MATRIX ENTRIES, UNIPOTENTS, AND LINEARITY OF AMALGAMS

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Abstract. We investigate linearity of amalgams of subgroups of algebraic groups along intersections with algebraic subgroups. In the process, we establish linearity of certain "doubles" of linear groups, and obtain new examples of finitely generated residually finite groups that fail to be linear.

1. Introduction

An abstract group Γ is *linear over* an integral domain R (or is R-*linear*) if Γ embeds in $\mathsf{GL}_n(\mathsf{R})$ for some $n \in \mathbb{N}$. One says Γ is *linear* if Γ is linear over some integral domain. The goal of the current paper is to isolate contexts in which certain groups constructed from linear groups remain, or cease to be, linear.

It has been known since the 40's that a free product of two linear groups is linear[1] [58, 68]. However, linearity of an amalgam of two linear groups may fail dramatically. For example, if Λ is a subgroup of a residually finite group Γ, then the *double* $\Gamma *_\Lambda \Gamma$ *of* Γ *along* Λ, that is, the amalgam given by the inclusion into either factor, is residually finite if and only if Λ is separable in Γ. (Recall that Λ is said to be *separable* in Γ if Λ is an intersection of finite-index subgroups of Γ, and that Γ is said to be *residually finite* if the trivial subgroup of Γ is separable.) On the other hand, finitely generated linear groups are residually finite by Mal'cev's theorem [49]. It is then not difficult to deduce from the Tits alternative [71], together with the abundance of finitely generated groups that are not residually finite [33], that any finitely generated linear group that is not virtually solvable possesses uncountably many subgroups the doubles along which fail to be residually finite, let alone linear.

If R is a finitely generated integral domain and G is an algebraic subgroup of GL_n, then $\mathsf{G}(\mathsf{R})$ is separable in $\mathsf{GL}_n(\mathsf{R})$; see [11, 47]. In particular, if R is an arbitrary integral domain and $\Gamma < \mathsf{GL}_n(\mathsf{R})$ is finitely generated, then any intersection Λ of Γ with an algebraic subgroup of $\mathsf{GL}_n(\mathsf{R})$ is separable in Γ. One may then ask whether there are examples of such pairs (Γ, Λ) for which the double $\Gamma *_\Lambda \Gamma$ nevertheless fails to be linear. Such examples were supplied by Wehrfritz [73, Cor. 2.4] and rediscovered by Druțu–Sapir [31]. In the present work, we use the superrigidity theorems of Margulis [51], Corlette [28], and Gromov–Schoen [35] to provide new examples via the following theorem.

Theorem 1.1. *Let* G *be a semisimple real algebraic group with no compact factors that is not locally isomorphic to* $\mathsf{O}(n,1)$ *or* $\mathsf{U}(n,1)$ *for any* $n \in \mathbb{N}$*, and let* $\Gamma < \mathsf{G}$ *be an irreducible lattice. If* Λ *is an infinite-index subgroup of* Γ *that is not cocompact in the Zariski-closure of* Λ *in* G*, then the double* $\Gamma *_\Lambda \Gamma$ *is not linear.*

Thus, for example, the double $\mathsf{SL}_n(\mathbb{Z}) *_{\mathsf{SL}_m(\mathbb{Z})} \mathsf{SL}_n(\mathbb{Z})$ is residually finite but fails to be linear for $2 \leqslant m \leqslant n-1$ (where $\mathsf{SL}_m(\mathbb{Z})$ is embedded in $\mathsf{SL}_n(\mathbb{Z})$, for instance, by adding 1's on

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[1] On the other hand, determining the smallest integral domain over which a free product of two linear groups remains linear is a much subtler problem. For example, it was only recently established that a free product of two $\mathbb{Z}$-linear groups is $\mathbb{Z}$-linear [29, Cor. 1.10].

the diagonal), as does the residually finite double $\mathsf{SL}_2(\mathbb{Z}[\sqrt{2}]) *_{\mathsf{SL}_2(\mathbb{Z})} \mathsf{SL}_2(\mathbb{Z}[\sqrt{2}])$. We refer to [34, 26, 27, 70] for other examples of nonlinear finitely generated residually finite groups.

On the other hand, the following positive result establishes linearity of a large class of doubles of the form $\Gamma *_\Lambda \Gamma$, where Γ is arithmetic and $\Lambda < \Gamma$ is cocompact in the Zariski-closure of Λ.

Theorem 1.2. *Let $n \geqslant 2$ be an integer and $\mathsf{C} < \mathsf{SL}_n(\mathbb{R})$ a compact subgroup. Let C be defined by a set of polynomials of degree at most r in the matrix entries and with coefficients in a subfield $K \subset \mathbb{R}$, and let $\Lambda := \mathsf{SL}_n(K) \cap \mathsf{C}$. Then there is a faithful representation of the double $\mathsf{SL}_n(K) *_\Lambda \mathsf{SL}_n(K)$ into $\mathsf{SL}_m(K[s,t])$, where $K[s,t]$ denotes the ring of polynomials in two variables over K and $m = 2^{4\binom{n^2+r}{r}}$.*

The following corollary is a consequence.

Corollary 1.3.
(1) Let V be a finite-dimensional real vector space, let $\Gamma < \mathsf{GL}(V)$, and let $\mathsf{L} < \mathsf{GL}(V)$ be compact. Then for every finite-index subgroup Λ of $\Gamma \cap \mathsf{L}$ that is separable[2] in Γ, the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$.
(2) Let $\Gamma < \mathsf{GL}_n(\mathbb{C})$ be a subgroup. Let G be a $\mathbb{C}$-algebraic subgroup of $\mathsf{GL}_n(\mathbb{C})$, and suppose Λ is a finite-index subgroup of $\Gamma \cap \mathsf{G}$ that is separable in Γ. If there is an embedding $\sigma : K \to \mathbb{C}$ such that $(\Gamma \cap \mathsf{G})^\sigma$ is precompact in $\mathsf{GL}_n(\mathbb{C})$, where K denotes the entry field of Γ, then the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$.

It thus follows, for example, that if Γ is a finitely generated subgroup of a compact Lie group, then the double $\Gamma *_\Lambda \Gamma$ is linear for any maximal abelian subgroup Λ of Γ.

While the stipulation that a finitely generated group embed in a compact Lie group may appear very restrictive, the authors are for instance not aware of any linear (Gromov-)hyperbolic group that does not embed in a compact Lie group. Indeed, celebrated work of Agol [3] established that any hyperbolic group that acts properly and cocompactly on a CAT(0) cube complex is virtually compact special. Following Haglund–Wise [37], a finitely generated group Γ is said to be *special* (respectively, *compact special*) if Γ embeds in a right-angled Coxeter group (resp., embeds as a quasiconvex subgroup of a right-angled Coxeter group W with respect to a Coxeter basis for W); see [37] for several equivalent definitions. Agol [4] showed that any finitely generated right-angled Coxeter group, and hence any finitely generated virtually special group, embeds in a compact Lie group. In §3, we apply the latter fact together with Corollary 1.3 to prove the following.

Theorem 1.4. *Let Γ be a virtually compact special Gromov-hyperbolic group and $\Lambda < \Gamma$ a quasiconvex subgroup. Then the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$.*

In the sequel, a generalization of Theorem 1.4 (Theorem 3.8) will be used, together with much existing knowledge, to show that the double of a finitely generated Kleinian group along an arbitrary finitely generated subgroup is linear (Corollary 3.10). Note that in the statement of Theorem 1.4 there is no malnormality assumption on Λ, so that the double $\Gamma *_\Lambda \Gamma$ need not be hyperbolic. Nevertheless, we expect that a double $\Gamma *_\Lambda \Gamma$ as in Theorem 1.4 remains virtually (compact) special (compare, for instance, [38]), giving an alternative proof of linearity of such doubles, though this statement does not seem to be available in the literature.

Given a subgroup Λ of a real algebraic[3] group G, we will denote by $\overline{\Lambda}^{\mathrm{Zar}}$ the Zariski-closure of Λ in G. As another application of Corollary 1.3, we establish the following.

[2]Note that here separability of Λ in Γ is not automatic even for finitely generated Γ. Indeed, if $m \geqslant 5$ and one takes $\Gamma := \mathsf{Spin}(f_m; \mathbb{Z}[\sqrt{2}])$, where f_m is the quadratic form $\sqrt{2}x_1^2 + \sqrt{2}x_2^2 + x_3^2 + \ldots + x_m^2$, and Λ to be the stabilizer in Γ of the standard basis vector e_1, then it follows from work of Kneser [43] and Millson [53] related to the congruence subgroup property that Λ possesses finite-index subgroups that fail to be separable in Γ. This example also demonstrates the importance of the separability assumptions in Theorems 1.5 and 1.6.

[3]Algebraic groups will be assumed throughout to be linear.

Theorem 1.5. *Let G be an absolutely almost simple real algebraic group, let $\Gamma < \mathsf{G}$ be a lattice in G with adjoint trace field $\neq \mathbb{Q}$, and let $\Lambda < \Gamma$ be a subgroup. Then*

(1) if Γ is arithmetic and Λ is cocompact in $\overline{\Lambda}^{\mathrm{Zar}}$ and separable in Γ, then the double $\Gamma *_\Lambda \Gamma$ is linear;

(2) if G is not locally isomorphic to $\mathsf{O}(n,1)$ or $\mathsf{U}(n,1)$ for any $n \in \mathbb{N}$, then the double $\Gamma *_\Lambda \Gamma$ is linear if and only if Λ is cocompact in $\overline{\Lambda}^{\mathrm{Zar}}$ and separable in Γ.

We remark for example that if $p \geqslant q \geqslant 2$ with $p + q \geqslant 5$ odd, then the adjoint trace field of any cocompact lattice in $\mathsf{O}(p,q)$ differs from $\mathbb{Q}$; see Remark 3.11.

Corollary 1.3 also has applications to doubles of certain S-arithmetic groups, since the latter groups often embed in compact Lie groups. Throughout this article, given a number field K and a place v of K, we denote by K_v the completion of K with respect to v, and given a finite set S of places of K, we denote by $\mathcal{O}_{K,S}$ the ring of S-integers of K.

Theorem 1.6. *Let $\mathbf{G}$ be an almost simple algebraic group over a number field K, and S be a finite set of places of K containing all archimedean places v of K such that $\mathbf{G}$ is isotropic over K_v. Suppose $\mathbf{H}$ is a reductive K-subgroup of $\mathbf{G}$ that is anisotropic over K_w for some archimedean place w of K. Set $\mathsf{G} = \prod_{v \in S} \mathbf{G}(K_v)$ and $\mathsf{H} = \prod_{v \in S} \mathbf{H}(K_v)$. Then for any lattice $\Gamma < \mathsf{G}$ commensurable with $\mathbf{G}(\mathcal{O}_{K,S})$ and any finite-index subgroup Λ of $\Gamma \cap \mathsf{H}$ that is separable in Γ, the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$.*

It is not clear to us whether in Theorem 1.6 one can replace the condition that $\mathbf{H}$ be anisotropic over K_w for some archimedean place v_w of K, with the weaker condition that $\mathbf{H}$ be anisotropic over K itself. We remark that if Γ is an irreducible lattice in a product G of almost simple groups over nonarchimedean local fields of characteristic 0, and if the sum of the ranks of the factors of G is $\geqslant 2$, then, by Margulis's arithmeticity theorem [51], we have that $\mathsf{G} = \prod_{v \in S} \mathbf{G}(K_v)$ and that Γ is commensurable with $\mathbf{G}(\mathcal{O}_{K,S})$ for some $\mathbf{G}$, K, and S as in Theorem 1.6, where S consists entirely of nonarchimedean places of K, and in this case, one knows that $\mathbf{G}$ is K_w-anisotropic for any archimedean place w of K, so that the latter also holds for any K-subgroup $\mathbf{H}$ of $\mathbf{G}$.

Our methods also allow us to establish linearity of doubles of certain matrix groups along unipotent-free subgroups in the following geometric setting. A submanifold Y of an Hadamard manifold X is said to be *reflective* if Y is the fixed point set of an isometric involution of X. Note that reflective submanifolds are automatically totally geodesic.

Theorem 1.7. *Let X be a Riemannian symmetric space of noncompact type and Y a reflective submanifold of X. Let $\Gamma < \mathrm{Isom}(X)$ be a discrete subgroup, and $\Lambda < \Gamma$ be the stabilizer of Y in Γ. If Λ acts cocompactly on Y, then the double $\Gamma *_\Lambda \Gamma$ embeds in $\mathsf{GL}_n(\mathbb{R})$ for some dimension $n = n(X)$ depending only on X.*

Note that in Theorem 1.7 there is no arithmeticity assumption on the ambient discrete group Γ, nor is Γ even required to be of finite covolume.

Reflective submanifolds of symmetric spaces, which appear elsewhere in the literature under the guise of affine symmetric pairs, have been classified [10, 45, 46]. For instance, if f is a non-degenerate quadratic form on $\mathbb{R}^3$, then $(\mathsf{SL}_3(\mathbb{R}), \mathsf{SO}(f;\mathbb{R}))$ is an affine symmetric pair. If f is moreover defined over $\mathbb{Q}$ and is $\mathbb{Q}$-anisotropic, then $\mathsf{SO}(f;\mathbb{Z})$ is cocompact in $\mathsf{SO}(f;\mathbb{R})$ [15, 56], and one deduces from Theorem 1.7 that in this case the double $\mathsf{SL}_3(\mathbb{Z}) *_{\mathsf{SO}(f;\mathbb{Z})} \mathsf{SL}_3(\mathbb{Z})$ is linear. Note that if f is in addition $\mathbb{R}$-isotropic[4] (i.e., indefinite), then it follows from Margulis super-rigidity that the image of $\mathsf{SO}(f;\mathbb{Z})$ under no faithful (indeed, infinite-image) finite-dimensional

[4] One can for instance take f to be the diagonal form $x_1^2 + x_2^2 - 7x_3^2$. This form is anisotropic over the 2-adics $\mathbb{Q}_2$ since -7 is a square in $\mathbb{Q}_2$ and the stufe of $\mathbb{Q}_2$ is 4.

real representation of Γ is precompact, so that linearity of this particular double cannot be deduced from Theorem 1.2.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, any totally geodesic $\mathbb{K}$-subspace of a $\mathbb{K}$-hyperbolic space is reflective. In particular, any totally geodesic subspace of a real hyperbolic space is reflective, and Theorem 1.7 is novel even in this setting, though Baker and Cooper [5] show in this case[5] that if one moreover assumes that the compact subspace $\Lambda\backslash Y \subset \Gamma\backslash X$ is embedded, that the normal bundle of $\Lambda\backslash Y$ in $\Gamma\backslash X$ has trivial holonomy, and that the length of a shortest orthogeodesic to $\Lambda\backslash Y$ in $\Gamma\backslash X$ is sufficiently large, then the double $\Gamma *_\Lambda \Gamma$ embeds discretely in the isometry group of some real hyperbolic space whose dimension depends only on the dimensions of X and Y, and hence only on the dimension of X; a version of this statement also follows from work of Danciger–Guéritaud–Kassel [29]. On the other hand, it follows from work of Agol [2], Belolipetsky–Thomson [8], and Bergeron–Haglund–Wise [12] that, for any $\epsilon > 0$, there are closed real hyperbolic manifolds $\Gamma\backslash X$ of arbitrary dimension, at least some of which are nonarithmetic, possessing compact embedded totally geodesic hypersurfaces $\Lambda\backslash Y$ the shortest orthogeodesics to which are of length $< \epsilon$. The latter examples Γ are known to be virtually compact special [12], so that the doubles $\Gamma *_\Lambda \Gamma$ are in this case again virtually compact special and hence linear [38], though the approach via specialness provides no control on the dimension of a faithful representation.

As another sample application of Theorem 1.7, if $\Gamma\backslash X$ is a Picard modular surface and $\Lambda\backslash Y \subset \Gamma\backslash X$ is any compact totally geodesic complex curve (such curves always exist; see [25, 61]), then the double $\Gamma *_\Lambda \Gamma$ is linear. Note that linearity of such a double cannot be deduced for instance from Theorem 1.5(1).

It is straightforward to check that a *maximal* totally complex subspace Y of a quaternionic hyperbolic space X is reflective. In this setting, if $\Gamma\backslash X$ is compact and the projection $\Lambda\backslash Y$ of Y to $\Gamma\backslash X$ is compact and embedded, then the double $\Gamma *_\Lambda \Gamma$, while Gromov-hyperbolic by the combination theorem of Bestvina and Feighn [13], fails to embed discretely in any simple Lie group of real rank one, as can be deduced from the superrigidity theorem of Corlette [28] by an argument similar to the proof of [69, Thm. 1.7]. On the other hand, it follows from Theorem 1.7 that such doubles are nevertheless linear,[6] and are thus perhaps good candidates for examples of linear Gromov-hyperbolic groups that fail to embed discretely in any connected Lie group. It is indeed open whether there is a linear Gromov-hyperbolic group that fails to admit an Anosov representation in the sense of Labourie [44] and Guichard–Wienhard [36]; see [22, pg. 5] and [14, Rmk. 3.6]. We remark however that it follows from work [30] of Dey with the second-named author that, even in the setting of this paragraph, one can always find a finite-index subgroup $\Gamma' < \Gamma$ containing Λ such that the double $\Gamma' *_\Lambda \Gamma'$ indeed admits an Anosov representation.

Organization of the paper. We postpone the proof of the negative statement (Theorem 1.1) to Section 5. In Section 2, we prove a key result (Theorem 2.1), to which all subsequent positive statements will be reduced, regarding linearity of doubles of matrix groups along certain subgroups for which membership is detected by the top-left entry. Section 3 is devoted to the proof of Theorem 1.2 and its consequences, while Section 4 contains the proof of Theorem 1.7.

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[5]The approach of Baker–Cooper should also yield a similar statement to their result in the case that X is a $\mathbb{K}$-hyperbolic space for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and $Y \subset X$ is a $\mathbb{K}$-subspace, where the target of the output discrete and faithful representation of the double $\Gamma *_\Lambda \Gamma$ is now the isometry group of some $\mathbb{K}$-hyperbolic space whose dimension depends only on those of X and Y. We thank Nicolas Tholozan for pointing this out to us.

[6]Linearity of such doubles in fact also follows from Theorem 1.5(1) in light of [32, Prop. 2.8].

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2. Doubling along subgroups detected by the top-left entry

This section is devoted to the proof of Theorem 2.1, to which many of the statements in the sequel will be reduced.

First, we establish some notation. Given an integral domain R, we denote by $\mathsf{Mat}_m(\mathsf{R})$ (respectively, by $\mathsf{Mat}_{m\times r}(\mathsf{R})$) the free R-module of $m \times m$ (resp., $m \times r$) matrices with entries in R. For $1 \leqslant i, j \leqslant m$, we will denote by $E_{ij}(m) \in \mathsf{Mat}_m(\mathsf{R})$ the matrix whose (i, j)-entry is 1 and the rest of whose entries are zero. For $d, k \in \mathbb{N}$, we view the product $\mathsf{GL}_k(\mathsf{R}) \times \mathsf{GL}_d(\mathsf{R})$ as a subgroup of $\mathsf{GL}_{k+d}(\mathsf{R})$ via the block-diagonal embedding $(A, B) \mapsto \begin{pmatrix} A & \\ & B \end{pmatrix}$. For a matrix $g = (g_{ij})_{i,j=1}^n$ in $\mathsf{GL}_n(\mathsf{R})$, we define the *top-left $k \times k$ block of g* to be the matrix $(g_{ij})_{i,j=1}^k$.

Theorem 2.1. *Let R be an integral domain, and Γ_1, Γ_2 be subgroups of $\mathsf{GL}_n(\mathsf{R})$, $n \geqslant 2$.*

(1) Suppose that $\Lambda < \Gamma_1 \cap \Gamma_2$ is a subgroup of $\{1\} \times \mathsf{GL}_{n-1}(\mathsf{R})$ such that for $i = 1, 2$, a matrix $\gamma \in \Gamma_i$ lies in Λ if and only if the top-left entry of γ is 1. Then the amalgam $\Gamma_1 *_\Lambda \Gamma_2$ admits a faithful representation into $\mathsf{GL}_{n+1}(\mathsf{R}[t])$, where $\mathsf{R}[t]$ denotes the ring of polynomials over R.

(2) Suppose that $\Lambda < \Gamma_1 \cap \Gamma_2$ is a subgroup of $\{I_k\} \times \mathsf{GL}_{n-k}(\mathsf{R})$, $2 \leqslant k \leqslant n-1$, such that for $i = 1, 2$, a matrix $\gamma \in \Gamma_i$ lies in Λ if and only if the top-left $k \times k$ block of γ is the identity. Then the amalgam $\Gamma_1 *_\Lambda \Gamma_2$ admits a faithful representation into $\mathsf{GL}_{n^2+1}(\mathsf{R}[s, t])$, where $\mathsf{R}[s, t]$ denotes the ring of polynomials over R in two variables.

Proof of Theorem 2.1(1). View $\mathsf{GL}_n(\mathsf{R})$ as the subgroup $\{1\} \times \mathsf{GL}_n(\mathsf{R}) < \mathsf{GL}_{n+1}(\mathsf{R}[t])$. Consider the matrices $u_t, \tau \in \mathsf{GL}_{n+1}(\mathsf{R}[t])$ given by

$$u_t := \begin{pmatrix} 1 & t & \\ 0 & 1 & \\ & & I_{n-1} \end{pmatrix}, \quad \tau := \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_{n-1} \end{pmatrix},$$

and let $\rho_t : \Gamma_1 *_\Lambda \Gamma_2 \to \mathsf{GL}_{n+1}(\mathsf{R}[t])$ be the representation induced by $\gamma_1 \mapsto u_t \gamma_1 u_t^{-1}$ on Γ_1 and $\gamma_2 \mapsto \tau u_t \gamma_2 u_t^{-1} \tau$ on Γ_2.

Say a matrix $M = (m_{ij})_{i,j=1}^{n+1} \in \mathsf{Mat}_{n+1}(\mathsf{R}[t])$ is *sufficiently transcendental* if $\deg(m_{ij}) \leqslant \deg(m_{11})$ for $1 \leqslant i, j \leqslant n+1$ and $\deg(m_{ij}) < \deg(m_{11})$ for $i > 1$. Note that a product of two sufficiently transcendental matrices is sufficiently transcendental. Note also that for $i = 1, 2$, and $\gamma_i \in \Gamma_i \smallsetminus \Lambda$, the product $(u_t \gamma_1 u_t^{-1})(\tau u_t \gamma_2 u_t^{-1} \tau)$ is sufficiently transcendental; indeed, a direct computation shows that the top-left entry of $(u_t \gamma_1 u_t^{-1})(\tau u_t \gamma_2 u_t^{-1} \tau)$ is a degree-2 polynomial in t whose leading coefficient is $\big((\gamma_1)_{11} - 1\big)\big((\gamma_2)_{11} - 1\big)$, while the remaining entries of $(u_t \gamma_1 u_t^{-1})(\tau u_t \gamma_2 u_t^{-1} \tau)$ (respectively, the entries of $(u_t \gamma_1 u_t^{-1})(\tau u_t \gamma_2 u_t^{-1} \tau)$ below the first column) are of degree $\leqslant 2$ (resp., are of degree $\leqslant 1$). It thus follows that for each element $\gamma \in \Gamma_1 *_\Lambda \Gamma_2$ of the form

$$\gamma = \gamma_1^{(1)} \gamma_2^{(1)} \cdots \gamma_1^{(r)} \gamma_2^{(r)}, \tag{1}$$

where $\gamma_i^{(1)}, \ldots, \gamma_i^{(r)} \in \Gamma_i \smallsetminus \Lambda$ for $i = 1, 2$, the matrix $\rho_t(\gamma)$ is sufficiently transcendental, and hence different from the identity. Since every element of $\Gamma_1 *_\Lambda \Gamma_2$ that is not conjugate into one of the factors is of, or has inverse of, the form (1) (see, for instance, [67, §1.2, Thm. 1]), we conclude that the representation ρ_t is faithful. ∎

Proof of Theorem 2.1 (2). Consider the left translation representation $L : \mathsf{GL}_n(\mathsf{R}[s]) \to \mathsf{GL}(\mathsf{Mat}_n(\mathsf{R}[s]))$,

$$L(h)M = hM,\ \ M \in \mathsf{Mat}_n(\mathsf{R}[s]), h \in \mathsf{GL}_n(\mathsf{R}[s]).$$

Let $W, W' \subset \mathsf{Mat}_n(\mathsf{R}[s])$ be the complementary free $\mathsf{R}[s]$-submodules given by

$$W := \left\{ \begin{pmatrix} X & \\ & 0 \end{pmatrix} : X \in \mathsf{Mat}_k(\mathsf{R}[s]) \right\},$$
$$W' := \left\{ \begin{pmatrix} 0 & * \\ * & Y \end{pmatrix} : Y \in \mathsf{Mat}_{n-k}(\mathsf{R}[s]) \right\}.$$

Fix the ordered basis $\mathcal{B}_1 := \big(\mathcal{E}, E_{22}(n), \ldots, E_{kk}(n), E_{12}(n), \ldots, E_{k(k-1)}(n)\big)$ for W, where $\mathcal{E} := \begin{pmatrix} I_k & \\ & 0 \end{pmatrix}$, and some ordered basis $\mathcal{B}_2$ for W'. Consider the unipotent element $g_s \in \mathsf{GL}(\mathsf{Mat}_n(\mathsf{R}[s]))$,

$$g_s := I_{n^2} + \sum_{j=1}^{k^2-1} s^j E_{1j}(k^2),$$

and the representation $L_s : \mathsf{GL}_n(\mathsf{R}[s]) \to \mathsf{GL}(\mathsf{Mat}_n(\mathsf{R}[s]))$ given by

$$L_s(h) := g_s L(h) g_s^{-1},\ \ \ h \in \mathsf{GL}_n(\mathsf{R}[s]).$$

For $h \in \mathsf{GL}_n(\mathsf{R})$, we may write

$$L(h)\mathcal{E} = h_{11}\mathcal{E} + \sum_{i=2}^{k}(h_{ii} - h_{11})E_{ii}(n) + \sum_{i \neq j} h_{ij}E_{ij}(n) + M_h$$

for some $M_h \in W'$. In particular, with respect to the ordered basis $(\mathcal{B}_1, \mathcal{B}_2)$ for $\mathsf{Mat}_n(\mathsf{R}[s])$, the top-left entry of the matrix $L_s(h) \in \mathsf{GL}(\mathsf{Mat}_n(\mathsf{R}[s]))$ is the polynomial

$$\big(L_s(h)\big)_{11} = h_{11} + \sum_{i=2}^{k} \big(h_{ii} - h_{11}\big)s^{i-1} + h_{12}s^k + \cdots + h_{k(k-1)}s^{k^2-1}.$$

Hence, for $h \in \mathsf{GL}_n(\mathsf{R})$, the top-left entry $\big(L_s(\gamma)\big)_{11} = 1$ if and only if the top-left $k \times k$ block of h is the identity. By assumption, for $\gamma \in \Gamma_i$, $i = 1, 2$, the latter holds if and only if $\gamma \in \Lambda$. Moreover, on Λ, the representations L and L_s coincide, act as the identity on W, and preserve the subspace W'. Therefore, by applying Theorem 2.1 (1) for the groups $L_s(\Gamma_1)$ and $L_s(\Gamma_2)$ and their common subgroup $L_s(\Lambda)$, we obtain a faithful representation $\rho : \Gamma_1 *_\Lambda \Gamma_2 \to \mathsf{GL}_{n^2+1}(\mathsf{R}[s,t])$. This completes the proof of the theorem. □

Example 2.2. Let K be a number field whose stufe is 4, so that K is in particular totally complex, and denote by $\mathcal{O}_K$ the ring of integers of K. Then, since the quadratic form $x_1^2 + x_2^2 + x_3^2$ is anisotropic over K, the Borel–Harish-Chandra theorem [15, 56] implies that the group $\Gamma := \mathsf{SO}_3(\mathcal{O}_K)$ embeds as an irreducible cocompact lattice in a product of $d/2$ copies of $\mathsf{SO}_3(\mathbb{C}) \cong \mathsf{PSL}_2(\mathbb{C})$. Moreover, it follows immediately from Theorem 2.1(1) that the double $\Gamma *_\Lambda \Gamma$ is linear if Λ is taken to be the stabilizer in Γ of the standard basis vector e_1. Note that for $d > 2$ this is an example to which Theorem 1.2 will not apply, since it is indeed not difficult to verify via Margulis superrigidity [51] that for such d the image of Λ under no faithful finite-dimensional real representation of Γ is precompact.[7]

[7]For an example with $d > 2$, one can for instance take $K = \mathbb{Q}(\sqrt{-7}, \sqrt{17})$. Indeed, on the one hand, it is known that the stufe of a totally complex number field is $\leqslant 4$; see [62] for the quartic case. On the other, the 2-adics $\mathbb{Q}_2$ contain square roots of -7 and 17 by Hensel's lemma, and the stufe of $\mathbb{Q}_2$ is 4.

3. Doubling along intersections with compact subgroups

In this section we prove Theorem 1.2 and establish some consequences. We first deduce from Theorem 2.1 the following technical lemma, which will also be useful in the following section.

Lemma 3.1. *Let Γ_1, Γ_2 be groups with a common subgroup Λ. Suppose we are given for some integer $p \geqslant 2$, for $i = 1, \ldots, p$, for $j = 1, 2$, and for some integral domain R representations $\rho_i^j : \Gamma_j \to \mathsf{GL}_{n_i}(\mathsf{R})$, at least one of which is faithful, satisfying the following properties:*

- *$\rho_i^j(\Lambda) \subset \{1\} \times \mathsf{GL}_{n_i-1}(\mathsf{R})$;*
- *the representations ρ_i^1 and ρ_i^2 coincide on Λ; and*
- *if $\gamma \in \Gamma_j$ and the top-left entry of each of the $\rho_i^j(\gamma)$ is equal to 1, then $\gamma \in \Lambda$.*

Then there is a faithful representation of $\Gamma_1 *_\Lambda \Gamma_2$ into $\mathsf{GL}_r(\mathsf{R}[s,t])$, where $r = (\sum_{i=1}^p n_i)^2 + 1$.

Proof. For $j = 1, 2$, let $\times_{i=1}^p \rho_i^j : \Gamma_j \to \mathsf{GL}_d(\mathsf{R})$ be the p-fold product of the representations $\rho_1^j, \ldots, \rho_p^j$, where $d = \sum_{i=1}^p n_i$. For $i = 1, 2$, let $(e_{i1}, \ldots, e_{in_i})$ be the standard basis of R^{n_i}. By re-ordering the basis $(e_{11}, \ldots, e_{1n_1}, \ldots, e_{p1}, \ldots, e_{pn_p})$ of $\bigoplus_{i=1}^p \mathsf{R}^{n_i}$ as $(e_{11}, \ldots, e_{p1}, \ldots, e_{p2}, \ldots, e_{pn_p})$, we obtain a faithful represention $\psi_j : \Gamma_j \to \mathsf{GL}_d(\mathsf{R})$, conjugate to $\times_{i=1}^p \rho_i^j$, with the property that for any $\gamma \in \Gamma_j$ the top left $p \times p$ block of $\psi_j(\gamma)$ is a diagonal matrix whose diagonal entries are the top-left entries of the $\rho_i^j(\gamma)$. Thus, by assumption, if $\gamma \in \Gamma_j$ and the top left $p \times p$ block of $\psi_j(\gamma)$ is the identity matrix, then $\gamma \in \Lambda$. Therefore, by applying Theorem 2.1(2) for the groups $\psi_1(\Gamma_1), \psi_2(\Gamma_2)$ and their common subgroup $\psi_1(\Lambda)$, we obtain a faithful representation of $\Gamma_1 *_\Lambda \Gamma_2$ into $\mathsf{GL}_r(\mathsf{R}[s,t])$, where $r = d^2 + 1$. □

For an integral domain R, we denote by $\mathrm{Sym}(\mathsf{R})$ the free R-module of $r \times r$ symmetric matrices over R, and by $\mathrm{Sym}_r : \mathsf{SL}_r(\mathsf{R}) \to \mathsf{SL}(\mathrm{Sym}(\mathsf{R}))$ the representation given by

$$\mathrm{Sym}_r(g)M := gMg^t$$

for $M \in \mathrm{Sym}(\mathsf{R})$. We first establish the following special case of Theorem 1.2, where the subgroup over which one doubles is the stabilizer of a line in the special orthogonal group.

Theorem 3.2. *Let R be a subring of $\mathbb{R}$. Let C be the stabilizer of the standard basis vector e_1 in $\mathsf{SO}(n)$, $n \geqslant 2$, and let $\Lambda := \mathsf{C} \cap \mathsf{SL}_n(\mathsf{R})$. Then there is a faithful representation of the amalgam $\mathsf{SL}_n(\mathsf{R}) *_\Lambda \mathsf{SL}_n(\mathsf{R})$ into $\mathsf{SL}_d(\mathsf{R}[s,t])$, where $d = \left(n^2 + 3n + 2\right)^2 + 1$.*

Proof. Consider the embedding $\iota : \mathsf{SL}_n(\mathsf{R}) \hookrightarrow \mathsf{SL}_{n+1}(\mathsf{R})$, $\iota(g) = \begin{pmatrix} g & \\ & 1 \end{pmatrix}$, and fix ordered bases $\mathcal{B}_n$ and $\mathcal{B}'_n$ for $\mathrm{Sym}(\mathsf{R}^{n+1})$ such that

- the first basis vector in $\mathcal{B}_n$ (respectively, in $\mathcal{B}'_n$) is the identity matrix I_{n+1} (resp., is the matrix $E_{11}(n+1)$), and
- the remaining basis vectors in $\mathcal{B}_n$ (respectively, in $\mathcal{B}'_n$) lie in the R-submodule $V_n := \{M \in \mathrm{Sym}(\mathbb{R}^{n+1}) : \mathrm{tr}(M) = 0\}$ (resp., in $V'_n := \{M \in \mathrm{Sym}(\mathbb{R}^{n+1}) : \mathrm{tr}(ME_{11}(n+1)) = 0\}$).

With respect to the bases $\mathcal{B}_n, \mathcal{B}'_n$, we obtain two faithful representations $\rho_n, \rho'_n : \mathsf{SL}_n(\mathsf{R}) \to \mathsf{SL}_r(\mathsf{R})$ conjugate to the representation $\mathrm{Sym}_{n+1} \circ \iota : \mathsf{SL}_n(\mathsf{R}) \to \mathsf{SL}(\mathrm{Sym}(\mathsf{R}^{n+1}))$, where $r = \frac{(n+2)(n+1)}{2}$, satisfying for any $g \in \mathsf{SL}_n(\mathsf{R})$,

$$\big(\rho_n(g)\big)_{11} = \frac{\mathrm{tr}(gg^t) + 1}{n+1}, \tag{2}$$

$$\big(\rho'_n(g)\big)_{11} = (g_{11})^2. \tag{3}$$

Moreover, observe that if $g \in \Lambda$, then $\mathrm{Sym}_{n+1}(\iota(g))$ fixes both $I_{n+1}, E_{11}(n+1) \in \mathrm{Sym}(\mathsf{R}^{n+1})$ and preserves the hyperplanes V_n and V'_n. In other words, $\rho_n(\Lambda)$ and $\rho'_n(\Lambda)$ are subgroups of $\{1\} \times \mathsf{SL}_{r-1}(\mathsf{R}) \subset \mathsf{SL}_r(\mathsf{R})$.

Now we check that ρ_n and ρ'_n satisfy the assumptions of Lemma 3.1. To check that, suppose $g \in \mathsf{SL}_n(\mathsf{R})$ satisfies

$$\big(\rho_n(g)\big)_{11} = \big(\rho'_n(g)\big)_{11} = 1.$$

By (2), we have $\operatorname{tr}(gg^t) = n$, and since gg^t is positive-definite and $\det(gg^t) = 1$, we conclude that $gg^t = I_n$. In addition, since $(g_{11})^2 = 1$ and the columns of g form an orthonormal basis of $\mathbb{R}^n$, we also have $g \in \mathsf{C}$. Thus, by applying Lemma 3.1 for ρ_n and ρ'_n, we obtain a faithful representation of $\mathsf{SL}_n(\mathsf{R}) *_\Lambda \mathsf{SL}_n(\mathsf{R})$ into $\mathsf{SL}_d(\mathsf{R}[s,t])$, where $d = (2r)^2 + 1$. □

Now let K be a subfield of $\mathbb{R}$, and for $n, r \in \mathbb{N}$, let $V_{n,r} \subset K[x_{11}, \dots, x_{nn}]$ denote the vector subspace of polynomials over K of degree at most r. To prove Theorem 1.2, we will use the following proposition due to Chevalley; see also [40, 11.2, Ch. IV].

Proposition 3.3. *Let $\mathsf{H} < \mathsf{SL}_n(K)$ be a K-subgroup defined by polynomials of degree at most r in $K[x_{11}, \dots, x_{nn}]$. There is a faithful K-polynomial representation $\rho_\mathsf{H} : \mathsf{SL}_n(K) \to \mathsf{SL}(V_{n,r})$ and a K-subspace $V_\mathsf{H} \subset V_{n,r}$ such that $\mathsf{H} = \big\{g \in \mathsf{SL}_n(K) : \rho_\mathsf{H}(g)V_\mathsf{H} = V_\mathsf{H}\big\}$.*

Proof of Theorem 1.2. By Proposition 3.3, there is a faithful K-polynomial representation $\rho_\mathsf{C} : \mathsf{SL}_n(K) \to \mathsf{SL}_{d_n}(K)$ satisfying

$$\Lambda = \big\{g \in \mathsf{SL}_n(K) : \rho_\mathsf{C}(g)\mathrm{span}_K(e_1, \dots, e_s) = \mathrm{span}_K(e_1, \dots, e_s)\big\}$$

and $\rho_\mathsf{C}(\Lambda) \subset \mathsf{O}(d_n; K)$, where $d_n := \dim(V_{n,r})$ and $1 \leqslant s \leqslant d_n - 1$. By composing ρ_C first with the embedding $\mathsf{SL}(K^{d_n}) \hookrightarrow \mathsf{SL}(K^{d_n}) \times \{1\}$ and then with $\wedge^s : \mathsf{SL}(K^{d_n+1}) \to \mathsf{SL}(\wedge^s K^{d_n+1})$, we obtain a faithful representation $\rho_{\mathsf{C},s} : \mathsf{SL}_n(K) \to \mathsf{SL}(\wedge^s K^{d_n+1})$ satisfying

$$\rho_{\mathsf{C},s}(\Lambda) = \rho_{\mathsf{C},s}(\mathsf{SL}_n(K)) \cap \big(\{\pm 1\} \times \mathsf{O}(W; K)\big),$$

where $W := \mathrm{span}_K\big(\{e_{i_1} \wedge \cdots \wedge e_{i_s} : \{i_1, \dots i_s\} \neq \{1, \dots, s\}\}\big)$. Therefore, there is an embedding

$$\mathsf{SL}_n(K) *_\Lambda \mathsf{SL}_n(K) \hookrightarrow \mathsf{SL}(\wedge^s K^{d_n+1}) *_{\Lambda'} \mathsf{SL}(\wedge^s K^{d_n+1}),$$

where $\Lambda' := \mathsf{SL}(\wedge^s K^{d_n}) \cap (\{\pm 1\} \times \mathsf{O}(W; K))$. Since $\binom{d_n+1}{\lceil \frac{d_n+1}{2} \rceil} \geqslant \binom{d_n}{s}$, by Theorem 3.2 there is a faithful representation $\varphi_\mathsf{C} : \mathsf{SL}_n(K) *_\Lambda \mathsf{SL}_n(K) \to \mathsf{SL}_q(K[s,t])$ with $q = ((d'_n)^2 + 3d'_n + 2)^2 + 1$, $d'_n = \binom{d_n+1}{\lceil \frac{d_n+1}{2} \rceil}$ and $d_n = \dim(V_{n,r}) = \binom{n^2+r}{r} - 1$. Finally, note that $d'_n \leqslant 2^{d_n}$ and hence $q \leqslant 2^{4\binom{n^2+r}{r}-3}$. □

Proof of Corollary 1.3(1). We first claim there is a finite-index subgroup Γ_Λ of Γ such that $\Gamma_\Lambda \cap \mathsf{L} = \Lambda$. Indeed, let $1, \gamma_1, \dots, \gamma_r$ be a full set of representatives of $(\Gamma \cap \mathsf{L})/\Lambda$. We may then take Γ_Λ to be any finite-index subgroup of Γ containing Λ but excluding $\gamma_1, \dots, \gamma_r$; the existence of such a finite-index subgroup of Γ is guaranteed by separability of Λ in Γ.

Viewing V now as a representation of Γ_Λ, let $\rho : \Gamma \to \mathsf{GL}(W)$ be the representation induced on Γ, so that V is a sub-Γ_Λ-representation of W. Let C be the Zariski-closure of $\rho(\Lambda)$ in $\mathsf{GL}(W)$, and let H be the algebraic subgroup of $\mathsf{GL}(W)$ consisting of those elements preserving V and whose restriction to V belongs to L. Since $\mathsf{L} \subset \mathsf{H}$ and $\rho(\Gamma) \cap \mathsf{H} = \rho(\Lambda)$, we have $\rho(\Gamma) \cap \mathsf{L} = \rho(\Lambda)$. Moreover, from the description of the induced representation, one sees that since Λ preserves a metric on V, the same is true of $\rho(\Lambda)$ on W, so that L is compact. The conclusion of Corollary 1.3 (1) now follows from Theorem 1.2, together with the fact that, for any subfield $K \subset \mathbb{R}$, the domain $K[s,t]$ embeds[8] in $\mathbb{C}$, so that any group that is $K[s,t]$-linear is $\mathbb{C}$-linear and thus $\mathbb{R}$-linear via restriction of scalars. □

[8]That $K[s,t]$ embeds in $\mathbb{C}$ is clear if K is, for instance, countable, but indeed any characteristic-zero domain of cardinality at most that of $\mathbb{C}$ embeds in $\mathbb{C}$.

Proof of Corollary 1.3(2). We may extend the embedding $\sigma : K \to \mathbb{C}$ to an automorphism of $\mathbb{C}$, which we continue to denote by σ. We have assumed that $\mathsf{G} = \mathbf{G}(\mathbb{C})$ for some $\mathbb{C}$-subgroup $\mathbf{G} < \mathsf{SL}_n$, and we will write $\mathsf{G}^\sigma := \mathbf{G}^\sigma(\mathbb{C})$. Observe that G^σ is a $\mathbb{C}$-algebraic subgroup and $\Gamma^\sigma \cap \mathsf{G}^\sigma = (\Gamma \cap \mathsf{G})^\sigma$ is by assumption precompact in $\mathsf{GL}_n(\mathbb{C})$. Let $j : \mathsf{GL}_n(\mathbb{C}) \to \mathsf{GL}_{2n}(\mathbb{R})$ be the standard embedding given by restriction of scalars, i.e., given by $g \mapsto \begin{pmatrix} \mathrm{Re}(g) & -\mathrm{Im}(g) \\ \mathrm{Im}(g) & \mathrm{Re}(g) \end{pmatrix}$ for $g \in \mathsf{GL}_n(\mathbb{C})$. Observe moreover that, since $j(\mathsf{G}^\sigma)$ is an $\mathbb{R}$-algebraic subgroup of $\mathsf{GL}_{2n}(\mathbb{R})$, we have that $j(\Lambda^\sigma)$ is of finite index in $j(\Gamma^\sigma) \cap \mathsf{L}$, where L denotes the (Zariski-)closure of $j(\Gamma^\sigma) \cap j(\mathsf{G}^\sigma)$ in $\mathsf{GL}_{2n}(\mathbb{R})$. Since H is compact and Λ^σ is separable in Γ^σ, we conclude from Corollary 1 that the double $\Gamma^\sigma *_{\Lambda^\sigma} \Gamma^\sigma$ is linear over $\mathbb{R}$. The latter double is isomorphic to the double $\Gamma *_\Lambda \Gamma$ and the corollary follows. ☐

The following lemmas will be used to establish Corollary 3.7, which will in turn be used to prove Theorem 1.4. These arguments can all be found in [48, 42], but we include them here for the convenience of the reader.

Lemma 3.4. *Let $\Gamma < \mathsf{GL}_d(\mathbb{R})$ and $\Lambda < \Gamma$. Let $\Gamma' < \Gamma$ be of finite index, and set $\Lambda' := \Gamma' \cap \Lambda$. If $\Gamma' \cap \overline{\Lambda'}^{\mathrm{Zar}} = \Lambda'$, then $\Gamma \cap \overline{\Lambda}^{\mathrm{Zar}} = \Lambda$.*

Proof. Let $\lambda_1, \ldots, \lambda_m \in \Lambda$ be a set of left coset representatives for Λ' in Λ. Then $\overline{\Lambda}^{\mathrm{Zar}} = \lambda_1\overline{\Lambda'}^{\mathrm{Zar}} \cup \cdots \cup \lambda_m\overline{\Lambda'}^{\mathrm{Zar}}$. Thus, given $\gamma \in \Gamma \cap \overline{\Lambda}^{\mathrm{Zar}}$, we have that $\gamma \in \lambda\overline{\Lambda'}^{\mathrm{Zar}}$ for some $\lambda \in \Lambda$. But then $\lambda^{-1}\gamma \in \Gamma \cap \overline{\Lambda'}^{\mathrm{Zar}} \subset \Gamma' \cap \overline{\Lambda'}^{\mathrm{Zar}}$, and so $\lambda^{-1}\gamma \in \Lambda'$. We conclude that $\gamma \in \lambda\Lambda' \subset \Lambda$. ☐

Lemma 3.5. *Let Γ' be a finite-index normal subgroup of a group Γ, and $\rho' : \Gamma' \to \mathsf{GL}_n(\mathbb{R})$ be a representation. Suppose $\Lambda' < \Gamma'$ is a subgroup satisfying $\rho'(\Gamma') \cap \overline{\rho'(\Lambda')}^{\mathrm{Zar}} = \rho'(\Lambda')$. Then for any subgroup $\Lambda < \Gamma$ such that $\Gamma' \cap \Lambda = \Lambda'$, the representation $\mathrm{ind}_{\Gamma'}$ of Γ induced by ρ' satisfies $\mathrm{ind}_{\Gamma'}(\Gamma) \cap \overline{\mathrm{ind}_{\Gamma'}(\Lambda)}^{\mathrm{Zar}} = \mathrm{ind}_{\Gamma'}(\Lambda)$.*

Proof. By Lemma 3.4, it suffices to show that $\mathrm{ind}_{\Gamma'}(\Gamma') \cap \overline{\mathrm{ind}_{\Gamma'}(\Lambda')}^{\mathrm{Zar}} = \mathrm{ind}_{\Gamma'}(\Lambda')$, but the latter is true since the restriction $\mathrm{ind}_{\Gamma'}\big|_{\Gamma'}$ extends to an ($\mathbb{R}$-algebraic) representation of $\mathsf{SL}_n(\mathbb{R})$. ☐

One says a subgroup Λ of a group Γ is a *virtual retract* of Γ if there is a finite-index subgroup $\Gamma' < \Gamma$ containing Λ such that Λ is a retract of Γ'. We will say a subgroup $\Lambda < \Gamma$ is *virtually a virtual retract* of Γ if there is a finite-index subgroup Γ' of Γ such that $\Lambda \cap \Gamma'$ is a retract of Γ'.

Lemma 3.6. *Let $\Gamma < \mathsf{SL}_n(\mathbb{R})$ be a subgroup, and suppose $\Lambda < \Gamma$ is virtually a virtual retract of Γ. Then there is a faithful representation $\rho : \Gamma \to \mathsf{SL}_d(\mathbb{R})$ for some $d \in \mathbb{N}$ such that $\rho(\Gamma) \cap \overline{\rho(\Lambda)}^{\mathrm{Zar}} = \rho(\Lambda)$. Moreover, if Λ is precompact in $\mathsf{SL}_n(\mathbb{R})$, then ρ can be chosen such that $\rho(\Lambda)$ is precompact in $\mathsf{SL}_d(\mathbb{R})$.*

Proof. By assumption, there is a finite-index subgroup $\Gamma' < \Gamma$ and a map $r : \Gamma' \to \Lambda' := \Gamma' \cap \Lambda$ such that $r\big|_{\Lambda'} = \mathrm{Id}_{\Lambda'}$. Up to replacing Γ' with $\bigcap_{\gamma\in\Gamma} \gamma\Gamma'\gamma^{-1}$ (and Λ' with $\Lambda \cap \bigcap_{\gamma\in\Gamma} \gamma\Gamma'\gamma^{-1}$), we can assume Γ' is normal in Γ. Now let $\rho' : \Gamma' \to \mathsf{SL}_{2n}(\mathbb{R})$ be the representation given by

$$\gamma \mapsto \begin{pmatrix} \gamma & \\ & r(\gamma) \end{pmatrix}$$

for $\gamma \in \Gamma'$. Then ρ' satisfies $\rho'(\Gamma') \cap \overline{\rho'(\Lambda')}^{\mathrm{Zar}} = \rho'(\Lambda')$, since indeed $\rho'(\Gamma') \cap \mathsf{\Lambda}(\mathsf{SL}_n(\mathbb{R})) = \rho'(\Lambda')$, where $\mathsf{\Lambda} : \mathsf{SL}_n(\mathbb{R}) \to \mathsf{SL}_{2n}(\mathbb{R})$ is the diagonal embedding $g \mapsto \mathrm{diag}(g, g)$. Hence, by Lemma 3.5, we may take ρ to be the representation $\mathrm{ind}_{\Gamma'}$ of Γ induced by ρ'.

Moreover, if Λ is contained in a compact subgroup $\mathsf{C} < \mathsf{SL}_n(\mathbb{R})$, then $\rho'(\Lambda') \subset \mathsf{\Lambda}(\mathsf{C})$ is precompact, and hence so is $\rho(\Lambda')$ since $\rho\big|_{\Gamma'}$ extends to a representation of $\mathsf{SL}_n(\mathbb{R})$. We conclude in this case that $\rho(\Lambda)$ is also precompact since Λ' is of finite index in Λ. ☐

The following is now immediate from Theorem 1.2 and Lemma 3.6.

Corollary 3.7. *Let $\Gamma < \mathsf{SL}_n(\mathbb{R})$ be a subgroup. Suppose that $\Lambda < \Gamma$ is virtually a virtual retract of Γ, and is precompact in $\mathsf{SL}_n(\mathbb{R})$. Then the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$.*

We are now ready to prove the following more general fact than Theorem 1.4 (see [39] for details on the notion of a relatively quasiconvex subgroup of a relatively hyperbolic group). Indeed, Theorem 1.4 is the particular case of Theorem 3.8 where the peripheral subgroups of Γ are trivial.

Theorem 3.8. *Let Γ be a virtually compact special group that is hyperbolic relative to a collection of finitely generated virtually abelian subgroups, and $\Lambda < \Gamma$ be a relatively quasiconvex subgroup. Then the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$.*

Proof. By work of Agol [4], there is an embedding of Γ in a compact Lie group. Theorem 1.4 now follows from Corollary 3.7, together with the fact that a residually finite group that is hyperbolic relative to a collection $\mathscr{P}$ of finitely generated virtually abelian subgroups possesses a finite-index subgroup whose intersection with each member of $\mathscr{P}$ is abelian, and the fact that relatively quasiconvex subgroups of compact special groups that are hyperbolic relative to a collection of finitely generated abelian subgroups are virtual retracts; see [37, Thm. 7.3] and [23, Thm. 1.3]. □

A similar argument where one instead uses that virtually abelian subgroups of arbitrary virtually special groups are virtual retracts [54], yields the following.

Corollary 3.9. *Let Γ be a finitely generated virtually special group and $\Lambda < \Gamma$ a virtually abelian subgroup. Then the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$.*

Using deep results about Kleinian groups, we also deduce the following.

Corollary 3.10. *Let Γ be a finitely generated discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$ *and $\Lambda < \Gamma$ an arbitrary finitely generated subgroup. Then the double $\Gamma *_\Lambda \Gamma$ is linear.*

Proof. By the resolution of the density conjecture [17, 18, 57, 59], one has that Γ is isomorphic to a geometrically finite subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ which we continue to denote by Γ. By Brooks' theorem [19], there is a discrete and faithful deformation of the inclusion $\Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ whose image is contained in a lattice within $\mathrm{Isom}(\mathbb{H}^3)$, so that we may assume Γ is itself a lattice in $\mathrm{Isom}(\mathbb{H}^3)$. One then has by work of Agol [3] and Wise [74, §17.c] that Γ is virtually compact special. If Λ is geometrically finite within $\mathrm{Isom}(\mathbb{H}^3)$, then Λ is relatively quasiconvex in Γ (with respect to the maximal parabolic subgroups of Γ; see [50, Thm. 1.3]), and hence the double $\Gamma *_\Lambda \Gamma$ is linear by Theorem 3.8. Otherwise, by Canary's covering theorem [21] and the resolution of the tameness conjecture [1, 20], there is a finite-index normal subgroup Γ' of Γ and a map $\varphi : \Gamma' \to \mathbb{Z}$ such that $\Lambda' := \Gamma' \cap \Lambda$ coincides with the kernel of φ. Now choose some faithful representation $\rho_0' : \Gamma' \to \mathsf{O}(d)$, as guaranteed by Agol [4], and define $\rho' : \Gamma' \to \mathsf{GL}_{d+1}(\mathbb{R})$ by

$$\rho(\gamma) = \begin{pmatrix} \gamma & \\ & e^{\varphi(\gamma)} \end{pmatrix}$$

for $\gamma \in \Gamma'$. Then $\rho(\Lambda')$ is the intersection of $\rho(\Gamma')$ with the compact subgroup

$$\left\{ \begin{pmatrix} g & \\ & 1 \end{pmatrix} \right\}_{g \in \mathsf{O}(d)}$$

of $\mathsf{GL}_{d+1}(\mathbb{R})$. It thus follows from Lemma 3.5 and Theorem 1.2 that the double $\Gamma *_\Lambda \Gamma$ is again linear in this case. □

We now proceed to the proofs of Theorems 1.5 and 1.6.

Proof of Theorem 1.5. Let $K \subset \mathbb{R}$ be the adjoint trace field of Γ, and denote by $\mathcal{O}_K$ the ring of integers of K. Since G is absolutely almost simple, if Γ is an arithmetic subgroup of G, we have (see, for instance, [66, Lem. 2.6]) that K is a totally real number field, and there exist a K-group $\mathbf{G} < \mathsf{SL}_d$, a local isomorphism $\phi : \mathsf{G} \to \mathbf{G}(\mathbb{R})$, and a finite-index normal subgroup $\Gamma' < \Gamma$ such that

- ϕ is injective on Γ';
- $\phi(\Gamma')$ is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_K)$;
- $\mathbf{G}^\sigma(\mathbb{R})$ is compact for each non-identity embedding $\sigma : K \to \mathbb{R}$.

Let $\Lambda' := \Gamma' \cap \Lambda$, and let $\varphi : \Gamma \to \mathsf{GL}_n(\mathbb{R})$ be the representation of Γ induced by $\phi : \Gamma' \to \mathsf{SL}_d(\mathbb{R})$. Since Λ' is of finite index in Λ and the latter is cocompact in $\overline{\Lambda}^{\mathrm{Zar}}$, we have that Λ' is cocompact in $\overline{\Lambda}^{\mathrm{Zar}}$ and hence cocompact in $\overline{\Lambda'}^{\mathrm{Zar}}$. It follows that Λ' is of finite index in $\Gamma \cap \overline{\Lambda'}^{\mathrm{Zar}}$. Since the restriction of φ to Γ' extends to an $\mathbb{R}$-algebraic representation of G, we have that $\varphi(\Lambda')$ is of finite index in $\varphi(\Gamma') \cap \overline{\varphi(\Lambda')}^{\mathrm{Zar}}$. Since the latter is of finite index in $\varphi(\Gamma) \cap \overline{\varphi(\Lambda)}^{\mathrm{Zar}}$, and since $\varphi(\Lambda)$ contains $\varphi(\Lambda')$, we conclude that $\varphi(\Lambda)$ is of finite index in $\varphi(\Gamma) \cap \overline{\varphi(\Lambda)}^{\mathrm{Zar}}$.

Since we are assuming that $K \neq \mathbb{Q}$, there is at least one non-identity embedding $\sigma_0 : K \to \mathbb{R}$. We may then postconjugate φ such that $\varphi(\Gamma) \subset \mathsf{GL}_n(K)$ and $\varphi(\Gamma')^{\sigma_0}$ is precompact in $\mathsf{GL}_n(\mathbb{R})$. The finite-index supergroup $\varphi(\Gamma)^{\sigma_0}$ of $\varphi(\Gamma')^{\sigma_0}$ is then also precompact in $\mathsf{GL}_n(\mathbb{R})$. We may now apply Corollary 1.3(2) to conclude that the double $\Gamma *_\Lambda \Gamma$ is linear. This proves Theorem 1.5(1).

Now suppose G is not locally isomorphic to $\mathsf{O}(n,1)$ or $\mathsf{U}(n,1)$ for any $n \in \mathbb{N}$. Then Γ is an arithmetic subgroup of G by the arithmeticity theorems of Margulis [51] and Gromov–Schoen [35], and hence we conclude from Theorem 1.5(1) that the double $\Gamma *_\Lambda \Gamma$ is linear as soon as Λ is cocompact in $\overline{\Lambda}^{\mathrm{Zar}}$ and separable in Γ. On the other hand, if Λ is not separable in Γ, then the double $\Gamma *_\Lambda \Gamma$ is not residually finite, let alone linear. Finally, if Λ is not cocompact in $\overline{\Lambda}^{\mathrm{Zar}}$, then since Γ is cocompact[9] in G, we have that Λ is of infinite covolume in $\overline{\Lambda}^{\mathrm{Zar}}$, and hence the double $\Gamma *_\Lambda \Gamma$ again fails to be linear by [69, Cor. 5.3]. ☐

Remark 3.11. By Margulis's arithmeticity theorem [51], any lattice $\Gamma < \mathsf{O}(p,q)$ for $p \geqslant q \geqslant 2$ and $p+q \geqslant 5$ is arithmetic. If moreover $p+q$ is odd, then it follows from the classification of arithmetic subgroups of $\mathsf{O}(p,q)$ that Γ is commensurable (in the wide sense) to $\mathsf{O}(f;\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of some totally real number field $K \subset \mathbb{R}$, and f is a quadratic form of signature (p,q) with coefficients in K; see [55, pp. 380]. In this case, the field K coincides with the adjoint trace field of Γ (again by [66, Lem. 2.6], for instance). If Γ is moreover cocompact, then f must be anisotropic over K and thus $K \neq \mathbb{Q}$ by Meyer's theorem.

Proof of Theorem 1.6. Let Γ' be a finite-index normal subgroup of Γ such that $\Gamma' \subset \mathbf{G}(\mathcal{O}_{K,S})$, and let $\Lambda' := \Gamma' \cap \Lambda$. Then the projection $\phi : \mathsf{G} \to \mathbf{G}(K_w)$ is faithful on $\mathbf{G}(\mathcal{O}_{K,S})$ and hence on Γ'. Let $\varphi : \Gamma \to \mathsf{GL}_n(\mathbb{R})$ be the representation induced by $\phi|_{\Gamma'}$. Then, since $\phi(H) = \mathbf{H}(K_w)$ is compact, we have as in the proof of Theorem 1.5 that the closure $\overline{\varphi(\Lambda)} \subset \mathsf{GL}_n(\mathbb{R})$ is compact and that $\varphi(\Lambda)$ is of finite index in $\varphi(\Gamma) \cap \overline{\varphi(\Lambda)}$. Since Λ was moreover assumed separable in Γ, we conclude from Corollary 1.3(1) that the double $\Gamma *_\Lambda \Gamma$ is linear over $\mathbb{R}$. ☐

4. Doubling along cocompact stabilizers

In this section, we prove Theorem 1.7. We will need the following consequence of Theorem 2.1.

Proposition 4.1. *Let Γ_1, Γ_2 be groups with a common subgroup Λ. Suppose we are given for $j = 1,2$ and for some integral domain R faithful representations $\phi^j : \Gamma_j \to \mathsf{GL}_n(\mathsf{R})$ and $\psi^j : \Gamma_j \to \mathsf{GL}_n(\mathsf{R})$ such that*

[9] Indeed, let $\Gamma' < \Gamma$ and ϕ be as in the proof of (1). Then since $\phi(\Gamma')^{\sigma_0}$ is precompact, we have that $\phi(\Gamma')^{\sigma_0}$ contains no nontrivial unipotent elements, and hence neither does Γ', so that Γ' is cocompact in G by Godement's criterion (see, for instance, [55, Prop. 5.3.1]).

- *the representations ϕ^1, ϕ^2, ψ^1, and ψ^2 all coincide on Λ; and*
- *for $j = 1, 2$ and $\gamma \in \Gamma_j$, if $\phi^j(\gamma)^{-1}\psi^j(\gamma)$ is unipotent, then $\gamma \in \Lambda$.*

Then, for $i = 1, \dots n+1$ and $j = 1, 2$, there are representations $\rho_i^j : \Gamma_j \to \mathsf{GL}_{n_i}(\mathsf{R})$, where $n_i = \binom{n+1}{i}$, all of which are faithful with the exception of the ρ_{n+1}^j, satisfying the hypotheses of Lemma 3.1. In particular, there is a faithful representation of $\Gamma_1 *_\Lambda \Gamma_2$ into $\mathsf{GL}_r(\mathsf{R}[s,t])$, where

$$r = \left(\binom{2n+2}{n+1} - 1\right)^2 + 1.$$

Proof. By composing the ϕ^j and the ψ^j with the embedding $\mathsf{GL}_n(\mathsf{R}) \times \{1\} \hookrightarrow \mathsf{GL}_{n+1}(\mathsf{R})$, we obtain representations $\Gamma_j \to \mathsf{GL}_{n+1}(\mathsf{R})$, which we continue to denote by ϕ^j and ψ^j, fixing the standard basis vector e_{n+1} and mapping no nontrivial matrix to a scalar matrix. Now for $i = 1, \dots n+1$ and $j = 1, 2$, let $\rho_i^j : \Gamma_j \to \mathsf{GL}(\mathrm{End}(\wedge^i \mathsf{R}^{n+1}))$ be the representation given by

$$\rho_i^j(\gamma)(M) = \left(\wedge^i \phi^j(\gamma)\right) M \left(\wedge^i \psi^j(\gamma)\right)^{-1}$$

for $M \in \mathrm{End}(\wedge^j \mathsf{R}^{n+1})$ and $\gamma \in \Gamma_j$. Note that, for $i = 1, \dots, n$, the ρ_i^j are faithful. Indeed, if $\rho_i^j(\gamma) = \mathrm{Id}$, then, by considering the action on the endomorphism $M = \mathrm{Id}$, we see that $\wedge^i \phi^j(\gamma) = \wedge^i \psi^j(\gamma)$, and hence $\wedge^i \phi^j(\gamma)$ commutes with all endomorphisms of $\wedge^j \mathsf{R}^{n+1}$, so that $\wedge^i \phi^j(\gamma)$ is scalar. For $i < n+1$, this implies that $\phi^j(\gamma)$ is itself scalar, so that $\gamma = \mathrm{Id}$.

Since ϕ^j and ψ^j coincide on Λ, we have that $\rho_i^j(\Lambda)$ fixes the identity map $\mathrm{Id} \in \mathrm{End}(\wedge^i \mathsf{R}^{n+1})$ and preserves the R-submodule $V_i := \{M \in \mathrm{End}(\wedge^i \mathsf{R}^{n+1}) : \mathrm{tr}(M) = 0\}$. In addition, for every $\gamma \in \Gamma_j$, we have

$$\rho_i^j(\gamma)(\mathrm{Id}) = \wedge^i \left(\phi^j(\gamma)\psi^j(\gamma)^{-1}\right) = \mathrm{tr}\left(\wedge^i \left(\phi^j(\gamma)\psi^j(\gamma)^{-1}\right)\right)\mathrm{Id} + M_{ij\gamma}$$

for some $M_{ij\gamma} \in V_i$. Thus, if $\mathcal{B}_i$ is an ordered basis for $\mathrm{End}(\wedge^i \mathsf{R}^{n+1})$ whose first basis vector is the identity endomorphism Id and whose remaining basis vectors are contained in V_i, then, writing $\rho_i^j(\gamma)$ with respect to the basis $\mathcal{B}_i$, we have that the top-left entry of $\rho_i^j(\gamma)$ is

$$\left(\rho_i^j(\gamma)\right)_{11} = \mathrm{tr}\left(\wedge^j \left(\phi^j(\gamma)\psi^j(\gamma)^{-1}\right)\right) \tag{4}$$

for any $\gamma \in \Gamma_j$.

We now check that the hypotheses of Lemma 3.1 are satisfied for the representations ρ_i^j. We have already observed that $\rho_i^j(\Lambda)$ fixes the identity endomorphism of $\wedge^i \mathsf{R}^{n+1}$ and preserves V_i, so that $\rho_i^j(\Lambda) \subset \{1\} \times \mathsf{GL}_{n_i-1}(\mathsf{R})$. That, for $i = 1, \dots, n+1$, the representations ρ_i^1 and ρ_i^2 coincide on Λ follows immediately from the fact that the representations ϕ^1, ϕ^2, ψ^1, and ψ^2 all coincide on Λ. Finally, if $\gamma \in \Gamma_j$ and $\left(\rho_i^j(\gamma)\right)_{11} = 1$ for $i = 1, \dots, n+1$, then, by (4), the characteristic polynomial of $\phi^j(\gamma)\psi^j(\gamma)^{-1}$ coincides with that of the identity matrix, i.e., the matrix $\phi^j(\gamma)\psi^j(\gamma)^{-1}$ is unipotent, and hence $\gamma \in \Lambda$ by assumption. Thus, by applying Lemma 3.1 to the representations ρ_i^j, we obtain a faithful representation of $\Gamma_1 *_\Lambda \Gamma_2$ into $\mathsf{GL}_r(\mathsf{R}[s,t])$ where $r = d^2 + 1$ and

$$d = \sum_{i=1}^{n+1} \dim_{\mathsf{R}} \left(\mathrm{End}(\wedge^i \mathsf{R}^{n+1})\right) = \sum_{i=1}^{n+1} \binom{n+1}{i}^2 = \binom{2n+2}{n+1} - 1. \qquad \square$$

One proves in precisely the same manner the following more technical statement.

Theorem 4.2. *Let Γ_1, Γ_2 be groups with a common subgroup Λ. Suppose we are given for $j = 1, 2$, for some integer $\ell \geqslant 1$, for $k = 1, \dots, \ell$, and for some integral domain R faithful representations $\phi_k^j : \Gamma_j \to \mathsf{GL}_n(\mathsf{R})$ and $\psi_k^j : \Gamma_j \to \mathsf{GL}_n(\mathsf{R})$ such that*

- *the representations ϕ_k^j and $\psi_{k'}^{j'}$ all coincide on Λ; and*
- *for $j = 1, 2$ and $\gamma \in \Gamma_j$, if $\phi_k^j(\gamma)^{-1}\psi_k^j(\gamma)$ is unipotent for every $k = 1, \dots, \ell$, then $\gamma \in \Lambda$.*

Then, for $i = 1, \dots n+1$, for $j = 1, 2$, and for $k = 1, \dots, \ell$, there are representations $\rho_i^{j,k} : \Gamma_j \to \mathsf{GL}_{n_i}(\mathsf{R})$, where $n_i = \binom{n+1}{i}$, all of which are faithful with the exception of the $\rho_{n+1}^{j,k}$, satisfying

- *$\rho_i^{j,k}(\Lambda) \subset \{1\} \times \mathsf{GL}_{n_i-1}(\mathsf{R})$;*
- *the representations $\rho_i^{1,k}$ and $\rho_i^{2,k}$ coincide on Λ for $i = 1, \dots, n+1$ and $k = 1, \dots, \ell$; and*
- *for $j = 1, 2$ and $\gamma \in \Gamma_j$, if the top-left entry of $\rho_i^{j,k}(\gamma)$ is equal to 1 for every i and k, then $\gamma \in \Lambda$.*

In particular, there is a faithful representation of $\Gamma_1 *_\Lambda \Gamma_2$ into $\mathsf{GL}_r(\mathsf{R}[s,t])$, where

$$r = \ell^2 \left(\binom{2n+2}{n+1} - 1 \right)^2 + 1.$$

We record the following consequence of Theorem 4.2.

Theorem 4.3. *Let R be an integral domain, let $\Gamma < \mathsf{GL}_m(\mathsf{R})$ be a subgroup, and suppose one has representations $\varphi_0, \dots, \varphi_\ell : \Gamma \to \mathsf{GL}_m(\mathsf{R})$ the subgroup generated by whose images contains no nontrivial unipotent elements. Then the double $\Gamma *_\Lambda \Gamma$ is linear, where*

$$\Lambda := \{\gamma \in \Gamma : \varphi_0(\gamma) = \dots = \varphi_\ell(\gamma)\}.$$

Proof of Theorem 4.3. For $\gamma \in \Gamma$, we have by our assumptions that $\gamma \in \Lambda$ if and only if $\varphi_0(\gamma)^{-1}\varphi_k(\gamma)$ is unipotent for every $k = 1, \dots, \ell$. We may now apply Theorem 4.2 with $\Gamma_1 = \Gamma_2 = \Gamma$, and with $\phi_k^1 = \phi_k^2 = \mathrm{id}_\Gamma \times \varphi_0$ and $\psi_k^1 = \psi_k^2 = \mathrm{id}_\Gamma \times \varphi_k$ for $k = 1, \dots, \ell$, where $\mathrm{id}_\Gamma \times \varphi_k : \Gamma \to \mathsf{GL}_{2m}(\mathsf{R})$ is the representation given by $\gamma \mapsto \begin{pmatrix} \gamma & \\ & \varphi_k(\gamma) \end{pmatrix}$ for $k = 0, \dots, \ell$. We obtain in this manner a faithful representation of the double $\Gamma *_\Lambda \Gamma$ into $\mathsf{GL}_r(\mathsf{R}[s,t])$, where where $r = \ell^2 \left(\binom{4m+2}{2m+1} - 1\right)^2 + 1$. ☐

Example 4.4. Let G be any semisimple real algebraic group, and fix a Langlands decomposition MAN of a minimal parabolic subgroup of G. Then for any cocompact lattice $\Gamma < \mathsf{G}$, if H is a Cartan subgroup of G, i.e., a conjugate of MA in G, such that $\Lambda := \Gamma \cap \mathsf{H}$ is cocompact[10] in H, then the double $\Gamma *_\Lambda \Gamma$ is linear. Indeed, since Λ is cocompact in H, there is some element $h \in \Lambda$ such that H is precisely the centralizer of h in G. Linearity of the double $\Gamma *_\Lambda \Gamma$ thus follows from Theorem 4.3 if one takes φ_0 to be the inclusion $\Gamma \to \mathsf{G}$ and φ_1 to be the inclusion postconjugated by h. Note that Theorem 1.2 cannot be applied to deduce linearity of some of these doubles; for example, if G is absolutely almost simple and of real rank $\geqslant 2$ and Γ has adjoint trace field $\mathbb{Q}$, then it follows from Margulis superrigidity [51] that the image of Λ under no faithful finite-dimensional real representation of Γ is precompact. Cocompact such Γ exist for instance within $\mathsf{G} = \mathsf{SL}_d(\mathbb{R})$ for any $d \geqslant 3$ (see [9]); note that Theorem 1.7 also does not apply to the latter examples since in those cases the pair $(\mathsf{G}, \mathsf{MA})$ fails to be affine symmetric.

Theorem 1.7 now follows immediately from the following more general statement.

Theorem 4.5. *Let X be a Riemannian symmetric space of noncompact type and $Y_1, \dots, Y_\ell$ be reflective submanifolds of X with nonempty intersection Y. Let $\Gamma < \mathrm{Isom}(X)$ be a discrete subgroup, let $\Lambda_k < \Gamma$ be the stabilizer of Y_k in Γ for $k = 1, \dots, \ell$, and let $\Lambda = \bigcap_{k=1}^{\ell} \Lambda_k$. If Λ_k acts cocompactly on Y_k for every k, then the double $\Gamma *_\Lambda \Gamma$ embeds in $\mathsf{GL}_r(\mathbb{R})$ for some dimension $r = r(X, \ell)$ depending only on X and ℓ.*

Proof. Denote by $\mathsf{G} = \mathrm{Isom}(X)$, by $\mathfrak{g}$ the Lie algebra of G, and by $\mathrm{Ad} : \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$ the adjoint representation of G. Note that Ad is faithful since the centralizer in G of the identity component

[10] That such H always abound follows from Poincaré recurrence on the compact homogeneous space $\Gamma\backslash\mathsf{G}$, together with a well-known argument of Selberg [64, Lem. 1.10].

of G is trivial. For $k = 1, \ldots, \ell$, denote by $\sigma_k \in \mathsf{G}$ the isometric involution of X whose fixed point set is Y_k.

Now fix $k \in \{1, \ldots, \ell\}$ and $\gamma \in \Gamma$. We claim that $\gamma \in \Lambda_k$ if $\mathrm{Ad}(\gamma^{-1}\sigma_k\gamma\sigma_k)$ is unipotent. Indeed, suppose $\gamma \notin \Lambda_k$. In the case that $Y_k \cap \gamma^{-1}Y_k \neq \varnothing$, we have that $\gamma^{-1}\sigma_k\gamma\sigma_k$ is a nontrivial isometry of X fixing $Y_k \cap \gamma^{-1}Y_k$ pointwise. In the case that $Y_k \cap \gamma^{-1}Y_k = \varnothing$, we have that the distance between Y_k and $\gamma^{-1}Y_k$ is bounded below by the length $\delta > 0$ of a shortest (positive-length) orthogeodesic to $\Lambda_k\backslash Y_k$ in $\Gamma\backslash X$. Since σ_k inverts every geodesic line in X orthogonal to Y_k, we have in this case that the isometry $\gamma^{-1}\sigma_k\gamma\sigma_k$ preserves each geodesic line L in X orthogonal to both Y_k and $\gamma^{-1}Y_k$ and translates by a distance of $\geqslant \delta$ along L. Thus, in either case, we have that $\gamma^{-1}\sigma_k\gamma\sigma_k$ is a nontrivial semisimple isometry of X, and hence that $\mathrm{Ad}(\gamma^{-1}\sigma_k\gamma\sigma_k)$ is a nontrivial semisimple matrix, so that $\mathrm{Ad}(\gamma^{-1}\sigma_k\gamma\sigma_k)$ in particular fails to be unipotent.

To conclude, we may now apply Theorem 4.2 with $\Gamma_1 = \Gamma_2 = \Gamma$, with $\phi_k^1 = \phi_k^2 = \mathrm{Ad}$, with $\psi_k^1 = \psi_k^2$ the representation given by precomposing Ad with conjugation by σ_k, and with R the entry field of the subgroup of $\mathsf{GL}(\mathfrak{g})$ generated by $\mathrm{Ad}(\Gamma), \mathrm{Ad}(\sigma_1\Gamma\sigma_1), \ldots, \mathrm{Ad}(\sigma_\ell\Gamma\sigma_\ell)$. We may in particular take

$$r(X, \ell) = \ell^2 \left(\binom{2\dim_{\mathbb{R}} \mathfrak{g} + 2}{\dim_{\mathbb{R}} \mathfrak{g} + 1} - 1 \right)^2 + 1. \qquad \square$$

Example 4.6. Let $\mathsf{G} = \mathsf{SL}_2(\mathbb{R}) \times \mathsf{SL}_2(\mathbb{R})$, and let $\Gamma < \mathsf{G}$ be any irreducible noncocompact lattice. Then it follows from Margulis's arithmeticity theorem, together with the fact that any quadratic field embeds in some quaternion division algebra over $\mathbb{Q}$, that there are conjugates H of the diagonal subgroup $\{(g,g)\}_{g \in \mathsf{SL}_2(\mathbb{R})} < \mathsf{G}$ such that $\Lambda := \Gamma \cap \mathsf{H}$ is cocompact in H. One then deduces from Theorem 1.7 that any such double $\Gamma *_\Lambda \Gamma$ is linear. These are examples to which Theorem 1.2 does not apply, since it again follows from Margulis superrigidity that in these cases the image of Λ under no faithful finite-dimensional real representation of Γ is precompact.

Example 4.7. Let M be an arithmetic $\mathbb{K}$-hyperbolic manifold of simplest type and of $\mathbb{K}$-dimension $d \geqslant 2$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Suppose also that the adjoint trace field of (the fundamental group of) M is quadratic, so that M is in particular compact. Then, as argued in [7, Cor. 4.5] in the case $\mathbb{K} = \mathbb{R}$, the manifold M is commensurable via restriction of scalars to an immersed totally geodesic $\mathbb{K}$-submanifold $\Lambda\backslash Y$ of a noncompact arithmetic $\mathbb{K}$-hyperbolic manifold $\Gamma\backslash X$, where X is of $\mathbb{K}$-dimension $2d+1$. It then follows from Theorem 1.7 that the double $\Gamma *_\Lambda \Gamma$ is linear. For $\mathbb{K} = \mathbb{H}$, this again gives examples to which Theorem 1.2 does not apply, this time by Corlette superrigidity [28].

Example 4.8. Let f_1 and f_2 be the $\mathbb{Q}$-anisotropic[11] rational quadratic forms $x_1^2 + x_2^2 - 7x_3^2$ and $17x_1^2 + x_2^2 - 7x_3^2$, respectively. Then $(\mathsf{SL}_3(\mathbb{R}), \mathsf{SO}(f_i;\mathbb{R}))$ is an affine symmetric pair and $\mathsf{SO}(f_i;\mathbb{Z})$ is cocompact in $\mathsf{SO}(f_i;\mathbb{R})$ for $i = 1, 2$. It then follows from Theorem 4.5 that the double $\Gamma *_\Lambda \Gamma$ is linear, where $\Gamma := \mathsf{SL}_3(\mathbb{Z})$, and where Λ is the virtually infinite cyclic subgroup

$$\Lambda := \mathsf{SO}(f_1;\mathbb{Z}) \cap \mathsf{SO}(f_2;\mathbb{Z}) = \mathsf{S}(\{\pm 1\} \times \mathsf{O}(f;\mathbb{Z}));$$

here f is the quadratic form in two variables given by $x^2 - 7y^2$. We include this example because the pair $(\mathsf{SL}_3(\mathbb{R}), \mathsf{S}(\{\pm1\} \times \mathsf{O}(f;\mathbb{R})))$ is *not* affine symmetric.

5. Doubling along noncocompact lattices

In this section, we use superrigidity to prove Theorem 1.1.

Proof of Theorem 1.1. If Λ is not of finite covolume in $\overline{\Lambda}^{\mathrm{Zar}}$, then $\Gamma *_\Lambda \Gamma$ is not linear by [69, Cor. 5.3]. We may thus assume that Λ is a noncocompact lattice in $\overline{\Lambda}^{\mathrm{Zar}}$, so that Γ is also not cocompact in G. By Margulis's arithmeticity theorem [51] (see also [55, Cor. 5.3.2]), there is then

[11] The f_i are indeed $\mathbb{Q}_2$-anisotropic again since -7 and 17 are both squares in $\mathbb{Q}_2$ and the stufe of $\mathbb{Q}_2$ is 4.

an algebraically connected and simply connected $\mathbb{Q}$-group $\mathbf{G}$ and a morphism $\phi : \mathbf{G}(\mathbb{R}) \to \mathsf{G}$ with finite kernel such that Γ is commensurable with $\phi(\mathbf{G}(\mathbb{Z}))$. Let Γ' be a finite-index subgroup of $\mathbf{G}(\mathbb{Z})$ such that ϕ is injective on Γ' and $\phi(\Gamma') \subset \Gamma$, and let $\Lambda' = \phi^{-1}(\Lambda) \cap \Gamma'$. Then it suffices to show that $\Gamma' *_{\Lambda'} \Gamma'$ is not linear.

By our assumption on Λ, we have that Λ' is of finite index in $\mathbf{H}(\mathbb{Z})$ for some isotropic $\mathbb{Q}$-subgroup $\mathbf{H} < \mathbf{G}$ of positive codimension. Let g be a nontrivial element of $\mathbf{T}(\mathbb{R})^\circ$ for some positive-dimensional $\mathbb{Q}$-split torus $\mathbf{T} < \mathbf{H}$, and denote by $U(g^{\pm 1}) < \mathbf{G}(\mathbb{R})$ the (stable) horospherical subgroup of $g^{\pm 1}$. Since $\mathbf{G}(\mathbb{R})^\circ$ is generated by $U(g)$ and $U(g^{-1})$ (see [51, Thm. I.2.3.1]), we must have that $\mathbf{H}(\mathbb{R}) \cap U(g^\epsilon)$ is of positive codimension in $U(g^\epsilon)$ for some $\epsilon \in \{\pm 1\}$; assume henceforth without loss of generality that $\epsilon = 1$. On the other hand, since $U(g)$ is defined over $\mathbb{Q}$, we have that $\Gamma' \cap U(g)$ is a lattice in $U(g)$. Thus, there is some $u \in \Gamma' \cap U(g)$ such that $\langle u \rangle \cap \mathbf{H}(\mathbb{R}) = \{1\}$.

Now suppose one has a faithful representation $\rho : \Gamma' *_{\Lambda'} \Gamma' \to \mathsf{GL}_d(\mathbb{F})$ for some field $\mathbb{F}$ and $d \in \mathbb{N}$. Since Γ' is itself not linear over any field of positive characteristic by our assumption on G, we must have that $\mathrm{char}(\mathbb{F}) = 0$. Since Γ' is finitely generated, and since $\mathsf{GL}_d(\mathbb{C})$ embeds into $\mathsf{GL}_{2d}(\mathbb{R})$, we may assume $\mathbb{F} = \mathbb{R}$. By Margulis superrigidity [51] in the case $\mathrm{rk}_{\mathbb{R}}(\mathsf{G}) \geqslant 2$, and by Corlette [28] and Gromov–Schoen [35] superrigidity in the case $\mathrm{rk}_{\mathbb{R}}(\mathsf{G}) = 1$, there exist finite-index subgroups $\Gamma_i < \Gamma'$ and continuous representations $\overline{\rho}_i : \mathbf{G}(\mathbb{R}) \to \mathsf{GL}_d(\mathbb{R})$ such that

$$\overline{\rho}_i(\gamma) = \rho(\gamma),\ \gamma \in \Gamma_i,$$
$$\overline{\rho}_1(h) = \overline{\rho}_2(h),\ h \in \Lambda' \cap \Gamma_1 \cap \Gamma_2.$$

Since $\mathbf{G}$ is algebraically simply connected, by [60, Thm. 3.3.4] the representations ρ_1, ρ_2 are $\mathbb{R}$-algebraic. Thus, there is a finite-index subgroup $\mathsf{H} < \mathbf{H}(\mathbb{R})$ containing $\Lambda' \cap \Gamma_1 \cap \Gamma_2$ with $\overline{\rho}_1(h) = \overline{\rho}_2(h)$ for every $h \in \mathsf{H}$. In particular, we have $\overline{\rho}_1(g) = \overline{\rho}_2(g)$. Denoting by $\mathcal{U}(g)$ the horospherical subgroup of $\overline{\rho}_1(g) = \overline{\rho}_2(g)$ in $\mathsf{GL}_d(\mathbb{R})$, we then have that $\overline{\rho}_i(u^m) \in \mathcal{U}(g)$ for $i = 1, 2$ and for $m > 0$ such that $u^m \in \Lambda' \cap \Gamma_1 \cap \Gamma_2$. Now since ρ was assumed faithful, we have that $\langle \overline{\rho}_1(u^m), \overline{\rho}_2(u^m) \rangle < \mathcal{U}(g)$ decomposes as the free product $\langle \overline{\rho}_1(u^m) \rangle * \langle \overline{\rho}_2(u^m) \rangle$. But this is absurd since $\mathcal{U}(g)$ is a nilpotent group. □

Remark 5.1. Note that if Λ is any finite-index subgroup of a finitely generated linear group Γ, then, since amalgams of finite groups are virtually free and hence linear, the amalgam $\Gamma *_\Lambda \Gamma$ is also linear.

Remark 5.2. What follows is a straightforward demonstration of the failure of Theorem 1.1 for certain pairs (Γ, Λ), where Γ is a lattice in $\mathsf{O}(n,1)$, $3 \leqslant n \leqslant 8$, and $\Lambda = \Gamma \cap \mathsf{H}$ is of finite covolume in some conjugate H of $\mathsf{O}(n-1,1)$ in $\mathsf{O}(n,1)$. Indeed, for any such n, there is a noncompact finite-volume hyperbolic polyhedron Q of dimension n; see [63]. Let P be the polyhedron obtained from Q by doubling P along some noncompact codimension-1 face F of Q. Let $\Gamma < \mathsf{O}(n,1)$ be the group generated by the reflections in the walls of P, and let $\Lambda < \Gamma$ be the group generated by the reflections in those walls of P that are orthogonal to F. Then the pair (Γ, Λ) is as above, but the double $\Gamma *_\Lambda \Gamma$ abstractly remains a (right-angled) Coxeter group and is thus linear by classical work of Tits [16] and Vinberg [72].

Corollary 5.3. *Let* G *and* $\Gamma < \mathsf{G}$ *be as in Theorem 1.1. If* Γ *is not cocompact in* G*, then* Γ *contains a subgroup* Λ *such that the double* $\Gamma *_\Lambda \Gamma$ *is residually finite but not linear.*

Proof. As in the proof of Theorem 1.1, it follows from Margulis's arithmeticity theorem that there is a $\mathbb{Q}$-group $\mathbf{G}$ and a local isomorphism $\phi : \mathbf{G}(\mathbb{R}) \to \mathsf{G}$ such that Γ is commensurable with $\phi(\mathbf{G}(\mathbb{Z}))$. Since Γ is not cocompact in G, Godement's criterion (see, for instance, [55, Prop. 5.3.1]) guarantees the existence of a unipotent element $u \in \mathbf{G}(\mathbb{Z})$. It now follows from the Jacobson–Morosov lemma (over $\mathbb{Q}$; see [41, Ch. 3, Thm. 17]) that there is a $\mathbb{Q}$-subgroup $\mathbf{H} < \mathbf{G}$ such that $\mathbf{H}(\mathbb{R})$ is locally isomorphic to $\mathsf{SL}_2(\mathbb{R})$ and $u \in \mathbf{H}(\mathbb{Z})$. Let $\mathsf{H} := \overline{\phi(\mathbf{H}(\mathbb{R}))}^{\mathrm{Zar}} < \mathsf{G}$, and

$\Lambda := \Gamma \cap \mathsf{H}$. Then Λ is separable in Γ by [11, Lemme principal], so that the double $\Gamma *_\Lambda \Gamma$ is residually finite [6, Prop. 1]. On the other hand, since H remains locally isomorphic to $\mathsf{SL}_2(\mathbb{R})$ and Λ contains the unipotent element $\phi(u)$, we have that Λ is not cocompact in Γ, so that the double $\Gamma *_\Lambda \Gamma$ is not linear by Theorem 1.1. $\square$

Remark 5.4. In the case that the real rank of G is $\geqslant 2$, Corollary 5.3 is implicit in the work of Tholozan–Tsouvalas [69]. Indeed, as observed in [69], it follows from work of Prasad–Rapinchuk [65] that in this case Γ contains semisimple elements γ such that $\overline{\langle\gamma\rangle}^{\mathrm{Zar}}$ is of dimension $\geqslant 2$. One then concludes from [69, Cor. 5.3] that the double $\Gamma *_{\langle\gamma\rangle} \Gamma$ is not linear. However, any such double is residually finite, essentially because arithmetic subgroups of tori satisfy the congruence subgroup property [24]; see [52, §3].

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