ANOSOV GROUPS THAT ARE INDISCRETE IN RANK ONE (ERRATUM)

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We correct here the proof of Lemma 2.1 in our note [2], which we restate here for the convenience of the reader.

Lemma 1. ([2, Lemma 2.1]) *Suppose a finite-index normal subgroup* $\Gamma_0 < \Gamma$ *embeds as a* P_1*-Anosov subgroup of* $\mathsf{SL}_d(\mathbb{R})$*. Then* Γ *embeds as a* P_1*-Anosov subgroup of* $\mathsf{SL}_r(\mathbb{R})$ *for some* $r \in \mathbb{N}$.

The proof relied on the following incorrect claim: If Γ_0 is a normal subgroup of Γ of index $m < \infty$, and $\rho : \Gamma_0 \to \mathsf{SL}_d(\mathbb{R})$ is a P_1-Anosov representation, then the induced representation $\rho^{\mathrm{ind}} : \Gamma \to \mathsf{SL}^{\pm}_{dm}(\mathbb{R})$ is P_m-Anosov. To see that this claim is incorrect, one can take Γ_0 to be a P_1-Anosov free subgroup of $\mathsf{SL}_3(\mathbb{R})$ generated by conjugates $a, b \in \mathsf{SL}_3(\mathbb{R})$ of the matrices

$$\begin{pmatrix} 3 & & \\ & 2 & \\ & & \frac{1}{6} \end{pmatrix}^q, \begin{pmatrix} 2 & & \\ & 1 & \\ & & \frac{1}{2} \end{pmatrix}^q,$$

respectively, where q is some large integer; take ρ to be the inclusion $\Gamma_0 \hookrightarrow \mathsf{SL}_3(\mathbb{R})$; and take Γ to be the abstract semidirect product $\Gamma_0 \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the generator of the $\mathbb{Z}/2\mathbb{Z}$ factor acts on Γ_0 by interchanging a and b. Then the induced representation $\rho^{\mathrm{ind}} : \Gamma \to \mathsf{SL}^{\pm}_6(\mathbb{R})$ fails to be P_2-Anosov since, for instance, the second- and third-largest eigenvalues of $\rho^{\mathrm{ind}}(a)$ coincide. We thank Balthazar Fléchelles for pointing out this flaw in our logic.

We now proceed to a corrected proof of Lemma 1 which instead relies on the following fact.

Lemma 2. *If* $\left\{\rho_i : \Gamma_0 \to \mathsf{SL}_d(\mathbb{R})\right\}_{i=1}^m$, $d \geqslant 2$, *is a finite collection of* P_1*-Anosov representations, then the tensor product* $\otimes_{i=1}^m \rho_i : \Gamma_0 \to \mathsf{SL}(\bigotimes_{i=1}^m \mathbb{R}^d) \cong \mathsf{SL}_{d^m}(\mathbb{R})$ *is* P_1*-Anosov.*

Proof. Note first that Γ_0 is a hyperbolic group by [4, 1].

For a matrix $g \in \mathsf{SL}_n(\mathbb{R})$, denote by $\lambda_1(g) \geqslant \cdots \geqslant \lambda_n(g)$ the moduli of the eigenvalues of g in nonincreasing order. Then, by the definition of the tensor product, for every $g_1, \ldots, g_m \in \mathsf{SL}_d(\mathbb{R})$,

$$\begin{aligned} \lambda_1(g_1 \otimes \cdots \otimes g_m) &= \lambda_1(g_1) \cdots \lambda_1(g_m), \\ \lambda_2(g_1 \otimes \cdots \otimes g_m) &= \max_{1 \leqslant i \leqslant m} \Big(\lambda_2(g_i) \prod_{j \neq i} \lambda_1(g_j)\Big). \end{aligned}$$

In particular, for every $\gamma \in \Gamma_0$ we have $\frac{\lambda_1(\rho_1(\gamma) \otimes \cdots \otimes \rho_m(\gamma))}{\lambda_2(\rho_1(\gamma) \otimes \cdots \otimes \rho_m(\gamma))} = \min_i \frac{\lambda_1(\rho_i(\gamma))}{\lambda_2(\rho_i(\gamma))}$. Since ρ_i is P_1-Anosov for each i, the conclusion now follows from the eigenvalue characterization of the P_1-Anosov condition [5, Prop. 1.2].

□

Proof of Lemma 1. Let $\rho : \Gamma_0 \hookrightarrow \mathsf{SL}_d(\mathbb{R})$ be the inclusion which is P_1-Anosov and let $m := [\Gamma : \Gamma_0]$. Let $H := \underbrace{\mathsf{SL}_d(\mathbb{R}) \times \cdots \times \mathsf{SL}_d(\mathbb{R})}_{m}$ and consider the semidirect product

$$G := H \rtimes \Sigma_m,$$

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where Σ_m is the symmetric group on m letters acting on H by permuting the factors. We may think of G as a subgroup of $\mathsf{SL}^{\pm}_{dm}(\mathbb{R})$ by writing elements of H (respectively, Σ_m) as block-diagonal (resp., block-permutation) matrices. We assume that the induced representation $\rho^{\mathrm{ind}} : \Gamma \to \mathsf{SL}^{\pm}_{dm}(\mathbb{R})$ of ρ (see, for instance, [3, Section 3.3]) has been conjugated so that $\rho^{\mathrm{ind}}(\Gamma) \subset G$ and $\rho^{\mathrm{ind}}(\Gamma_0) \subset H$. Note that, by the definition of ρ^{ind}, and since Γ_0 is normal in Γ, the restriction $\rho^{\mathrm{ind}}\big|_{\Gamma_0}$ is a direct sum of representations $\rho \circ \varphi_1, \dots, \rho \circ \varphi_m : \Gamma_0 \to \mathsf{SL}_d(\mathbb{R})$, where φ_i is an automorphism of Γ_0 for $i = 1, \dots, m$. Each of the factors $\rho \circ \varphi_i$ is moreover P_1-Anosov since automorphisms of Γ_0 are quasi-isometries.

Now observe that G admits a representation $\tau : G \to \mathsf{SL}^{\pm}(\bigotimes_{i=1}^m \mathbb{R}^d)$, where $\tau\big|_H$ is the usual m-fold tensor product, and Σ_m acts on $\tau(H)$ by permuting the factors. We now have that $(\tau \circ \rho^{\mathrm{ind}})\big|_{\Gamma_0}$ is the tensor product of the $\rho \circ \varphi_i$, and is thus P_1-Anosov by Lemma 2. The representation $(\tau \circ \rho^{\mathrm{ind}})\big|_{\Gamma} : \Gamma \to \mathsf{SL}^{\pm}(\bigotimes_{i=1}^m \mathbb{R}^d) \cong \mathsf{SL}^{\pm}_{d^m}(\mathbb{R})$ is then P_1-Anosov since its restriction to the finite-index subgroup Γ_0 is P_1-Anosov. By composing $\tau \circ \rho^{\mathrm{ind}}$ with the embedding $\mathsf{SL}^{\pm}_{d^m}(\mathbb{R}) \hookrightarrow \mathsf{SL}_{d^m+1}(\mathbb{R})$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix},$$

we obtain a representation of Γ into $\mathsf{SL}_{d^m+1}(\mathbb{R})$ that remains P_1-Anosov; the latter again follows easily, for instance, from the eigenvalue characterization of the P_1-Anosov condition [5, Prop. 1.2]. ☐

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