BORDERED HYPERBOLIC MANIFOLDS WITH A FIXED PERIMETER-TO-VOLUME RATIO

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Abstract. We exhibit, for any $n \geq 3$, infinitely many pairwise incommensurable complete finite-volume hyperbolic n-manifolds, both compact and noncompact, each with nonempty totally geodesic boundary and all sharing the same perimeter-to-volume ratio.

Two Riemannian manifolds (possibly with boundary) are said to be *commensurable* if they share a common finite Riemannian cover. Given a complete Riemannian manifold M with finite-area boundary, one defines the *perimeter* of M to be the area of ∂M. For such M that are moreover of finite volume, the perimeter-to-volume ratio is evidently a commensurability invariant. The goal of this note is to show that this invariant is rather crude even among hyperbolic manifolds with totally geodesic boundary.

Theorem 1. *For every $n \geq 3$, there are infinitely many pairwise incommensurable compact (respectively, pairwise incommensurable noncompact complete finite-volume) hyperbolic n-manifolds with nonempty totally geodesic boundary all sharing the same perimeter-to-volume ratio.*

Note that if M is a compact Riemannian manifold with boundary, then ∂M is compact and hence automatically of finite area. That the area of ∂M remains finite in the case that M is a complete finite-volume hyperbolic manifold of dimension ≥ 3 with totally geodesic boundary follows from a recurrence theorem of Dani [6].

Using elementary hyperbolic geometry, one easily produces uncountably many geometrically distinct complete finite-volume hyperbolic surfaces, both compact and noncompact, each with nonempty compact totally geodesic boundary and all sharing the same perimeter-to-volume ratio. On the other hand, by Mostow–Prasad rigidity [17, 20], there are, up to isometry, only countably many complete finite-volume hyperbolic manifolds of dimension ≥ 3 with totally geodesic boundary.

The manifolds we exhibit in the proof of Theorem 1 are obtained by gluing together pieces of incommensurable arithmetic hyperbolic manifolds in the style of Gromov and Piatetski-Shapiro [11]. It was observed by Raimbault [21] that the resulting hybrids are sensitive, even up to commensurability, to the pattern in which the subarithmetic building blocks are glued.

Given a complete hyperbolic manifold M with totally geodesic boundary, denote by $2M$ the double of M along ∂M. The following argument is due to Raimbault [21, Lem. 3.2]; see also [10, Lem. 3.5]. We sketch it for the convenience of the reader.

Lemma 2. *Let $n \geq 3$, and let N_1 and N_2 be complete finite-volume hyperbolic n-manifolds with totally geodesic boundary. Suppose each of the N_j admits a decomposition into two submanifolds N_j^1 and N_j^2 with nonempty interior and totally geodesic boundary such that*

- *the double $2N^i_j$ is arithmetic for $i, j \in \{1, 2\}$;*
- *the doubles $2N^1_1$ and $2N^1_2$ are commensurable; and*
- *the doubles $2N^1_j$ and $2N^2_j$ are incommensurable for $j = 1, 2$.*

In the case $n = 3$, suppose further that ∂N^1_1 and ∂N^1_2 are compact. If N_1 and N_2 are commensurable, then so are N^1_1 and N^1_2.

Proof of Lemma 2. Suppose there is a common finite cover N of the N_j with covering maps $p_j : N \to N_j$. Then $p_j^{-1}(N^1_j) \subset p_k^{-1}(N^1_k)$ for $j \neq k$, since otherwise at least one of the components of the common submanifold $p_j^{-1}(N^1_j) \cap p_k^{-1}(N^2_k)$ of the two incommensurable arithmetic manifolds $2p_j^{-1}(N^1_j)$ and $2p_k^{-1}(N^2_k)$ would have Zariski-dense[1] fundamental group, and this is impossible since the commensurability class of an arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ is already detected by any Zariski-dense subgroup. Thus, we have that $p_1^{-1}(N^1_1) = p_2^{-1}(N^1_2)$ is a common finite cover of the N^1_j. $\square$

Proof of Theorem 1. We first handle the compact case. Fix $n \geq 3$. Set $K = \mathbb{Q}(\sqrt{2})$, and let $\mathcal{O}_K$ be the ring of integers of K (in this case, $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$). Let f_1 and f_2 be the quadratic forms in $n+1$ variables with coefficients in K given by

$$x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2,$$
$$17x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2,$$

respectively. We choose these forms because Gelander and Levit [10, §4.2] have shown that f_1 and f_2 are not similar over K. For $i = 1, 2$, denote by $\mathbb{H}_{f_i}$ the f_i-hyperboloid model of $\mathbb{H}^n$, and by $\mathrm{O}'_{f_i}(\mathbb{R}) = \mathrm{Isom}(\mathbb{H}_{f_i})$ the index-2 subgroup of $\mathrm{O}_{f_i}(\mathbb{R})$ preserving $\mathbb{H}_{f_i}$. Denote by $H_i \subset \mathbb{H}_{f_i}$ the geodesic hyperplane $\{x_1 = 0\}$. For a nonzero ideal $\mathfrak{n} \subset \mathcal{O}_K$, denote by $\Gamma_i(\mathfrak{n})$ the principal congruence subgroup of level $\mathfrak{n}$ in $\Gamma_i := \mathrm{O}'_{f_i}(\mathcal{O}_K)$, and by $\Sigma_i(\mathfrak{n})$ the projection of H_i to the hyperbolic n-orbifold $M_i(\mathfrak{n}) := \Gamma_i(\mathfrak{n})\backslash\mathbb{H}_{f_i}$. Note that, since we have chosen forms f_1, f_2 with the same coefficients for $x_2, \ldots, x_{n+1}$, the immersed compact totally geodesic hypersurfaces $\Sigma_1(\mathfrak{n})$ and $\Sigma_2(\mathfrak{n})$ of $M_1(\mathfrak{n})$ and $M_2(\mathfrak{n})$, respectively, are isometric for any choice of nonzero ideal $\mathfrak{n} \subset \mathcal{O}_K$. Now choose such an ideal $\mathfrak{n}$ such that, for $i = 1, 2$, the group $\Gamma_i(\mathfrak{n})$ is torsion-free and $\Sigma_i(\mathfrak{n})$ is a properly embedded two-sided totally geodesic hypersurface in the manifold $M_i(\mathfrak{n})$. By an argument of Lubotzky [14] (see also [8]), there is then a smaller nonzero ideal $\mathfrak{m} \subset \mathfrak{n}$ such that the union of two lifts Σ_i and Σ'_i of $\Sigma_i(\mathfrak{n})$ to $M_i(\mathfrak{m})$ fails to separate $M_i(\mathfrak{m})$. Let M_i be the manifold with nonempty totally geodesic boundary obtained by cutting $M_i(\mathfrak{m})$ along Σ_i and Σ'_i. Color two of the boundary components of M_i red and the other two green.

We now define a sequence $(N_j)_{j\geq 1}$ of hyperbolic n-manifolds with entirely red boundary as follows. For $i = 1, 2$ and $j \geq 1$, let N^i_j be a (connected) manifold with precisely two green boundary components obtained by doubling M_i along a single green boundary component, then doubling the resulting manifold along a single green boundary component, and so on, until the resulting manifold consists of 2^{j-1} copies of M_i. Then the double $2N^i_j$ is commensurable to $M_i(\mathfrak{m})$. In particular, we have that $2N^1_j$ and $2N^2_j$ are incommensurable arithmetic manifolds since the forms f_1 and f_2 are not similar over K; see [11, §2.6]. Now set N_j to be any manifold

[1]The fundamental group of a complete finite-volume hyperbolic manifold of dimension $n \geq 4$ with (codimension-2) corners is Zariski-dense in $\mathrm{Isom}(\mathbb{H}^n)$. Zariski-density also holds for $n = 3$ in the case of compact boundary.

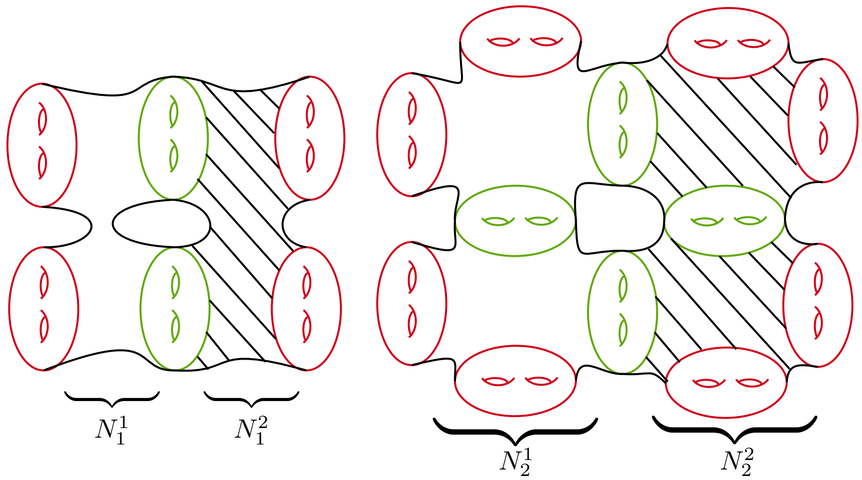

FIGURE 1. Schematic representation of N_1 and N_2, respectively.

with entirely red boundary obtained by gluing the N_j^i along the green boundary components via isometries (see Figure 1). Then $\mathrm{area}(\partial N_j) = 2^{j+1}\mathrm{area}(\Sigma_1)$ and $\mathrm{vol}(N_j) = 2^{j-1}(\mathrm{vol}(M_1) + \mathrm{vol}(M_2))$, so that the N_j all share the same perimeter-to-volume ratio $\frac{4\cdot\mathrm{area}(\Sigma_1)}{\mathrm{vol}(M_1)+\mathrm{vol}(M_2)}$. Moreover, as the perimeter-to-volume ratio of N_j^1 is $(1+2^{-j})\frac{\mathrm{area}(\Sigma_1)}{\mathrm{vol}(M_1)}$, we have that for $k > j \geq 1$ the perimeter-to-volume ratio of N_k^1 is strictly smaller than that of N_j^1, and hence N_k is not commensurable to N_j by Lemma 2.

Note that, by the theorem of Borel and Harish-Chandra [4, 18], the manifolds M_i, and hence also the manifolds N_j, obtained in the above manner are compact since the chosen forms f_i were K-anisotropic. We next explain how to adjust the above argument to obtain cusped examples in the case $n \geq 4$. In this case, we take $K = \mathbb{Q}$ (so that $\mathcal{O}_K = \mathbb{Z}$), and instead use the forms f_1 and f_2 given by

$$x_1^2 + x_2^2 + \ldots + x_n^2 - 2x_{n+1}^2,$$
$$5x_1^2 + x_2^2 + \ldots + x_n^2 - 2x_{n+1}^2,$$

respectively; Gelander and Levit [10] have again shown that f_1 and f_2 are not similar over $K = \mathbb{Q}$. We now proceed precisely as in the compact case, but the manifolds M_i one obtains in the process, and hence also the manifolds N_j, have cusps since the f_i are $\mathbb{Q}$-isotropic.

Finally, we explain how to obtain cusped examples in the case $n = 3$. We again take $K = \mathbb{Q}$, but we instead take f_1 and f_2 to be the forms

$$x_1^2 + x_2^2 + x_3^2 - 7x_4^2,$$
$$7x_1^2 + x_2^2 + x_3^2 - 7x_4^2,$$

respectively. That f_1 and f_2 are not similar over $K = \mathbb{Q}$ can be seen for instance from the fact that f_1 is $\mathbb{Q}$-anisotropic[2] while f_2 is evidently not. In particular, while the resulting manifold M_1 will be compact in this case, so that the surfaces

[2]Since -7 is a square in the 2-adics $\mathbb{Q}_2$ by Hensel's lemma, and since the stufe of $\mathbb{Q}_2$ is 4, the form f_1 is indeed $\mathbb{Q}_2$-anisotropic.

Σ_i and Σ_i' will also be compact, the manifold M_2 will be cusped. We may now proceed precisely as in the compact case, since Lemma 2 still applies to distinguish the commensurability classes of the resulting manifolds N_j. □

We conclude with some remarks. Observe that a pair of complete finite-volume manifolds with finite perimeters have the same isoperimetric ratio if and only if the ratio of their volumes agrees with the ratio of their perimeters, while commensurability moreover implies that this common ratio is rational. It is clear that the latter also holds for each family $\{N_j\}_j$ of incommensurable manifolds constructed above.

If M is a complete finite-volume hyperbolic n-manifold with nonempty totally geodesic boundary, then the ideal boundary of the universal cover $\widetilde{M}$ on the conformal sphere $\mathbb{S}^{n-1}$ at infinity, where $\widetilde{M}$ is viewed as a closed convex subset of $\mathbb{H}^n$, is a *Schottky set*, that is, the complement in $\mathbb{S}^{n-1}$ of a union of disjoint round open balls. Two such hyperbolic manifolds are then commensurable precisely when their associated Schottky sets differ by a conformal map of $\mathbb{S}^{n-1}$. It follows from a rigidity theorem of Bonk–Kleiner–Merenkov [3] that the sequences of hyperbolic manifolds sharing the same perimeter-to-volume ratio that are constructed in the proof of Theorem 1 indeed have the property that their associated Schottky sets pairwise fail to be quasisymmetric (in the compact case, this already follows from earlier work of Frigerio [9], as well as from unpublished work of Kapovich–Kleiner–Leeb–Schwartz; see the second footnote in [12]).

Besides the perimeter-to-volume ratio, another commensurability invariant of a complete finite-volume hyperbolic manifold with nonempty totally geodesic boundary is of course the Hausdorff dimension of the associated Schottky set. We however remain unaware of a single pair of incommensurable such manifolds of dimension ≥ 3, let alone an infinite family of pairwise incommensurable such manifolds, that demonstrably fail to be distinguished by the latter invariant. We are also not aware of a complete hyperbolic manifold M of dimension ≥ 3 sharing the same perimeter-to-volume ratio with an incommensurable such manifold of the same dimension but possessing no properly immersed totally geodesic hypersurface contained entirely in the interior of M. The absence of such a hypersurface within M is equivalent to the absence of nonperipheral round hyperspheres within the Schottky set associated to M, as was recently established in full generality by Lee–Oh [13, Cor. 1.2] using Ratner's measure classification theorem [22] and work of Dani–Margulis [7]; see also the previous works [15, Appendix B], [2, §10].

It follows essentially from strong approximation that, up to diminishing the ideal $\mathfrak{m}$ in the proof of Theorem 1, the common perimeter-to-volume ratio in Theorem 1 can be made arbitrarily small; see Belolipetsky–Weinberger [1, Thm. 4.1] (in particular the last page of the proof). One can then apply an inequality of Buser [5, Thm. 7.1] and identities of Elstrodt–Patterson–Sullivan [25, Thm. 2.17] and Sullivan [23, 24] to see that the Hausdorff dimensions of the associated Schottky sets can be made arbitrarily close to the dimension of the ambient sphere.

On the other hand, Miyamoto [16] showed that for $n = 3, 4$, there is a unique commensurability class of complete finite-volume hyperbolic n-manifolds with totally geodesic boundary maximizing the perimeter-to-volume ratio among all such manifolds. For $n = 3$, the maximizers are precisely those whose associated Schottky set in $\mathbb{S}^2$ is the Apollonian gasket; accordingly, we will call any such hyperbolic 3-manifold with totally geodesic boundary *Apollonian*.

J. Raimbault has informed us that it follows from his work [19] with B. Petri that one should expect the perimeter-to-volume ratio of a generic compact hyperbolic 3-manifold with nonempty totally geodesic boundary to be large, in the following sense. In [19], Petri and Raimbault introduced a model for random compact 3-manifolds with nonempty boundary; a manifold in their model is obtained by randomly gluing truncated tetrahedra along their hexagonal facets. Among other features, they show that, with probability tending to one as the number of tetrahedra tends to infinity, such a manifold is connected and admits a hyperbolic metric with totally geodesic boundary. Moreover, it follows from their hyperbolization argument that, given any $\epsilon > 0$, the perimeter-to-volume ratio of such a manifold is asymptotically almost surely within ϵ from that of an Apollonian manifold as the number of tetrahedra tends to infinity. Indeed, Petri and Raimbault show, in a sense that they make precise, that as the number of tetrahedra tends to infinity, a random 3-manifold in their model converges to the universal cover of an Apollonian manifold, that is, to the convex hull in $\mathbb{H}^3$ of the Apollonian gasket.

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References

[1] M. Belolipetsky and S. Weinberger, *Growth of k-dimensional systoles in congruence coverings*, Geom. Funct. Anal., 34 (2024), pp. 979–1005.

[2] Y. Benoist and H. Oh, *Geodesic planes in geometrically finite acylindrical 3-manifolds*, Ergodic Theory Dyn. Syst., 42 (2022), pp. 514–553.

[3] M. Bonk, B. Kleiner, and S. Merenkov, *Rigidity of Schottky sets*, Am. J. Math., 131 (2009), pp. 409–443.

[4] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2), 75 (1962), pp. 485–535.

[5] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. Éc. Norm. Supér. (4), 15 (1982), pp. 213–230.

[6] S. G. Dani, *On orbits of unipotent flows on homogeneous spaces*, Ergodic Theory Dyn. Syst., 4 (1984), pp. 25–34.

[7] S. G. Dani and G. A. Margulis, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, in I. M. Gelfand Seminar. Part 1: Papers of the Gelfand seminar in functional analysis held at Moscow University, Russia, September 1993, Providence, RI: American Mathematical Society, 1993, pp. 91–137.

[8] S. Douba and F. V. Pallete, *On reflections of congruence hyperbolic manifolds*. Preprint, arXiv:2506.23994 [math.GT] (2025), 2025.

[9] R. Frigerio, *Commensurability of hyperbolic manifolds with geodesic boundary*, Geom. Dedicata, 118 (2006), pp. 105–131.

[10] T. Gelander and A. Levit, *Counting commensurability classes of hyperbolic manifolds*, Geom. Funct. Anal., 24 (2014), pp. 1431–1447.

[11] M. Gromov and I. Piatetski-Shapiro, *Nonarithmetic groups in Lobachevsky spaces*, Inst. Hautes Études Sci. Publ. Math., (1988), pp. 93–103.

[12] M. Kapovich and B. Kleiner, *Hyperbolic groups with low-dimensional boundary*, Ann. Sci. Éc. Norm. Supér. (4), 33 (2000), pp. 647–669.

[13] M. Lee and H. Oh, *Properness and finiteness of totally geodesic submanifolds in the convex core*. Preprint, arXiv:2511.09377 [math.GT] (2025), 2025.

[14] A. Lubotzky, *Free quotients and the first Betti number of some hyperbolic manifolds*, Transform. Groups, 1 (1996), pp. 71–82.
[15] C. T. McMullen, A. Mohammadi, and H. Oh, *Geodesic planes in hyperbolic 3-manifolds*, Invent. Math., 209 (2017), pp. 425–461.
[16] Y. Miyamoto, *Volumes of hyperbolic manifolds with geodesic boundary*, Topology, 33 (1994), pp. 613–629.
[17] G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Publications Mathématiques de l'IHÉS, 34 (1968), pp. 53–104.
[18] G. D. Mostow and T. Tamagawa, *On the compactness of arithmetically defined homogeneous spaces*, Ann. of Math. (2), 76 (1962), pp. 446–463.
[19] B. Petri and J. Raimbault, *A model for random three-manifolds*, Comment. Math. Helv., 97 (2022), pp. 729–768.
[20] G. Prasad, *Strong rigidity of $\mathbb{Q}$-rank 1 lattices*, Inventiones mathematicae, 21 (1973), pp. 255–286.
[21] J. Raimbault, *A note on maximal lattice growth in* $\mathrm{SO}(1,n)$, Int. Math. Res. Not. IMRN, (2013), pp. 3722–3731.
[22] M. Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. (2), 134 (1991), pp. 545–607.
[23] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math., Inst. Hautes Étud. Sci., 50 (1979), pp. 171–202.
[24] ———, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math., 153 (1984), pp. 259–277.
[25] ———, *Related aspects of positivity in Riemannian geometry*, J. Differ. Geom., 25 (1987), pp. 327–351.

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