}^{n+1}$, respectively, then, in the case that H intersects H', the dihedral angle θ formed by H and H' satisfies $\cos\theta = |v^T B_n v'|$, and in the case that H and H' are disjoint, the distance δ between H and H' satisfies $\cosh\delta = |v^T B_n v'|$. Note that, in the latter case, one has $v^T B_n v' < -1$ precisely when the corresponding halfspaces of $\mathbb{H}^n$ are neither nested nor disjoint.

Let $\tau : \mathbb{Q}(\sqrt{5}) \to \mathbb{Q}(\sqrt{5})$ be the nontrivial automorphism of $\mathbb{Q}(\sqrt{5})$, and denote by B_n^τ the Galois conjugate of B_n, i.e., the matrix obtained by applying τ to B_n entry by entry. Using the fact that $\tau(\varphi) = 1-\varphi$, one verifies by the above computation that B_n^τ is positive-definite. It thus follows from the Borel–Harish-Chandra theorem [BHC62; MT62] that $\Delta_n := \mathrm{O}'(B_n, \mathbb{Z}[\varphi])$ is a cocompact arithmetic lattice in $\mathrm{O}'(B_n, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^n)$.

3.2. **Passing to a larger complex.** Now fix $n \geqslant 4$, and let L be a simplicial complex of dimension $\leqslant 3$ on n vertices $v_1, \dots, v_n$. We construct a right-angled reflection subgroup Γ_L of Δ_n with nerve $L^\#$ as follows. (Here $L^\#$ denotes the complex obtained by applying to L the subdivision procedure described in §2.1.) Let L_n be the 3-skeleton of the $(n-1)$-simplex with vertices $v_1, \dots, v_n$. We may view L as a subcomplex of L_n, and therefore $L^\#$ as a subcomplex of $L_n^\#$. (Note that $L^\#$ will then necessarily be a *full* subcomplex of $L_n^\#$, since the Przytycki–Świątkowski subdivision of a simplex of dimension $\geqslant 1$ is never a simplex; see [PS09, Lemma 2.10].) We will show that Δ_n contains a right-angled reflection subgroup Γ_n whose nerve is $L_n^\#$. We then take Γ_L to be the standard subgroup of Γ_n corresponding to the subcomplex $L^\# \subset L_n^\#$.

3.3. **Generating the reflection group Γ_n.** We now explain informally how to construct Γ_n. Choose a vertex of the right-angled 120-cell $P_{120} \subset \mathbb{H}^4$, and let $P_{116} \subset \mathbb{H}^4$ be the 116-sided right-angled polyhedron obtained from P_{120} by "forgetting" the four facets of P_{120} containing that vertex. We now view P_{116} as a polyhedron in $\mathbb{H}^n$, and for each 3-simplex σ of L_n, we choose a particular translate P_σ of P_{116} within $\mathbb{H}^n$. Miraculously, there is a natural (and highly symmetric) way to choose the P_σ such that they together form a polyhedron in $\mathbb{H}^n$ whose associated reflection group Γ_n is contained in Δ_n and has nerve $L_n^\#$.

Recall n is the number of vertices of L_n. We now proceed with the formal construction of Γ_n. We will first associate to each vertex of $L_n^\#$ a spacelike unit[2] vector in $\mathbb{R}^{n+1}$. Then, we will define Γ_n to be the subgroup of $O'(n,1)$ generated by the reflections in the geodesic hyperplanes of $\mathbb{H}^n$ determined by these vectors.

The vertices of $L_n^\#$ belong to a hierarchy:

- At the zeroth level of this hierarchy are the vertices $v_1, \dots, v_n$ that already belonged to the simplicial complex L_n prior to subdivision. It is helpful to think of the v_i as those vertices v_i of $L_n^\#$ such that v_i lies in the interior of a 0-simplex of L_n. For $i = 1, \dots, n$, we associate to v_i the standard basis vector $c_i e_i$, with $c_i = 1$; see Table 1. (Notice that $e_i^T B_n e_j = -\varphi$ for $i \neq j$.)
- Next in the hierarchy are the "new" vertices w of $L_n^\#$ such that w lies in the interior of a 1-simplex of L_n. There are $\binom{n}{2}$ such vertices of $L_n^\#$ (one for each 1-simplex, and hence 6 for each 3-simplex, of L_n). We associate the vector $f_{ij} = f_{ji} := c_i e_i + c_j e_j + c_{n+1} e_{n+1}$ to the vertex w of $L_n^\#$ lying in the interior of the 1-simplex of L_n determined by v_i and v_j; for the values of the coefficients, see Table 1. (Notice that $e_m^T B_n f_{ij} = 0$ for $m = i, j$.)
- Next are the vertices w of $L_n^\#$ such that w lies in the interior of a 2-simplex of L_n. There are $3\binom{n}{3}$ such vertices of $L_n^\#$ (three for each 2-simplex, and hence 12 for each 3-simplex, of L_n). We associate the vector $f_{jk}^i = f_{kj}^i := c_i e_i + c_j e_j + c_k e_k + c_{n+1} e_{n+1}$ to the unique vertex w_i of $L_n^\#$ lying in the interior of the 2-simplex of L_n determined by v_i, v_j, v_k and adjacent to v_i in $L_n^\#$ (see Figure 2); see Table 1 for the values of the coefficients. (Notice that $e_i^T B_n f_{jk}^i$, $f_{ij}^T B_n f_{jk}^i$, and $(f_{jk}^i)^T B_n f_{ki}^j$ all vanish.)
- At the final level are the vertices w of $L_n^\#$ such that w lies in the interior of a 3-simplex σ of L_n. There are $94\binom{n}{4}$ such vertices of $L_n^\#$ (94 for each 3-simplex of L_n). The assignment of vectors $c_i e_i + c_j e_j + c_k e_k + c_\ell e_\ell + c_{n+1} e_{n+1}$ to the vertices of $L_n^\#$ lying in the interior of the 3-simplex of L_n determined by v_i, v_j, v_k and v_ℓ is again described in Table 1, but, for brevity's sake, we will not be as explicit here as in the previous steps of the hierarchy. Note that in this case there is less symmetry, so we have multiple rows

[2]These will be unit vectors with respect to the form B_n.

in Table 1 corresponding to this level of the hierarchy. We will nevertheless verify that the collection of vectors thus obtained gives rise to a reflection group with the correct nerve.

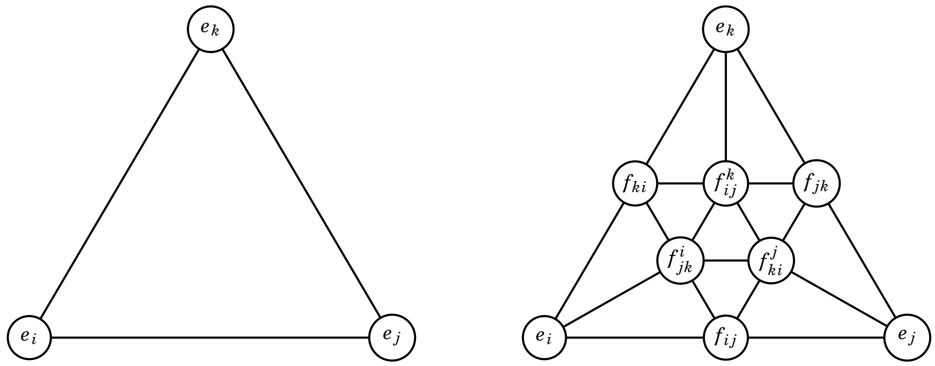

FIGURE 2. (Left) A 2-simplex σ of L_n; (Right) the subdivision $\sigma^\#$ of σ in $L_n^\#$.

Level in hierarchy	c_i	c_j	c_k	c_ℓ	c_{n+1}	sum	# of permutations of c_i, c_j, c_k, c_ℓ
0	1	0	0	0	0	1	4
1	1	1	0	0	$-(2-\varphi)$	φ	6
2	φ	1	1	0	-1	$1+\varphi$	12
3	$1+\varphi$	φ	1	1	-2	$1+2\varphi$	12
	$1+\varphi$	$1+\varphi$	φ	1	$-(1+\varphi)$	$2+2\varphi$	12
	$1+\varphi$	$1+\varphi$	$1+\varphi$	1	-3	$1+3\varphi$	4
	$2+\varphi$	$1+\varphi$	$1+\varphi$	φ	$-(2+\varphi)$	$2+3\varphi$	12
	$1+2\varphi$	$1+\varphi$	$1+\varphi$	$1+\varphi$	$-(1+2\varphi)$	$3+3\varphi$	4
	$1+2\varphi$	$2+\varphi$	$1+\varphi$	$1+\varphi$	$-(3+\varphi)$	$2+4\varphi$	12
	$1+2\varphi$	$1+2\varphi$	$2+\varphi$	$1+\varphi$	$-(2+2\varphi)$	$3+4\varphi$	12
	$2+2\varphi$	$1+2\varphi$	$1+2\varphi$	$2+\varphi$	$-(3+2\varphi)$	$3+5\varphi$	12
	$2+2\varphi$	$2+2\varphi$	$1+2\varphi$	$1+2\varphi$	$-(2+3\varphi)$	$4+5\varphi$	6
	$2+2\varphi$	$2+2\varphi$	$2+2\varphi$	$1+2\varphi$	$-(4+2\varphi)$	$3+6\varphi$	4
	$1+3\varphi$	$2+2\varphi$	$2+2\varphi$	$2+2\varphi$	$-(3+3\varphi)$	$4+6\varphi$	4

TABLE 1. The coefficients of the vectors corresponding to the vertices of $L_n^\#$. The rightmost column contains the number of vertices of each type within the subdivision of a single 3-simplex.

3.4. **The nerve of Γ_n is $L_n^\#$.** We now proceed to check that Γ_n is a right-angled reflection group with nerve $L_n^\#$. In other words, we will check that the reflections associated above to the vertices of $L_n^\#$ have an appropriate Gram matrix.

We will first consider the case of a pair of vertices of $L_n^\#$ that arise from a common 3-simplex of L_n. We will then consider the case of a pair of vertices of $L_n^\#$ arising from distinct 3-simplices of L_n.

3.4.1. *Pairs of vertices from a single 3-simplex of L_n.* Fix four distinct indices $i,j,k,\ell \in \{1,\dots,n\}$ corresponding to a single 3-simplex $\sigma_{ijk\ell}$ of L_n. Consider the following four vectors

$$x_p = -(\varphi-1)e_m + (\varphi-1)e_{n+1} \text{ for } p = 1,2,3,4,$$

corresponding to the four facets of P_{120} that we "forgot" in order to obtain P_{116}. For $p = 5,\dots,120$, let x_p be the vectors in the span of $e_i, e_j, e_k, e_\ell, e_{n+1}$ (written in that basis) assigned previously to the 116 vertices of $\sigma_{ijk\ell}^\#$.

Note that the coordinates of x_p are zero outside of $i,j,k,\ell,n+1$. We now restrict to the 5-dimensional subspace of $\mathbb{R}^{n,1}$ given by the span of $e_i,e_j,e_k,e_\ell,e_{n+1}$. Note that the matrix representing the restriction of B_n to the span of $e_i,e_j,e_k,e_\ell,e_{n+1}$, with respect to the latter basis, is the matrix B_4; see (3.1). Moreover, we have

$$g^T B_4 g = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -\varphi \end{pmatrix},$$

where

$$g = \begin{pmatrix} -\varphi & -1 & 1-\varphi & 0 & \varphi \\ -\varphi & -1 & \varphi-1 & 0 & \varphi \\ -1 & -\varphi & 0 & 1-\varphi & \varphi \\ -1 & -\varphi & 0 & \varphi-1 & \varphi \\ 2 & 2 & 0 & 0 & 1-2\varphi \end{pmatrix}.$$

Then we have $\varphi^{-1}g^{-1}x_p = (z_p,1) \in \mathbb{R}^4 \times \mathbb{R}$, where $z_p \in \mathbb{R}^4$ give the Cartesian coordinates of the vertices of a 600-cell (the combinatorial dual of the 120-cell) of unit radius, centered at the origin of Euclidean 4-space and with edges of length φ^{-1}.

More precisely, the corresponding vectors z_p in Euclidean 4-space are as follows:

- 8 vectors obtained by taking permutations of
$$(\pm 1, 0, 0, 0);$$
- 16 vectors of the form
$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right);$$
- the remaining 96 vectors, obtained by taking even permutations of
$$\left(\pm\frac{1}{2}\varphi, \pm\frac{1}{2}, \pm\frac{1}{2}\varphi^{-1}, 0\right)$$

(see Coxeter [Cox48, §8.7]).

Let z_q and z_r be vectors in $\mathbb{R}^4$ corresponding to vertices of the 600-cell, and let θ denote the angle between them. Then, using the coordinates, one gets that $\cos\theta \in \left\{1, \frac{\varphi}{2}, \frac{1}{2}, \frac{\varphi^{-1}}{2}, 0, -\frac{\varphi^{-1}}{2}, -\frac{1}{2}, -\frac{\varphi}{2}, -1\right\}$. Hence, if z_q and z_r correspond to adjacent vertices of the 600-cell, then

$$\cos\theta = \frac{\varphi}{2}.$$

Therefore,

$$x_q^T B_4 x_r = \varphi^2(\varphi^{-1}g^{-1}x_q)^T(g^T B_4 g)(\varphi^{-1}g^{-1}x_r) = \varphi^2(2\cos\theta - \varphi) = 0.$$

This implies that the vectors x_q and x_r are orthogonal with respect to the Lorentzian form B_4, so that the corresponding generators of Γ_n commute.

Moreover, if z_q and z_r are not adjacent, then the angle θ between z_q and z_r satisfies $\cos\theta \leqslant \frac{1}{2}$. It follows that in this case

$$x_q^T B_4 x_r = \varphi^2(2\cos\theta - \varphi) \leqslant -\varphi.$$

Consequently, if the vectors x_q and x_r are not orthogonal, their Lorentzian inner product satisfies

$$x_q^T B_4 x_r \leqslant -\varphi < -1.$$

This shows that the associated hyperplanes of $\mathbb{H}^n$ are disjoint, and that the associated halfspaces are neither nested nor disjoint.

We have at this point verified that, for any two vertices of $L_n^\#$ lying in a common 3-simplex of L_n, the associated vectors in $\mathbb{R}^{n+1}$ have the correct combinatorics, i.e., we have shown that the list of reflections associated above to vertices of $L_n^\#$ arising from a single 3-simplex of L_n have an appropriate Gram matrix.

3.4.2. *Pairs of vertices from distinct 3-simplices of L_n.* We will now have to consider pairs of vertices of $L_n^\#$ that do not lie in a common 3-simplex of L_n. Note that necessarily such vertices are not adjacent in $L_n^\#$, so we are going to check that the corresponding geodesic hyperplanes of $\mathbb{H}^n$ have positive distance (and, moreover, that the corresponding halfspaces of $\mathbb{H}^n$ are neither nested nor disjoint).

Let

$$x = c_i e_i + c_j e_j + c_k e_k + c_\ell e_\ell + c_{n+1} e_{n+1} \quad \text{and} \quad y = c'_{i'} e_{i'} + c'_{j'} e_{j'} + c'_{k'} e_{k'} + c'_{\ell'} e_{\ell'} + c'_{n+1} e_{n+1}$$

be the vectors corresponding to a pair of vertices v and w of $L_n^\#$, respectively. Assume that v and w do not lie in a common 3-simplex of L_n. Let σ_v (respectively, σ_w) be the simplex of L_n containing v (resp., w) in its interior.

Case 1: σ_v and σ_w do not intersect. In this case,

$$\{i,j,k,\ell\} \cap \{i',j',k',\ell'\} = \varnothing,$$

and

$$x^T B_n y = -\varphi\big(c_i + c_j + c_k + c_\ell + c_{n+1}\big)\big(c'_{i'} + c'_{j'} + c'_{k'} + c'_{\ell'} + c'_{n+1}\big) \leqslant -\varphi < -1.$$

since $c_i + c_j + c_k + c_\ell + c_{n+1} \geqslant 1$, as one can easily check in Table 1.

Case 2: The intersection of σ_v and σ_w is a 0-simplex of L_n. Up to permuting the indices, it suffices to consider the case where $i = i'$ and

$$\{j,k,\ell\} \cap \{j',k',\ell'\} = \varnothing.$$

In this case, we have

$$x^T B_n y = -\varphi\big(c_i + c_j + c_k + c_\ell + c_{n+1}\big)\big(c'_i + c'_{j'} + c'_{k'} + c'_{\ell'} + c'_{n+1}\big) + (1+\varphi) c_i c'_i. \tag{3.2}$$

If $c_i = 0$ or $c'_i = 0$, then (3.2) $\leqslant -\varphi$. Hence, we may assume that $c_i, c'_i \neq 0$, that is, $c_i, c'_i \geqslant 1$ (see Table 1). Since v and w do not lie in a common 3-simplex of L_n, and since v (respectively, w) lies in the interior of σ_v (resp., σ_w), the level of each of v and w in the hierarchy is greater than 0. Hence, one can check using Table 1 and the identity $\varphi^2 = \varphi + 1$ that

$$\varphi c_i \leqslant c_i + c_j + c_k + c_\ell + c_{n+1}.$$

It follows that (3.2) is bounded above by

$$-\varphi \cdot (\varphi c_i)(\varphi c'_i) + (1+\varphi) c_i c'_i = -\varphi c_i c'_i \leqslant -\varphi < -1.$$

Case 3: The intersection of σ_v and σ_w is a 1-simplex of L_n.

It suffices to consider the case where $i = i'$, $j = j'$, and

$$\{k,\ell\} \cap \{k',\ell'\} = \varnothing.$$

In this case, we have

$$x^T B_n y = -\varphi\big(c_i + c_j + c_k + c_\ell + c_{n+1}\big)\big(c'_i + c'_j + c'_{k'} + c'_{\ell'} + c'_{n+1}\big) + (1+\varphi)\big(c_i c'_i + c_j c'_j\big). \tag{3.3}$$

By the Cauchy–Schwarz inequality, (3.3) is bounded above by

$$-\varphi\big(c_i + c_j + c_k + c_\ell + c_{n+1}\big)\big(c'_i + c'_j + c'_{k'} + c'_{\ell'} + c'_{n+1}\big) + (1+\varphi)\sqrt{c_i^2 + c_j^2}\,\sqrt{(c'_i)^2 + (c'_j)^2}. \tag{3.4}$$

As in the previous case, since v and w do not lie in a common 3-simplex of L_n, their level in the hierarchy is greater than 1. Looking at Table 1, one sees that the worst-case scenario is that c_i, c'_i are the first coefficients; that c_j, c'_j are the second coefficients; that c_k, $c'_{k'}$ are the third coefficients; and that c_ℓ, $c'_{\ell'}$ are the last coefficients in Table 1. An exhaustive computation, provided in the SageMath file (*exhaustive-PS-subdivision-Limit-Set*.ipynb) and available on the webpage [Dou+], of the $\binom{13}{2} = 78$ possibilities then shows that

$$(3.4) \leqslant -\varphi < -1.$$

Case 4: The intersection of σ_v and σ_w is a 2-simplex of L_n.

It suffices to consider the case where $i = i'$, $j = j'$, $k = k'$, and $\ell \neq \ell'$. In this case, we have

$$x^T B_n y = -\varphi\big(c_i + c_j + c_k + c_\ell + c_{n+1}\big)\big(c'_i + c'_j + c'_k + c'_{\ell'} + c'_{n+1}\big) + (1+\varphi)\big(c_i c'_i + c_j c'_j + c_k c'_k\big). \tag{3.5}$$

Again, by the Cauchy–Schwarz inequality, (3.5) is bounded above by

$$-\varphi\big(c_i + c_j + c_k + c_\ell + c_{n+1}\big)\big(c'_i + c'_j + c'_k + c'_{\ell'} + c'_{n+1}\big) + (1+\varphi)\sqrt{c_i^2 + c_j^2 + c_k^2}\,\sqrt{(c'_i)^2 + (c'_j)^2 + (c'_k)^2}. \tag{3.6}$$

As in the previous cases, since v and w do not lie in a common 3-simplex of L_n, their level in the hierarchy is equal to 3. Looking at Table 1, one sees that the worst-case scenario is that c_i, c'_i are the first coefficients; that c_j, c'_j are the second coefficients; that c_k, $c'_{k'}$ are the third coefficients; and that c_ℓ, $c'_{\ell'}$ are the last coefficients in Table 1. An exhaustive computation, provided in the SageMath file (*exhaustive-PS-subdivision-Limit-Set*.ipynb) and available on the webpage [Dou+], of the $\binom{12}{2} = 66$ possibilities then shows that

$$(3.6) \leqslant -\varphi < -1.$$

Denoting by $x_v \in \mathbb{R}^{n+1}$ the unit vector we have associated to the vertex $v \in L_n^{\#(0)}$, we have checked that, for every pair of vertices $v, w \in L_n^{\#(0)}$, one has that $B_n(x_v, x_w) = 0$ if and only if v and w are adjacent in $L_n^{\#}$; and that, if v and w are not adjacent, then $B_n(x_v, x_w) < -1$. Hence, the group Γ_n generated by the reflections in the linear hyperplanes of $\mathbb{R}^{n+1}$ orthogonal to the x_v with respect to the form B_n is indeed a reflection group in $\mathrm{Isom}(\mathbb{H}^n)$ whose nerve is $L_n^{\#}$.

Remark 3.1. The strict inequality $B_n(x_v, x_w) < -1$ whenever v and w are not adjacent moreover shows that Γ_n is convex cocompact (see, for instance, [DH13, Theorem 4.7]). Since the reflection group Γ_n is automatically geometrically finite, convex cocompactness will also follow once we establish in §3.5 that Γ_n is a subgroup of the cocompact lattice Δ_n). We obtain in particular that Γ_n is Gromov-hyperbolic, and hence recover the result of Przytycki–Świątkowski [PS09, Proposition 2.13] that the complex $L_n^{\#}$ is flag-no-square.

3.5. Γ_n **is a subgroup of** Δ_n**.** To see that $\Gamma_n \subset \Delta_n$, note that, for each vertex v of $L_n^{\#}$ with associated unit vector $x_v \in \mathbb{R}^{n+1}$ as specified above, the reflection in the linear hyperplane of $\mathbb{R}^{n+1}$ orthogonal to x_v with respect to the form B_n is given by

$$x \longmapsto x - (x^T B_n x_v) x_v$$

for $x \in \mathbb{R}^{n+1}$, and this map has image in $\Delta_n = \mathrm{O}'(B_n, \mathbb{Z}[\varphi])$ since the entries of B_n and x_v all lie in $\mathbb{Z}[\varphi]$; see Table 1.

3.6. Γ_L **is Zariski-dense in** $\mathrm{O}(B_n, \mathbb{R})$**.** Finally, recall from §3.2 that $L \subseteq L_n$ and that Γ_L is the standard subgroup of Γ_n corresponding to the full subcomplex $L^{\#} \subseteq L_n^{\#}$.

We now show that the reflection group Γ_L is Zariski-dense in $\mathrm{O}(B_n, \mathbb{R})$. We first show that if $(L^{\#})^{(1)}$ is the join of two disjoint subsets V_1 and V_2 of $(L^{\#})^{(0)}$ whose union is $(L^{\#})^{(0)}$, then either V_1 or V_2 is empty. First, since no two vertices in $L^{(0)}$ are adjacent in $(L^{\#})^{(1)}$, the vertices of $L^{(0)}$ all lie in precisely one of the V_i, say V_1. Now a hypothetical vertex v in V_2 that is adjacent to every vertex in V_1, and hence to every vertex in $L^{(0)}$, cannot lie in the interior of a 2- or 3-simplex of L. Moreover, since $n \geqslant 3$, such a vertex v also cannot lie in the interior of a 1-simplex of L. We conclude that V_2 must indeed be empty.

Now since $(L^{\#})^{(1)}$ is not a join, any Γ_L-invariant subspace of $\mathbb{R}^{n+1}$ either contains the span U of the unit vectors that we have associated to $(L^{\#})^{(0)}$, or is contained in the intersection U' of their orthogonal complements with respect to the form B_n; see, e.g., Vinberg [Vin72, Proposition 19] or [Dan+25, Proposition 3.23].

Since L has dimension $d \geqslant 1$, there are vertices in a positive level of the hierarchy, so we have that U contains $e_1, \ldots, e_n$ as well as a vector of the form $\sum_{i=1}^{n+1} c_i e_i$ with $c_{n+1} \neq 0$, and hence coincides with $\mathbb{R}^{n+1}$. Moreover, since B_n is nondegenerate, we have that U' is trivial. Therefore, the action of Γ_L on $\mathbb{R}^{n+1}$ is irreducible.

Since B_n is Lorentzian, the Zariski closure of any irreducible subgroup of $\mathrm{O}(B_n, \mathbb{R})$ contains $\mathrm{SO}(B_n, \mathbb{R})$; see, e.g., [BH04, Proposition 1]. It follows that Γ_L is Zariski-dense in $\mathrm{O}(B_n, \mathbb{R})$.

This completes the proof of Theorem 1.2.

4. Proofs of Corollaries 1.4, 1.5, and 1.7

Proof of Corollary 1.4. Let L be a triangulation of N with n vertices. By Theorem 1.2, there is a convex cocompact right-angled reflection group $\Gamma_L < \mathrm{Isom}(\mathbb{H}^n)$ whose nerve is the flag-no-square triangulation $L^\#$ of N. By Lemma 2.2, the Gromov boundary of Γ_L is homeomorphic to $\mathscr{X}(N)$ (if N is non-orientable) or $\mathscr{X}(N\#\overline{N})$ (if N is orientable), and hence so is the limit set Λ_{Γ_L} by convex cocompactness of Γ_L. □

Proof of Corollary 1.5. Item (**1.**) follows Corollary 1.4, together with the existence of a 6-vertex triangulation of the real projective plane. Item (**2.**) follows again from Corollary 1.4, together with the existence of a 7-vertex triangulation of the torus. Item (**3.**) follows from Theorem 1.2 and Lemma 2.1, together with the existence of a 16-vertex triangulation of the Poincaré homology 3-sphere due to Björner and Lutz [BL00]. □

The latter authors conjecture [BL00, Conjecture 6] that 16 is the minimal number of vertices in any triangulation of the Poincaré homology sphere. It is known [BD05] that at least 12 vertices are needed to triangulate any homology 3-sphere differing from S^3.

Proof of Corollary 1.7. This follows from Theorem 1.2 and Lemma 2.3, together with the existence of a triangulation of the p-fold dunce hat $\widehat{M}_p$ with $\lceil \frac{3p}{2} \rceil + 3$ vertices. Figure 3 illustrates the cases $p = 2$, $p = 3$, and $p = 5$. □

Note that the leftmost triangulation in Figure 3 is the triangulation of the real projective plane $\mathbb{RP}^2 = \widehat{M}_2$ arising from the complete graph on 6 vertices, and 6 is the minimal possible number of vertices in any triangulation of $\mathbb{RP}^2$; see [Bar82].

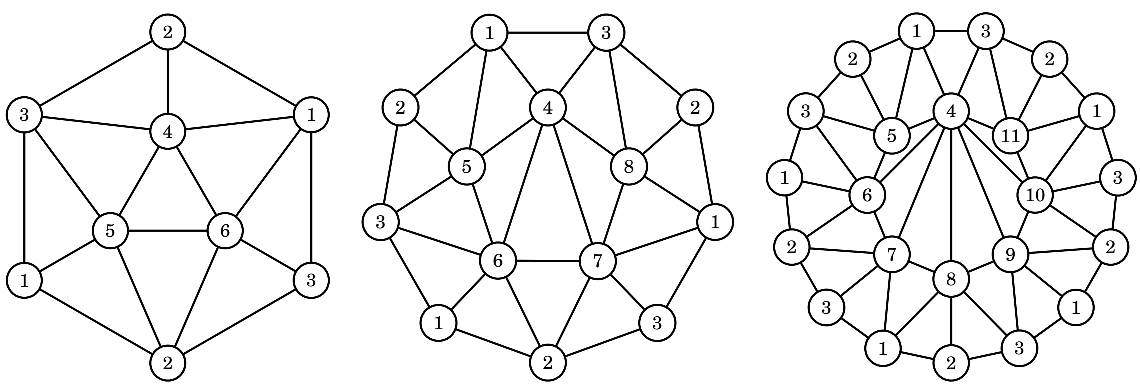

FIGURE 3. Triangulations of $\widehat{M}_2$, $\widehat{M}_3$, and $\widehat{M}_5$, shown from left to right.

References

[BD05] B. Bagchi and B. Datta. “Combinatorial triangulations of homology spheres”. In: *Discrete Math.* 305.1-3 (2005), pp. 1–17.

[Bar82] D. Barnette. “Generating the triangulations of the projective plane”. In: *J. Combin. Theory Ser. B* 33.3 (1982), pp. 222–230.

[BH04] Y. Benoist and P. de la Harpe. “Adhérence de Zariski des groupes de Coxeter”. In: *Compos. Math.* 140.5 (2004), pp. 1357–1366.

[BM91] M. Bestvina and G. Mess. “The boundary of negatively curved groups”. In: *J. Amer. Math. Soc.* 4.3 (1991), pp. 469–481.

[BL00] A. Björner and F. H. Lutz. “Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere.” English. In: *Exp. Math.* 9.2 (2000), pp. 275–289.

[BHC62] A. Borel and Harish-Chandra. "Arithmetic subgroups of algebraic groups". In: *Ann. of Math. (2)* 75 (1962), pp. 485–535.

[Bou97] M. Bourdon. "Sur la dimension de Hausdorff de l'ensemble limite d'une famille de sous-groupes convexes co-compacts". In: *C. R. Acad. Sci. Paris Sér. I Math.* 325.10 (1997), pp. 1097–1100.

[BO14] E. Breuillard and H. Oh, eds. *Thin groups and superstrong approximation*. Vol. 61. Mathematical Sciences Research Institute Publications. Selected expanded papers from the workshop held in Berkeley, CA, February 6–10, 2012. Cambridge University Press, Cambridge, 2014, pp. xii+362.

[Cox48] H. S. M. Coxeter. *Regular polytopes*. English. London: Methuen & Co., Ltd., XX, 322 p. 8 plates. (1948). 1948.

[DGK18] J. Danciger, F. c. Guéritaud, and F. Kassel. "Convex cocompactness in pseudo-Riemannian hyperbolic spaces". In: *Geom. Dedicata* 192 (2018), pp. 87–126.

[Dan+25] J. Danciger, F. Guéritaud, F. Kassel, G.-S. Lee, and L. Marquis. "Convex cocompactness for Coxeter groups". English. In: *J. Eur. Math. Soc. (JEMS)* 27.1 (2025), pp. 119–181.

[DKŚ24] D. Danielski, M. Kapovich, and J. Świątkowski. "Complete characterizations of hyperbolic Coxeter groups with Sierpiński curve boundary and with Menger curve boundary". English. In: *Fundam. Math.* 267.2 (2024), pp. 117–128.

[Dav02] M. W. Davis. "Exotic aspherical manifolds". In: *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*. Vol. 9. ICTP Lect. Notes. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, pp. 371–404.

[Dav08] M. W. Davis. *The geometry and topology of Coxeter groups*. Vol. 32. London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2008, pp. xvi+584.

[Dav24] M. W. Davis. *Infinite group actions on polyhedra*. Vol. 77. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, [2024] ©2024, pp. xi+271.

[DH13] M. Desgroseilliers and F. Haglund. "On some convex cocompact groups in real hyperbolic space". In: *Geom. Topol.* 17.4 (2013), pp. 2431–2484.

[Dou+] S. Douba, G.-S. Lee, L. Marquis, and L. Ruffoni. *SageMath files for the computation in this paper*. Available at https://marquis.pages.math.cnrs.fr/ludo-marquis/sagemath/SageMath.html.

[Dou+26] S. Douba, G.-S. Lee, L. Marquis, and L. Ruffoni. "Convex cocompact groups in real hyperbolic spaces with limit set a Pontryagin sphere". In: *Bull. Lond. Math. Soc.* 58.4 (2026).

[Dra97] A. N. Dranishnikov. "On the virtual cohomological dimensions of Coxeter groups". In: *Proc. Amer. Math. Soc.* 125.7 (1997), pp. 1885–1891.

[Dra99] A. N. Dranishnikov. "Boundaries of Coxeter groups and simplicial complexes with given links". In: *J. Pure Appl. Algebra* 137.2 (1999), pp. 139–151.

[Dra05] A. N. Dranishnikov. "Cohomological dimension theory of compact metric spaces". In: *arXiv Mathematics e-prints* (Jan. 2005). arXiv: math/0501523 [math.GN].

[DO07] J. Dymara and D. Osajda. "Boundaries of right-angled hyperbolic buildings". In: *Fund. Math.* 197 (2007), pp. 123–165.

[Fis03] H. Fischer. "Boundaries of right-angled Coxeter groups with manifold nerves". In: *Topology* 42.2 (2003), pp. 423–446.

[Fre82] M. H. Freedman. "The topology of four-dimensional manifolds". In: *J. Differential Geometry* 17.3 (1982), pp. 357–453.

[Gro87] M. Gromov. "Hyperbolic groups". In: *Essays in group theory*. Vol. 8. Math. Sci. Res. Inst. Publ. Springer, New York, 1987, pp. 75–263.

[Jak91] W. Jakobsche. "Homogeneous cohomology manifolds which are inverse limits". eng. In: *Fundamenta Mathematicae* 137.2 (1991), pp. 81–95.

[Kap08] M. Kapovich. "Kleinian groups in higher dimensions". In: *Geometry and dynamics of groups and spaces*. Vol. 265. Progr. Math. Birkhäuser, Basel, 2008, pp. 487–564.

[Kap09] M. Kapovich. *Hyperbolic manifolds and discrete groups*. Modern Birkhäuser Classics. Reprint of the 2001 edition. Birkhäuser Boston, Boston, MA, 2009, pp. xxviii+467.

[KK00] M. Kapovich and B. Kleiner. "Hyperbolic groups with low-dimensional boundary". In: *Ann. Sci. École Norm. Sup. (4)* 33.5 (2000), pp. 647–669.

[Ma26] J. Ma. "The Universal 2D Menger Space as Limit Sets of Higher-Dimensional Kleinian Groups". In: *Int. Math. Res. Not. IMRN* 4 (2026), Paper No. rnag026.

[MZ25] J. Ma and F. Zheng. "Discrete embeddings of hyperbolic groups with Pontryagin surfaces as boundaries". In: (2025). In preparation.

[Mil75] J. Milnor. "On the 3-dimensional Brieskorn manifolds $M(p,q,r)$". In: *Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox)*. Vol. No. 84. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1975, pp. 175–225.

[MT62] G. D. Mostow and T. Tamagawa. "On the compactness of arithmetically defined homogeneous spaces". In: *Ann. of Math. (2)* 76 (1962), pp. 446–463.

[Mou88] G. Moussong. *Hyperbolic Coxeter groups*. Thesis (Ph.D.)–The Ohio State University. ProQuest LLC, Ann Arbor, MI, 1988, p. 55.

[PV05] L. Potyagailo and E. Vinberg. "On right-angled reflection groups in hyperbolic spaces". In: *Comment. Math. Helv.* 80.1 (2005), pp. 63–73.

[PS09] P. Przytycki and J. Świątkowski. "Flag-no-square triangulations and Gromov boundaries in dimension 3". In: *Groups Geom. Dyn.* 3.3 (2009), pp. 453–468.

[Rad03] D. G. Radcliffe. "Rigidity of graph products of groups". In: *Algebr. Geom. Topol.* 3 (2003), pp. 1079–1088.

[Ran+96] A. A. Ranicki, A. J. Casson, D. P. Sullivan, M. A. Armstrong, C. P. Rourke, and G. E. Cooke. *The Hauptvermutung book*. Vol. 1. *K*-Monographs in Mathematics. A collection of papers of the topology of manifolds. Kluwer Academic Publishers, Dordrecht, 1996, pp. vi+190.

[Sul79] D. Sullivan. "The density at infinity of a discrete group of hyperbolic motions". In: *Inst. Hautes Études Sci. Publ. Math.* 50 (1979), pp. 171–202.

[Ś20a] J. Świątkowski. "Trees of manifolds as boundaries of spaces and groups". In: *Geom. Topol.* 24.2 (2020), pp. 593–622.

[Ś20b] J. Świątkowski. "Trees of metric compacta and trees of manifolds". In: *Geom. Topol.* 24.2 (2020), pp. 533–592.

[Vin72] E. B. Vinberg. "Discrete linear groups generated by reflections". English. In: *Math. USSR, Izv.* 5 (1972), pp. 1083–1119.

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