Michigan Math. J. (Advance Publication)

2-Systems of Arcs on Spheres with Prescribed Endpoints

Sami Douba

Abstract. Let S be an n-punctured sphere with $n \geq 3$. We prove that $\binom{n}{3}$ is the maximum size of a family of pairwise nonhomotopic simple arcs on S joining a fixed pair of distinct punctures of S and pairwise intersecting at most twice. On the way, we show that a square annular diagram A has a corner on each of its boundary paths if A contains at least one square and the dual curves of A are simple arcs joining the boundary paths of A and pairwise intersecting at most once.

1. Introduction

A *k-system* $\mathcal{A}$ of arcs on a punctured surface S is a collection of essential simple arcs on S such that no two arcs of $\mathcal{A}$ are homotopic or intersect more than k times. We begin with the following observation.

Remark 1.1. If $k = 0$, S is an n-punctured sphere with $n \geq 3$, and the arcs of $\mathcal{A}$ all join a fixed pair of distinct punctures p, q of S, then $|\mathcal{A}| \leq n-2$. To see this, fix a complete hyperbolic metric on S of area $2\pi(n-2)$ and realize the arcs of $\mathcal{A}$ as geodesics on S. Cutting S along $\mathcal{A}$, we obtain a collection of hyperbolic punctured strips. Since the boundary of each strip consists of two arcs of $\mathcal{A}$, and since each arc of $\mathcal{A}$ appears twice as a boundary component of some strip, we count precisely $|\mathcal{A}|$ strips. The bound on $|\mathcal{A}|$ now follows from the fact that each of these strips has area at least 2π. Moreover, this bound is tight since we can easily devise a 0-system on S whose complement consists entirely of once-punctured strips (see Figure 1). An area argument also shows that the maximum size of $\mathcal{A}$ is $2n-5$ if we assume instead that $p = q$.

Problems involving bounding the size of a k-system of arcs, of which Remark 1.1 serves as a trivial example, originated in similar problems for curves. Juvan, Malnič, and Mohar [JMM96] introduced the term "k-system" and showed that the maximum size $N(k, \Sigma)$ of a k-system of essential simple closed curves on a fixed compact surface Σ is finite. Independently, Farb and Leininger inquired about $N(k, g) := N(k, \Sigma)$ for Σ closed and oriented of genus g and $k = 1$. In response, Malestein, Rivin, and Theran [MRT14] provided an upper bound exponential in g and showed that $N(1, 2) = 12$. Also for $k = 1$, Przytycki [Prz15, Thm. 1.4] produced an upper bound on the order of g^3; since then, tighter bounds on $N(1, g)$ have been found by Aougab, Biringer, and Gaster [ABG17] and more recently by

Received September 6, 2019. Revision received March 30, 2020.
The author was supported in part by the grant #346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund.

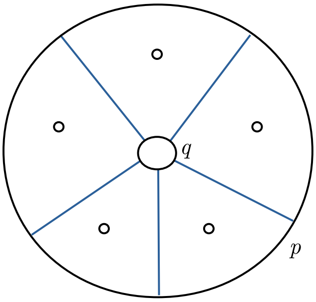

Figure 1 A maximum-size 0-system joining distinct punctures p, q of the 7-punctured sphere.

Greene [Gre18a]. Moreover, Przytycki [Prz15, Cor. 1.6] provided an upper bound on $N(k,g)$ for arbitrary k that grows like g^{k^2+k+1}. This bound was subsequently improved by Greene [Gre18b] to one that grows like $g^{k+1}\log g$.

In [Prz15], so as to prove the aforementioned results about k-systems of curves, Przytycki first proved stronger results about k-systems of arcs; he showed, for example, that the maximum size of a k-system of arcs on a punctured surface S of Euler characteristic $\chi < 0$ (where distinct arcs are not required to have the same endpoints) grows like $|\chi|^{k+1}$. In the same paper, Przytycki proved the following:

THEOREM 1.2 ([Prz15, Theorem 1.7]). *Let p, q be punctures of an n-punctured sphere S, where $n \geq 3$. The maximum size of a 1-system $\mathcal{A}$ of arcs on S joining p and q is $\binom{n-1}{2}$.*

Note that p and q are not assumed to be distinct in the statement of Theorem 1.2. More recently, Bar-Natan [Bar17] showed that for S, p, q as in Theorem 1.2, if $p = q$, then the maximum size of a 2-system of arcs on S joining p and q is $\binom{n}{3}$. The main result of this paper is that Bar-Natan's maximum holds for p, q distinct:

THEOREM 1.3. *Let p, q be distinct punctures of an n-punctured sphere S, where $n \geq 3$. The maximum size of a 2-system $\mathcal{A}$ of arcs on S joining p and q is $\binom{n}{3}$.*

It is worth noting that the natural analogue of Theorem 1.3 does not hold in positive genus. More precisely, it is not true that if S is an n-punctured surface of genus g with $n \geq 2$ and $g > 0$, then the maximum size of a 2-system of arcs joining a fixed pair of distinct punctures of S is, or is even bounded above by, $\binom{|\chi(S)|+2}{3} = \binom{2g+n}{3}$; see Figure 2 for a counterexample in the case $g = 1$ and $n = 2$, found with Przytycki.

Organization

In Section 3, we provide an example of a 2-system of size $\binom{n}{3}$ joining a fixed pair of distinct punctures of an n-punctured sphere for $n \geq 3$. The remaining sections

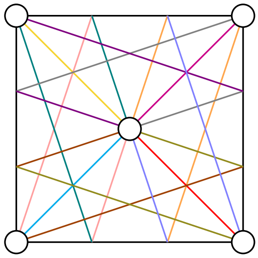

Figure 2 A 2-system of size 12 joining the punctures of a twice-punctured torus.

are concerned with proving that $\binom{n}{3}$ is an upper bound on the size of such a 2-system $\mathcal{A}$. This is proved by induction on n; we prove that the number of arcs of $\mathcal{A}$ that become homotopic after forgetting a puncture of S is not too large. This, in turn, is proved by induction via the following:

LEMMA 1.4. *Let S be an n-punctured sphere, and let p, q, r be distinct punctures of S. Let $\mathcal{P}$, $\mathcal{Q}$ be 1-systems of arcs starting at r and ending at p, q, respectively, so that no arc of $\mathcal{P}$ intersects an arc of $\mathcal{Q}$. Suppose $\mathcal{R} \subset \mathcal{P} \times \mathcal{Q}$ such that for any $(\alpha, \beta), (\alpha', \beta') \in \mathcal{R}$,*

(i) *we have $|\alpha \cap \alpha'| + |\beta \cap \beta'| \leq 1$;*
(ii) *if $|\alpha \cap \alpha'| + |\beta \cap \beta'| = 1$ with $\alpha \neq \alpha'$, $\beta \neq \beta'$, then the cyclic order around r of the r-ends of $\alpha, \alpha', \beta, \beta'$ is given by $(\alpha, \alpha', \beta, \beta')$ or $(\alpha', \alpha, \beta', \beta)$.*

Then $|\mathcal{R}| \leq \binom{n-1}{2}$.

Section 6 is devoted to the proof of Lemma 1.4. The inductive step again involves forgetting a puncture s of S, but this time, we choose s with care to control the subsequent behavior of the arcs of $\mathcal{P}$ and $\mathcal{Q}$. More precisely, we require that s be *p-isolated* (see Section 2 for definitions).

To guarantee that a puncture with this property exists, we take a detour into annular square diagrams. A *k-system annular diagram A* is an annular diagram whose dual curves constitute a k-system of arcs joining the boundary paths of A. Such a diagram arises as the dual square complex to a k-system $\mathcal{A}$ on a punctured sphere with distinct prescribed endpoints. In Section 5, we prove the following:

THEOREM 1.5. *Let A be a 1-system annular diagram. Then either A is a cycle, or A has a corner on each of its boundary paths.*

Roughly speaking, a corner corresponds to an isolated puncture. Note that Theorem 1.5 does not hold for $k = 2$; see Figure 5 (bottom left) for a counterexample suggested by Przytycki. We have also included a direct proof found by the referee of the existence of an isolated puncture (Proof 2 of Corollary 5.3); this proof does not use annular square diagrams.

2. Definitions

2.1. Arc Systems

A *puncture* is a topological end of a space S obtained from a connected oriented compact surface Σ by removing finitely many points $p_1, \ldots, p_n$ from Σ. Note that the punctures of S are in bijection with $p_1, \ldots, p_n$ and that we allow punctures on the boundary of Σ. If $p_1, \ldots, p_n$ are taken from the interior of Σ, then we call S an *n-punctured* Σ.

An *arc* on S is a proper map $\alpha : (0, 1) \to S$. A proper map induces a map between ends of topological spaces; in this sense, α “maps” each endpoint of $(0, 1)$ to a puncture p of S. We call p an *end* of α. If p, q are ends of α, then we say that α *starts* at p and *ends* at q, or that α *joins* p and q. A *segment* of α is the restriction of α to some positive-length subinterval of $(0, 1)$.

An arc α is *simple* if it is an embedding, in which case we identify α and its segments with their images in S. If J is a subinterval of $(0, 1)$ with endpoints t_1, t_2 and α is a simple arc mapping t_i to x_i for $i = 1, 2$, then we denote the segment $\alpha|_J$ by $(x_1 x_2)_\alpha$. If $R \subset S$ is a subset and p is an end of α corresponding to an endpoint $t_0 = 0, 1$ of $(0, 1)$, then we say that the *p-end of α lies in R* if $\alpha^{-1}(R)$ is a neighborhood of t_0 in $(0, 1)$.

A *homotopy* between arcs α_1 and α_2 is a proper map $(0, 1) \times [0, 1] \to S$ whose restrictions to $(0, 1) \times \{0\}$ and $(0, 1) \times \{1\}$ are α_1 and α_2, respectively. In particular, a homotopy preserves ends. If r is a puncture of S, then we say that two arcs on S are *r-homotopic* if they are homotopic on the surface $\bar{S}$ obtained from S by forgetting r. Two arcs are in *minimal position* if the number of their intersection points cannot be decreased by a homotopy. Note that if a pair of arcs have a point of intersection that is not transversal, then they are not in minimal position. An arc α is *essential* if it cannot be homotoped into a puncture in the sense that there is no proper map $(0, 1) \times [0, 1) \to S$ whose restriction to $(0, 1) \times \{0\}$ is α. Unless otherwise stated, all arcs in the paper are simple and essential, and all intersections between arcs are transversal. Note that an arc joining distinct punctures of a punctured surface is automatically essential.

Let R be a closed disc with at most two punctures on its boundary and possibly with punctures in its interior. A *region* between arcs α_1 and α_2 on S is a properly embedded $R \subset S$ such that ∂R is a union of exactly two segments σ_1 and σ_2, where σ_i is a segment of α_i for $i = 1, 2$ (see Figure 3). We say that α_1 and α_2 (or, more specifically, σ_1 and σ_2) *form* or *bound* R. If R has no punctures in its interior, then we say that R is *empty.* If R has exactly 0 (respectively, 1, 2) punctures on its boundary and $R \cap (\alpha_1 \cup \alpha_2) = \partial R$, then we call R a *bigon* (respectively, *half-bigon, strip*). We say that R is *adjacent* to a puncture p of S if p lies on the boundary of R. If p, s are distinct punctures of S and $\mathcal{A}$ is a collection of arcs on S with s contained in a half-bigon or strip H adjacent to p formed by a pair of arcs of $\mathcal{A}$ such that H is a component of $S - \bigcup \mathcal{A}$, then we say that s is *p-isolated by $\mathcal{A}$.*

We will make frequent use of the following lemma.

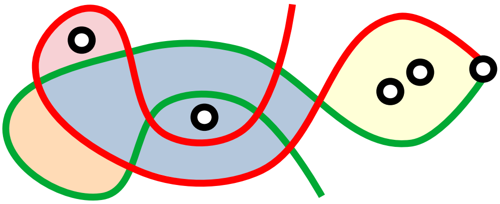

Figure 3 Regions formed by arcs. The yellow region is a half-bigon. The pink and orange regions are bigons. The orange bigon is empty.

LEMMA 2.1 (The bigon criterion, [FM12, Proposition 1.7]). *Two intersecting arcs on a punctured surface are in minimal position if and only if they form no empty regions.*

Since an empty region bounded by intersecting arcs must contain an empty bigon or half-bigon, we immediately obtain the following corollary.

COROLLARY 2.2. *Two intersecting arcs on a punctured surface are in minimal position if and only if they form no empty bigons or half-bigons.*

A *k-system* of arcs on a punctured surface S is a collection $\mathcal{A}$ of essential simple arcs on S such that no two arcs of $\mathcal{A}$ are homotopic or have more than k points of intersection. We will mainly consider the case where S is a sphere punctured at least thrice and $\mathcal{A}$ is a 2-system of arcs joining a fixed pair of distinct punctures p, q of S. Note that for any two arcs α_1, α_2 of such a collection $\mathcal{A}$, a region bounded by α_1, α_2 that contains neither p nor q must be a bigon, a half-bigon, or a strip.

If a punctured surface S has Euler characteristic $\chi < 0$, then S admits a complete hyperbolic metric of area $2\pi|\chi|$. Under such a metric, the homotopy class of any arc contains a unique geodesic representative, and any two distinct geodesic arcs are in minimal position. Thus, for the purposes of determining the size of a k-system $\mathcal{A}$ of arcs on S, we may assume that $\mathcal{A}$ consists of geodesics.

2.2. *Combinatorial Complexes*

A map $X \to Y$ between CW complexes X and Y is *combinatorial* if its restriction to each open cell of X is a homeomorphism onto an open cell of Y. A CW complex X is *combinatorial* if the attaching map of each cell in X is combinatorial for some subdivision of the sphere. A cell of dimension 0 is a *vertex,* and a cell of dimension 1 is an *edge.* The *degree* of a vertex v of X is the number of edges in X incident to v, with loops counted twice.

2.3. *Square Complexes*

An n-cube is a copy of $[-1, 1]^n$. A *square complex* X is a combinatorial complex whose cells are n-cubes with $n \leq 2$; that is, X is a combinatorial 2-complex each of whose 2-cells is attached via a combinatorial map from a 4-cycle into the 1-skeleton of X. The cells of X are called *cubes,* and its 2-cells are called *squares.*

A *midcube* is a subspace of a cube $[-1, 1]^n$ obtained by restricting one coordinate to 0. Let U be a new square complex whose cells are midcubes of X and whose attaching maps are restrictions of attaching maps in X to midcubes. A *dual curve* α of a cube c in X is a connected component of U containing a midcube of c. If c is an edge, then we say that α is *dual* to c. We call the dual curve α an *arc* if it is homeomorphic to an interval (possibly of length 0). There is a natural immersion $\alpha \to X$; if this map is an embedding, then we say that α is *simple.* In this case, we identify α with its image in X. Note that if c is a square whose dual curves are simple, then it has exactly two dual curves.

2.4. *Annular Diagrams*

An *annular diagram* A is a finite combinatorial cell decomposition of S^2 minus two disjoint open 2-cells (see Figure 4). The attaching map of each of these 2-cells is a *boundary path* of A.

We call A a *square annular diagram,* or simply a *diagram,* if it is also a square complex (see Figure 5). A *corner* on a boundary path P of a diagram A is a vertex v on P of degree 2 that is contained in some square of A.

Let c be a square of a diagram A with boundary path P, and let x be the center of c. Suppose the dual curves α, β of c are dual to consecutive edges a, b on P with shared vertex v. Let γ be the loop obtained from the subarcs of α, β joining x and P and from the half-edges of a, b containing v. If γ is homotopic in A to a constant path, then we call c a *cornsquare* with *outerpath* ab.

A *hexagon move* on a diagram A is the replacement of three squares forming a subdivided hexagon by an alternate three squares forming a subdivided hexagon (see Figure 6). A hexagon move can be visualized as a benign "sliding" operation on one of the dual curves of A, so that if A' is obtained from A by a hexagon move, then there is a natural correspondence between the dual curves of A and those of A'. Note that the number of squares of A is preserved under hexagon moves.

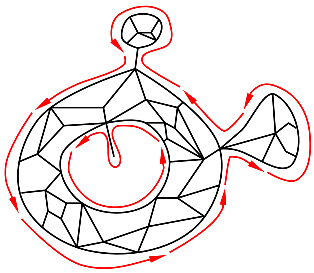

Figure 4 An annular diagram. The boundary paths are indicated in red.

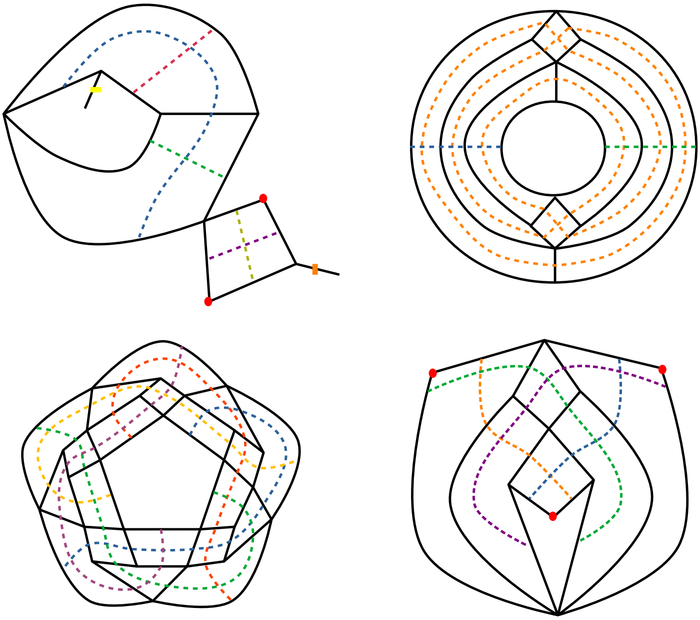

Figure 5 Square annular diagrams. Dual curves are dashed and colored. Corners are colored red. The bottom left (respectively, bottom right) diagram is a 2-system (respectively, 1-system) annular diagram. Note that the square at which the blue and orange dual curves meet in the bottom right diagram is a cornsquare with an outerpath on each boundary path, whereas the square at which the green and purple dual curves meet is not a cornsquare, even though the latter two dual curves are dual to consecutive edges on each boundary path.

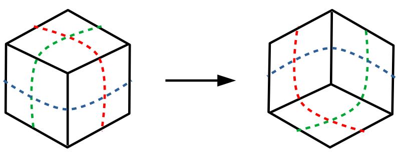

Figure 6 A hexagon move and its effect on dual curves.

A square annular diagram A is a k-*system annular diagram* if its dual curves are simple arcs joining the boundary paths of A and pairwise intersecting at most k times in A. Note that the number of intersections between any pair of dual curves of A is preserved under a hexagon move. Thus, if A' is obtained from a k-system annular diagram A by a hexagon move, then A' is also a k-system annular diagram.

3. A 2-System of Maximum Size

We provide an example of a 2-system of arcs of size $\binom{n}{3}$ joining a fixed pair of distinct punctures of an n-punctured sphere S. This collection was independently discovered by Assaf Bar-Natan.

We think of S as $\mathbb{R}^2$ punctured at $p = (-1, 0)$ and at the points $r_i = (i - \frac{1}{2}, 0)$ for $i = 1, \dots, n-2$. We construct a 2-system $\mathcal{A}$ joining p and the puncture q at infinity.

Let $\alpha_{<-1}$ be the arc given by the ray $\{(x, 0) : x < -1\}$. For $a, b, c \in \{0, 1, \dots, n-2\}$ with $a < b < c$ or $0 < a < b = c = n-2$, let α_{abc} be the graph of the polynomial function $f_{abc} : (-1, \infty) \to \mathbb{R}$ given by $x \mapsto (x+1)(x-a)(x-b)(x-c)$.

Note that for distinct triples (a, b, c) and (a', b', c'), the difference $f_{abc} - f_{a'b'c'}$ is a cubic polynomial, one of whose roots is -1. Thus, the α_{abc} pairwise intersect at most twice. Furthermore, the α_{abc} are pairwise nonhomotopic [Bar17, proof of Lemma 4.2], and $\alpha_{<-1}$ is not homotopic to any of the α_{abc} since the complement of $\alpha_{<-1} \cup \alpha_{abc}$ is a pair of punctured strips.

Now fix $M > 0$ such that $M > |f_{abc}(x)|$ for all $x \in (-1, n-2]$. For each $i, j \in \{1, \dots, n-2\}$ with $i < j$, let α_{ij} be the union of the following horizontal and vertical segments: the segment joining $(-1, 0)$ and $(-1, M)$, the segment joining $(-1, M)$ and $(i - \frac{1}{2} + \frac{1}{4}, M)$, the segment joining $(i - \frac{1}{2} + \frac{1}{4}, M)$ and $(i - \frac{1}{2} + \frac{1}{4}, -M)$, the segment joining $(i - \frac{1}{2} + \frac{1}{4}, -M)$ and $(-2, -M)$, the segment joining $(-2, -M)$ and $(-2, M+1)$, the segment joining $(-2, M+1)$ and $(j - \frac{1}{2} - \frac{1}{4}, M+1)$, and the vertical ray traveling down from $(j - \frac{1}{2} - \frac{1}{4}, M+1)$. Note that each α_{ij} intersects $\alpha_{<-1}$ exactly once, and each α_{abc} exactly twice (see Figure 7). Furthermore, each α_{ij} is in minimal position with $\alpha_{<-1}$ by Corollary 2.2; since $\alpha_{<-1}$ is disjoint from the α_{abc}, this shows that none of the α_{ij} is homotopic to any of the α_{abc}.

We claim that the α_{ij} are pairwise nonhomotopic. Indeed, for $k \in \{1, \dots, n-3\}$ let γ_k be the horizontal arc joining the punctures at $x = k - \frac{1}{2}$ and $x = k + \frac{1}{2}$, and note that α_{ij} and γ_k are in minimal position by Corollary 2.2. Since no two of the α_{ij} share the same number of intersection points with each of the γ_k, the α_{ij} must be pairwise nonhomotopic.

We claim further that the α_{ij} pairwise intersect at most twice. Indeed, if $i, j, i', j' \in \{1, \dots, n-2\}$ with $i \le i'$, then the number of intersection points between $\alpha_{ij}, \alpha_{i'j'}$ is determined by the order of j, i', j'. If $i' < j \le j'$, then α_{ij} and $\alpha_{i'j'}$ are disjoint (see Figure 8, top). If $i' < j' < j$, then α_{ij} and $\alpha_{i'j'}$ intersect once (see Figure 8, middle). Otherwise, $j \le i'$, and there are two points of intersection between α_{ij} and $\alpha_{i',j'}$ (see Figure 8, bottom). Thus the family $\mathcal{A}$ consisting of $\alpha_{<-1}$, the α_{abc}, and the α_{ij} is a 2-system of size $1 + (n-3) + \binom{n-1}{3} + \binom{n-2}{2} = \binom{n}{3}$.

4. Properties of r-Homotopic Arcs Intersecting at Most Twice

Let p, q, r be distinct punctures of a punctured sphere S, and let $\mathcal{A}$ be a 2-system of arcs on S joining p and q. Let $\bar{S}$ be the surface obtained from S by forgetting

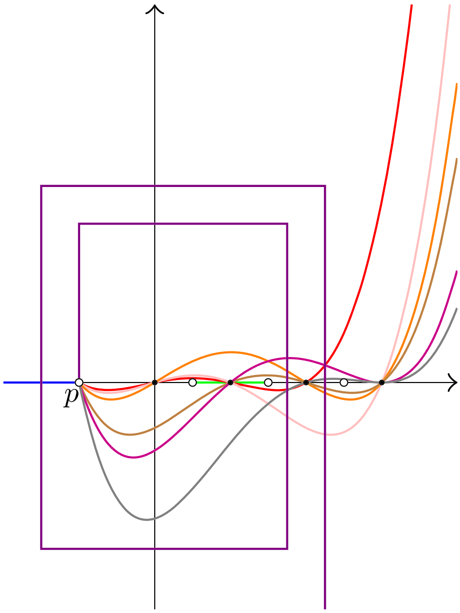

Figure 7 The arcs α_{abc} on the 5-punctured sphere, together with arc $\alpha_{<-1}$, drawn in blue, arc α_{23}, drawn in violet, and arc γ_1, drawn in green.

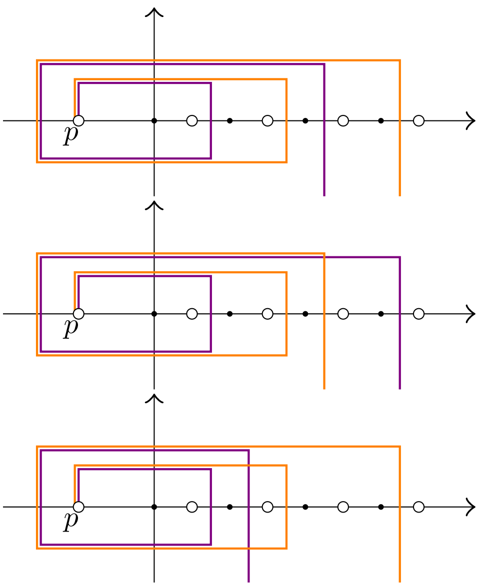

Figure 8 The α_{ij} pairwise intersect at most twice.

r, and for each arc $\alpha \in \mathcal{A}$, let $\bar{\alpha}$ be the homotopy class of the corresponding arc on $\bar{S}$. To bound the size of $\mathcal{A}$ from above, we will need to examine to what extent the map $\alpha \mapsto \bar{\alpha}$ is injective. In this section, we collect some facts about the fibers of this map. Together, the results of this section show that we can extend $\mathcal{A}$ so that the size of each fiber is 1 larger than the number of pairs of disjoint arcs in that fiber.

The main results of this section are Lemmas 4.5, 4.6, and 4.7. The proofs are rather technical and may be skipped on an initial reading.

LEMMA 4.1. *Let p, q, r be distinct punctures of a punctured sphere S, and let α_1, α_2 be a pair of r-homotopic arcs joining p and q and intersecting at most twice. If the α_i are in minimal position, then they are in one of the configurations shown in Figure 9, up to relabeling p and q.*

Proof. If α_1 and α_2 are disjoint, then they bound a strip whose only puncture is r (see Figure 9, top left). Otherwise, by Corollary 2.2, α_1 and α_2 bound a half-bigon or bigon R whose only puncture is r. If α_1 and α_2 intersect exactly once, then R is a half-bigon, and since the α_i are r-homotopic, all punctures of S distinct from p, q, r lie in the other half-bigon formed by α_1 and α_2 (see Figure 9, top right).

If the α_i intersect twice and R is a half-bigon, then the α_i do not form a bigon, since otherwise they would not be r-homotopic (see Figure 10). Thus, in this case,

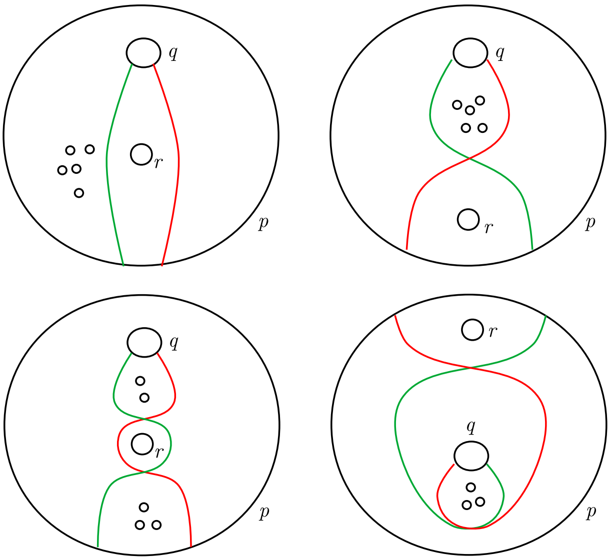

Figure 9 The possible configurations of a pair of r-homotopic arcs in minimal position and intersecting at most twice, up to relabeling p and q.

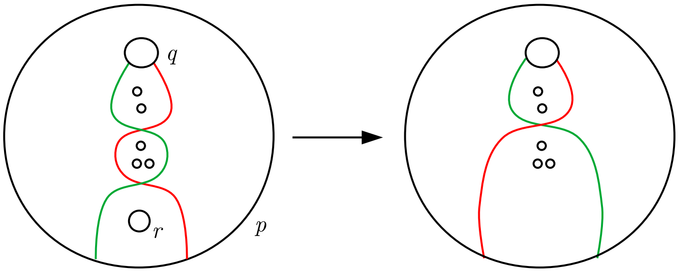

Figure 10 If α_1 and α_2 are in minimal position, intersect exactly twice, and form a bigon that does not contain r, then they cannot be r-homotopic. The fact that the bigon and half-bigons bounded by the α_i prior to forgetting r are punctured and the fact that the arcs on the right are in minimal position are consequences of Corollary 2.2.

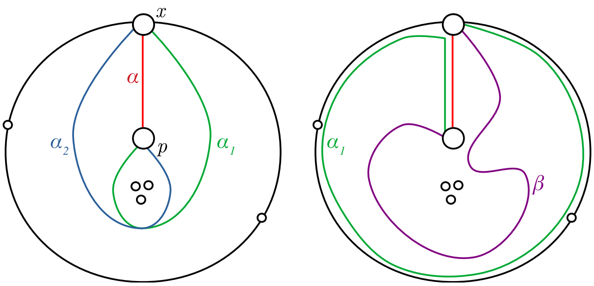

Figure 11 Reconstructing the α_i in Lemma 4.2.

the α_i must be as in the bottom right diagram of Figure 9, and since the α_i are r-homotopic, all punctures of S distinct from p, q, r must lie in the other half-bigon formed by the α_i.

Otherwise, R is a bigon, and the α_i also bound a pair of punctured half-bigons. These half-bigons must contain all the remaining punctures of S since the α_i are r-homotopic (see Figure 9, bottom left). □

The corollary of the following lemma will be useful in the proof of Lemma 4.4. The former tells us that, in a particular context, if we have a portion of an arc, then we can trace out the remainder of that arc.

Lemma 4.2. *Let D be a disc with at least two punctures in its interior and at least one puncture on its boundary, and let α be an arc joining an interior puncture p of D to a puncture x on ∂D. If β is another arc joining p and x such that α and β bound a strip containing all interior punctures of D distinct from p, then β is homotopic to exactly one of the arcs α_1 and α_2 shown in Figure 11 (left).*

Proof. Suppose that the x-end of β lies to the right of α. Then we may homotope α_1 so that it bounds an empty strip with β, as in Figure 11 (right). Thus, in this case, β is homotopic to α_1. Similarly, if the x-end of β lies to the left of α, then β is homotopic to α_2. □

COROLLARY 4.3. *Let p, q, r be distinct punctures of an n-punctured sphere S with $n \geq 4$, and let α, β be r-homotopic arcs in minimal position joining p, q and intersecting once or twice. Let $x_1, \ldots, x_m$ be the points of intersection of α, β in the order that β traverses them as β travels from p to q, and set $x_0 = p$, $x_{m+1} = q$. For $i = 0, \ldots, m$, let β_i be the segment of β joining x_i and x_{i+1}. If $m = 2$, then the homotopy types of β_0 and β_1 determine that of β. If α and β do not bound a bigon, then the homotopy type of β_0 determines that of β for $m = 1, 2$.*

Proof. For $i = 0, \ldots, m$, let α_i be the segment of α joining x_i and x_{i+1}. We puncture S at $x_1, \ldots, x_m$.

Case 1: α and β intersect exactly once. Cutting S along α_0 and β_0 yields two punctured strips. Let D be the strip containing q. Note that x_1 is now a puncture on ∂D and that α_1 and β_1 are arcs joining q and x_1 and bounding a strip containing all the interior punctures of D distinct from q. Thus by Lemma 4.2 the homotopy type of β_1 is uniquely determined, since only one of the arcs described in Lemma 4.2 produces a β that intersects α transversally at x_1 (in fact, the only other candidate homotopy class of β_1 produces a β that is homotopic to α).

Case 2: α and β form a bigon. Let D be the square containing the puncture q obtained by cutting S along α_0, β_0, α_1, β_1. Now x_2 is a puncture on ∂D, and α_2 and β_2 are arcs joining q and x_2 and bounding a strip containing all the interior punctures of D distinct from q, so we may apply Lemma 4.2 as in Case 1. Again, only one of the two homotopy classes to which β_2 must belong by Lemma 4.2 produces a β that intersects α transversally at x_2 (see Figure 12).

Case 3: α and β intersect exactly twice but do not form a bigon. Let D be the strip containing q obtained by cutting S along α_0 and β_0. Since x_1 is a puncture

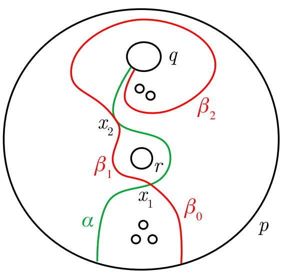

Figure 12 By Lemma 4.2, given segments β_0 and β_1 of β, there are at most two homotopy classes of arcs joining q and x_2 to which β_2 can belong. One such homotopy class produces a β that intersects α nontransversally at x_2, as shown above.

on ∂D, and α_1 and β_1 are arcs joining x_1 and x_2 and bounding a strip containing all the interior punctures of D distinct from x_2, the homotopy type of β_1 is uniquely determined by Lemma 4.2 as in the previous cases. Now let D' be the strip containing q obtained by cutting D along α_1 and β_1. Since x_2 is a puncture on $\partial D'$, and α_2 and β_2 are arcs joining q and x_2 and bounding a strip containing all interior punctures of D' distinct from q, the homotopy type of β_2 is uniquely determined by Lemma 4.2. □

LEMMA 4.4. *Let p, q, r be distinct punctures of an n-punctured sphere S with $n \geq 4$. Let $\mathcal{A}_r$ be a maximal 2-system of r-homotopic arcs on S that join p and q and are pairwise in minimal position. Then, up to homotopy and relabeling p and q, $\mathcal{A}_r$ is as in Figures 13 or 14, depending on whether or not there is a pair of arcs in $\mathcal{A}_r$ that form a bigon.*

We divide the proof of Lemma 4.4 into the following two lemmas. Note that if $\mathcal{A}_r$ is as in Lemma 4.4, then $\mathcal{A}_r$ necessarily contains a pair of intersecting arcs.

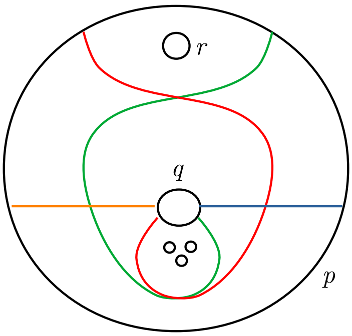

Figure 13 The configuration of $\mathcal{A}_r$ in Lemma 4.4 if no pair of arcs in $\mathcal{A}_r$ form a bigon.

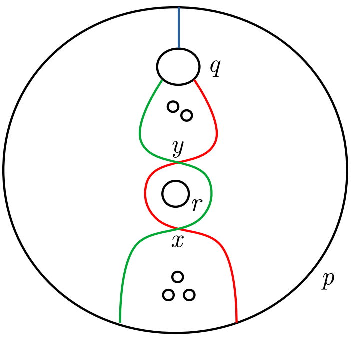

Figure 14 The configuration of $\mathcal{A}_r$ if a pair of arcs in $\mathcal{A}_r$ form a bigon.

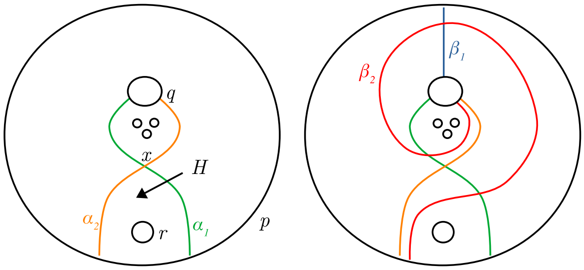

Figure 15 Case 1 of Lemma 4.5.

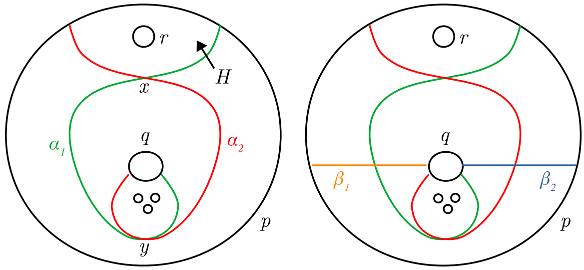

Figure 16 Case 2 of Lemma 4.5.

LEMMA 4.5. *Let $\mathcal{A}_r$ be as in Lemma 4.4, and suppose further that $\mathcal{A}_r$ contains intersecting arcs $\alpha_1, \alpha_2 \in \mathcal{A}_r$ that do not form a bigon. Then $\mathcal{A}_r$ is as in Figure 13 up to homotopy and relabeling p and q.*

Proof. Let H be the half-bigon formed by the α_i containing r, and assume that H is adjacent to p (see Figures 15, 16, left). Let $\beta \in \mathcal{A}_r$.

Case 1: The α_i intersect exactly once. Let x be their unique point of intersection. If β is disjoint from the α_i, then β is homotopic to arc β_1 in Figure 15 (right). Now suppose β is not disjoint from α_i, and let z be the first point of intersection of β and the α_i as β travels from p to q. Suppose that z lies on α_1. If the p-end of β lies in H, then β forms a half-bigon with α_1 whose only puncture is r, since otherwise α_1 and β would form an empty half-bigon, contradicting our assumption that α_1 and β are in minimal position (Corollary 2.2). Thus, by Lemma 4.1 and Corollary 4.3, β is either homotopic to α_2 or to arc β_2 in Figure 15 (right). If the p-end of β lies outside H, then z cannot lie on the segment $(px)_{\alpha_1}$, since otherwise β and α_1 would form an empty half-bigon. Thus z lies on $(xq)_{\alpha_1}$, and so β again forms a half-bigon with α_1 whose only puncture is r. Thus, by Corollary 4.3, β is again homotopic to one of α_2 or β_2. Note that, by the above, $\mathcal{A}_r$ cannot contain an additional arc β' intersecting α_2 first as β' travels from p to q. This is because the reflection of β_2 across the vertical diameter in Figure 15 (right) intersects β_2 thrice.

Case 2: The α_i intersect exactly twice. Let x, y be the points of intersection of the α_i in the order that α_1 traverses them as it travels from p to q. In this case, β intersects at least one of the α_i since p, q are in distinct components of the complement of $\alpha_1 \cup \alpha_2$. Let z be the first point of intersection of β and the α_i as β travels from p to q. Suppose that z lies on α_1. If the p-end of β lies in H, then, as in Case 1, β forms a half-bigon with α_1 whose only puncture is r. Thus, by Corollary 4.3, β is either homotopic to α_2 or to arc β_1 in Figure 16 (right). If the p-end of β lies outside H, then, as in Case 1, z cannot lie on the segment $(px)_{\alpha_1}$. Thus z lies on $(xy)_{\alpha_1}$. But then β again forms a half-bigon with α_1 whose only puncture is r, and so β is either homotopic to α_2 or to β_1 as before. Similarly, if β intersects α_2 first as it travels from p to q, then β is either homotopic to α_1 or to arc β_2 in Figure 16 (right). □

Lemma 4.6. *Let $\mathcal{A}_r$ be as in Lemma 4.4, and suppose further that $\mathcal{A}_r$ contains arcs $\alpha_1, \alpha_2 \in \mathcal{A}_r$ that form a bigon. Then $\mathcal{A}_r$ is as in Figure 14 up to homotopy and relabeling p and q.*

Proof. Let H be the half-bigon adjacent to p formed by the α_i. Let x, y be the points of intersection of the α_i in the order that α_1 traverses them as it travels from p to q.

Let $\beta \in \mathcal{A}_r$. If β is disjoint from the α_i, then β is homotopic to the blue arc in Figure 14. Now suppose β is not disjoint from the α_i, and let $z_1, z_2, \ldots$ be the points of intersection of β and the α_i in the order that β traverses them as it travels from p to q. We assume that z_1 lies on α_1. Note that, by Lemma 4.5, β forms a bigon (containing only the puncture r) with each of the α_i that it intersects.

Case 1: z_1 lies on the segment $(yq)_{\alpha_1}$. In this case, α_1 and β form a half-bigon whose only puncture is r. As remarked above, this is impossible.

Case 2: z_1 lies on the segment $(xy)_{\alpha_1}$. In this case, z_2 does not lie on $(xy)_{\alpha_2}$. Otherwise, since α_2 and β are in minimal position, they would form a half-bigon whose only puncture is r (as in Case 1 of Lemma 4.5), but this is impossible. Thus z_2 lies on $(xy)_{\alpha_1}$, and so β is homotopic to α_2 by Corollary 4.3.

Case 3: z_1 lies on the segment $(px)_{\alpha_1}$. In this case, since α_1, β are in minimal position, β forms a half-bigon H' with α_1 adjacent to p and containing at least one of the punctures of H.
Observe that z_2 cannot lie on the segment $(yq)_{\alpha_2}$, since otherwise α_2 and β would form a half-bigon containing r. We also have that z_2 cannot lie on $(xy)_{\alpha_1}$, since otherwise α_1 and β would not form a bigon. Furthermore, if z_2 lies on $(px)_{\alpha_2}$, then so must z_3, since otherwise α_1 and β would not form a bigon. But if z_2 and z_3 both lie on $(px)_{\alpha_2}$, then α_2 and β form a bigon that does not contain r, which is impossible.

Now, if H' contains all the punctures of H, then z_2 cannot lie on $(xy)_{\alpha_2}$ since α_2, β are in minimal position, and if z_2 lies on $(yq)_{\alpha_1}$, then β is homotopic to α_2 by Corollary 4.3. Thus we may assume that H' contains some but not all of the punctures of H.

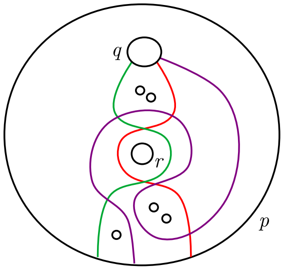

Figure 17 The arc β in the proof of Lemma 4.6 is ultimately forced to intersect one of the α_i thrice.

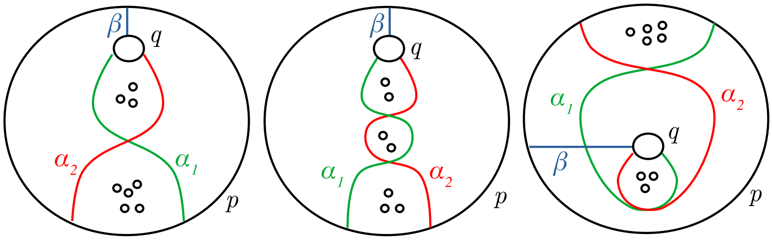

Figure 18 The arc β can be added to a 2-system of arcs joining p and q containing the α_i.

Under this assumption, z_2 cannot lie on $(yq)_{\alpha_1}$, since otherwise β would be homotopic to the purple arc in Figure 17 (left) by Corollary 4.3, and so β would intersect α_2 thrice. For the same reason, z_3 cannot lie on $(xy)_{\alpha_1}$ if z_2 lies on $(xy)_{\alpha_2}$. The only case left to consider is that z_2 and z_3 both lie on $(xy)_{\alpha_2}$. But then β is homotopic to the orange arc in Figure 17 (right) by Corollary 4.3, and so β intersects α_1 thrice. □

LEMMA 4.7. *Let p, q be distinct punctures of an n-punctured sphere S with $n \geq 4$. Let α_1, α_2, and β be arcs on S joining p and q in one of the configurations shown in Figure 18. Then an arc γ joining p and q that is in minimal position with β and intersects β at least thrice must intersect α_1 or α_2 at least thrice.*

Proof. Set $x_0 = p$, and let x_1, x_2, x_3 be the first three points of intersection of β and γ in the order that γ traverses them as γ travels from p to q. For $i = 0, 1, 2$, let $\beta_i = (x_i x_{i+1})_\beta$ and $\gamma_i = (x_i x_{i+1})_\gamma$, and let R_i be the region not containing p bounded by β_i and γ_i. If γ_0 does not intersect the α_i, then R_0 contains no punctures, and so β and γ are not in minimal position by Lemma 2.1, contradicting our assumption. Thus γ_0 has at least one point of intersection with the α_i. Similarly, R_1 and R_2 must each contain at least one puncture of S, and so each of γ_1 and γ_2 has at least two points of intersection with the α_i. Thus, γ has at least three points of intersection with α_1 or α_2. □

5. 1-System Annular Diagrams

In this section, we prove Theorem 1.5, which will be useful in the inductive step of the proof of Lemma 1.4.

LEMMA 5.1. *Let S be a twice-punctured sphere, and let p and q be its punctures. Let $\mathcal{A}$ be a finite collection of simple arcs joining p and q. If there is a pair of intersecting arcs of $\mathcal{A}$, then there is a pair of intersecting arcs of $\mathcal{A}$ forming a region R adjacent to p such that no other arc of $\mathcal{A}$ has its p-end in R.*

Proof. Pick $\alpha, \beta \in \mathcal{A}$ such that α and β intersect. Since neither α nor β has both of its ends at p, there is a unique region R adjacent to p formed by α and β. If R is as in the statement of the lemma, then we are done. Otherwise, there is an arc $\beta' \in \mathcal{A}$ whose p-end lies in R. Since the q-end of β' is outside R, the arc β' must intersect one of α or β, say α. We now repeat the above steps with arcs α and β'. Since there are finitely many arcs in $\mathcal{A}$, this process must terminate. $\square$

The following corollary follows immediately.

COROLLARY 5.2. *Let A be a square annular diagram whose dual curves are simple arcs joining its two boundary paths, and let P be a boundary path of A. If A has at least one square, then A has a cornsquare with outerpath on P.*

We now proceed to the proof of Theorem 1.5.

Proof of Theorem 1.5. We proceed by induction on the number of squares of A. If A has no squares, then A is a cycle. Now suppose A has at least one square, that there is a boundary path P of A without a corner, and that the theorem holds for any annular square complex with fewer squares than A. Since A contains a square, A contains a cornsquare with outerpath on P by Corollary 5.2. Since A is a 1-system annular diagram, we may thus produce a corner on P via a series of hexagon moves [Wis12, Figure 3.17]. Note that a single hexagon move cannot produce two corners on P; otherwise, there would be a dual curve beginning and terminating at P (see Figure 19).

We perform hexagon moves until the first corner v on P is produced. Note that each neighbor of v has degree at least 4. Indeed, since P had no corners, we had to have performed at least one hexagon move to obtain v, but a neighbor of v of degree 3 would correspond to a corner prior to performing that move, contradicting our assumption that v is the first corner produced on P (see Figure 20). Thus

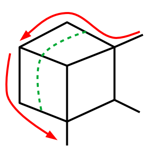

Figure 19 If a single hexagon move produces two corners on P, then there is a dual curve beginning and ending at P.

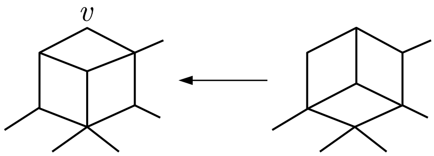

Figure 20 If a corner v produced by a hexagon move has a neighbor of degree 3, then that neighbor had to have been a corner prior to performing that move.

by deleting v as well as the two edges and the square incident to v we obtain a 1-system annular diagram with one fewer square than A and without any corners on one of its boundary paths, contradicting the induction hypothesis. □

COROLLARY 5.3. *Let p and q be punctures of an n-punctured sphere S, and let $\mathcal{A}$ be a 1-system of arcs joining p and q such that $|\mathcal{A}| \geq 2$ and the arcs of $\mathcal{A}$ are pairwise in minimal position. There is a puncture s of S distinct from p and q that is p-isolated by $\mathcal{A}$. If $\mathcal{A}$ contains a pair of intersecting arcs, then s can be chosen so that the component H of $S - \bigcup \mathcal{A}$ containing s is a half-bigon.*

Proof 1. The dual square complex A to $\mathcal{A}$ is an annular square complex as in Theorem 1.5. Let P be the boundary path corresponding to p. Note that A has at least two vertices since $|\mathcal{A}| \geq 2$ and that A has at least one square if and only if $\mathcal{A}$ contains at least one pair of intersecting arcs. If A is a cycle, then we may take H to be the strip corresponding to any vertex of A. Otherwise, A has a corner v on P, and we may take H to be the half-bigon corresponding to v. In either case, H is punctured since the arcs of $\mathcal{A}$ are pairwise nonhomotopic and in minimal position. □

What follows is a direct proof of Corollary 5.3 found by the referee that does not employ square complexes.

Proof 2. Note that since $|\mathcal{A}| \geq 2$, we must have $n \geq 4$. If $\mathcal{A}$ is a 0-system, then the components of $S - \bigcup \mathcal{A}$ are punctured strips, and we may take s to be any puncture of S distinct from p and q. Otherwise, by Lemma 5.1 there is a pair of intersecting arcs $\alpha, \beta \in \mathcal{A}$ forming a region R adjacent to p such that no other arc of $\mathcal{A}$ has its p-end in R. Since α and β are in minimal position and intersect exactly once, R is a punctured half-bigon. Let $\alpha' \in \mathcal{A}$ (resp., $\beta' \in \mathcal{A}$) be the arc of $\mathcal{A}$ whose intersection with α (resp., β) is closest to p along α (resp., along β). If $\alpha' = \beta$ (resp., $\beta' = \alpha$), then we may take s to be any puncture of R. Otherwise, since α' (resp., β') does not have its p-end in R, it must intersect $(px)_\beta$ (resp., $(px)_\alpha$), where x is the unique intersection point of α and β. The p-end of α' (resp., β') lies in one of the components of $\alpha' - \alpha$ (resp., $\beta' - \beta$). Observe that since $\mathcal{A}$ is a 1-system, either α' has its p-end in the component of $\alpha' - \alpha$ disjoint from R, or β' has its p-end in the component of $\beta' - \beta$ disjoint from R. Without loss of generality, assume that the former is true, and set $\alpha_1 = \alpha'$.

Now α_1 and α form a punctured half-bigon H_1 adjacent to p, and no other arcs of $\mathcal{A}$ may intersect $(px_1)_\alpha$, where x_1 is the unique intersection point of α and α_1. In particular, because $\mathcal{A}$ is a 1-system, any arc of $\mathcal{A}$ that enters H_1 (necessarily through $(px_1)_{\alpha_1}$) must have its p-end in H_1. If no arc of $\mathcal{A}$ enters H_1, then we may take s to be any puncture of H_1. Otherwise, let α_2 be the arc whose intersection with α_1 is closest to p along α_1. Now α_1 and α_2 form a half-bigon H_2, and we choose the arc α_3 whose intersection with α_2 is closest to p along α_2, and so on. Since $\mathcal{A}$ is necessarily finite, this process must terminate. $\square$

6. Proof of Lemma 1.4

In this section, we prove Lemma 1.4, which essentially constitutes the inductive step in the proof of Theorem 1.3. We will need the following:

LEMMA 6.1 ([Erd46]). *A set of pairwise intersecting straight line segments between ℓ points on a circle in $\mathbb{R}^2$ has size at most ℓ.*

Proof of Lemma 1.4. We fix a complete hyperbolic metric on S of area $2\pi(n-2)$. We may assume that $\mathcal{P}$ and $\mathcal{Q}$ are nonempty and that the arcs of $\mathcal{P} \cup \mathcal{Q}$ are pairwise in minimal position. We divide the proof into steps.

Step 0: The arcs of $\mathcal{P}$ (and hence the arcs of $\mathcal{Q}$) are consecutive at r. Indeed, suppose $\alpha, \alpha' \in \mathcal{P}$ are distinct, and suppose there is an arc $\beta \in \mathcal{Q}$ whose r-end lies in the strip or half-bigon H bounded by α and α' and adjacent to r. Since β does not intersect α or α', the puncture q must lie in H. Since no arc of $\mathcal{Q}$ intersects α or α', it follows that the r-end of every arc of $\mathcal{Q}$ must also lie in H.

We fix an orientation on S. This induces a cyclic order C of the arcs of $\mathcal{P} \cup \mathcal{Q}$ around r. By Step 0 this order in turn induces a linear order $<$ on $\mathcal{P}$, where the minimum and maximum arcs of $\mathcal{P}$ are those with a successor or predecessor in $\mathcal{Q}$ under C.

We proceed by induction. If $n = 3$, then, up to homotopy, there is a unique arc joining r to each of p, q, and the statement of the lemma holds. Now let $n \geq 4$, and assume that the lemma holds if S has fewer punctures. If $\mathcal{P}$ consists of a single arc, then the lemma is trivially satisfied since $|\mathcal{Q}| \leq \binom{n-1}{2}$ by Theorem 1.2. Thus we may assume that $|\mathcal{P}| \geq 2$.

Step 1: There is a puncture s of S distinct from p, q, r that is p-isolated by $\mathcal{P}$. Indeed, if the arcs of $\mathcal{P}$ are pairwise disjoint, then since $|\mathcal{P}| \geq 2$, we have that $S - \mathcal{P}$ consists of at least two punctured strips adjacent to p and r, and so we may take s to be a puncture of any such strip that does not contain q (see Figure 21, left). Otherwise, by Corollary 5.3 there is a puncture s distinct from p, r that is p-isolated by $\mathcal{P}$ such that the component of $S - \bigcup \mathcal{P}$ containing s is a half-bigon (see Figure 21, right). In this case, s is necessarily distinct from q since we are assuming $\mathcal{Q}$ to be nonempty, and so there is at least one arc disjoint from the arcs of $\mathcal{P}$ joining q and r.

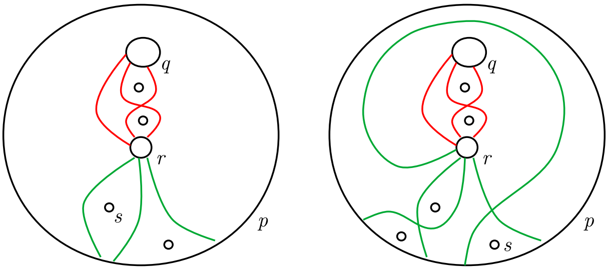

Figure 21 An illustration of Step 1.

Let $\bar{S}$ be the surface obtained from S by forgetting the puncture s (endowed with a complete finite-area hyperbolic metric), and for each arc $\alpha \in \mathcal{P} \cup \mathcal{Q}$, let $\bar{\alpha}$ be the corresponding arc on $\bar{S}$. Let $\bar{\mathcal{P}}$, $\bar{\mathcal{Q}}$ be the collection of all $\bar{\alpha}$ for $\alpha \in \mathcal{P}, \mathcal{Q}$, respectively. We tighten the arcs of $\bar{\mathcal{P}} \cup \bar{\mathcal{Q}}$ to geodesics, thereby identifying arcs that correspond to s-homotopic arcs on S.

The orientation on S induces an orientation on $\bar{S}$. As above, this gives us a linear order $\prec$ on $\bar{\mathcal{P}}$.

Step 2: Two distinct arcs in $\mathcal{Q}$ cannot be s-homotopic. Otherwise, they would form a strip adjacent to q, r or a half-bigon adjacent to one of q, r whose only puncture is s, which cannot happen since the component of $S - \bigcup(\mathcal{P} \cup \mathcal{Q})$ containing s is adjacent to p. Thus we may identify $\bar{\mathcal{Q}}$ with $\mathcal{Q}$.

Step 3: Arcs in $\mathcal{P}$ that are s-homotopic must be consecutive at r. Indeed, suppose that $\alpha, \alpha' \in \mathcal{P}$ are s-homotopic. If α, α' bound a strip whose only puncture is s, then the r-end of any arc $\beta \in \mathcal{P}$ distinct from α, α' cannot lie inside this strip, since otherwise β and one of α, α' would necessarily form a half-bigon adjacent to r whose only puncture is s, contradicting the fact that s is p-isolated.
Otherwise, α and α' form a half-bigon adjacent to p whose only puncture is s, and a half-bigon adjacent to r containing all punctures of S except p, r, s (see Figure 22). Thus, any arc α'' in $\mathcal{P}$ distinct from α and α' with r-end outside the latter half-bigon must be disjoint from α and α', in which case α, α'', and α' are in fact s-homotopic and consecutive at r.

In the case that $\mathcal{P}$ contains a (necessarily unique) pair of intersecting s-homotopic arcs α, α', we extend $\mathcal{P}$ and $\mathcal{R}$ as follows. As discussed in Step 3, there is a unique arc α'' up to homotopy joining p and r and disjoint from α and α'. The arc α'' is s-homotopic to α and α' and lies between α and α' in the linear order on $\mathcal{P}$. If $\mathcal{P}$ contains an arc α''' homotopic to α'', then we rename the former arc α''. Otherwise, we add α'' to $\mathcal{P}$. At this stage, if there is an arc $\beta \in \mathcal{Q}$ such that $(\alpha, \beta), (\alpha', \beta) \in \mathcal{R}$ but $(\alpha'', \beta) \notin \mathcal{R}$, then we add the pair (α'', β) to $\mathcal{R}$. Since α'' is disjoint from all arcs in $\mathcal{P} \cup \mathcal{Q}$, we have not violated any of the conditions of the lemma.

Note that if $\mathcal{P}$ does not contain a pair of intersecting s-homotopic arcs, then the fibers of the map $\mathcal{P} \to \bar{\mathcal{P}}$ have size at most 2.

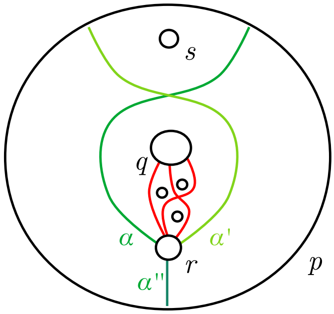

Figure 22 The only possible configuration of two intersecting s-homotopic arcs $\alpha, \alpha' \in \mathcal{P}$ and the unique arc α'' up to homotopy joining p and r and disjoint from α and α'.

Step 4: For any $\alpha, \alpha' \in \mathcal{P}$, if $\alpha < \alpha'$ then $\bar{\alpha} \preceq \bar{\alpha'}$. Indeed, if $\bar{\alpha} \succ \bar{\alpha'}$, then α and α' form a half-bigon adjacent to r whose only puncture is s. This cannot happen since s is p-isolated.

Let $\bar{\mathcal{R}}$ be the image of $\mathcal{R}$ under the map $\mathcal{P} \times \mathcal{Q} \to \bar{\mathcal{P}} \times \mathcal{Q}$, $(\alpha, \beta) \mapsto (\bar{\alpha}, \beta)$. It is clear that $\bar{\mathcal{R}}$ satisfies condition (i), and it follows from Step 4 that $\bar{\mathcal{R}}$ satisfies (ii) as well, so that $|\bar{\mathcal{R}}| \le \binom{n-2}{2}$. It remains to show that $|\mathcal{R}| - |\bar{\mathcal{R}}| \le n-2$.

Let $\mathcal{I}$ be the subset of $\bar{\mathcal{R}}$ consisting of elements with more than one preimage under the map $\mathcal{R} \to \bar{\mathcal{R}}$.

Step 5: If $(\bar{\alpha}, \beta), (\bar{\alpha'}, \beta') \in \mathcal{I}$ with $\beta \neq \beta'$, then β and β' are disjoint. Indeed, let $(\alpha_1, \beta), (\alpha_2, \beta)$ be preimages of $(\bar{\alpha}, \beta)$, and let $(\alpha'_1, \beta'), (\alpha'_2, \beta')$ be preimages of $(\bar{\alpha'}, \beta')$ with $\alpha_1 \neq \alpha_2$ and $\alpha'_1 \neq \alpha'_2$. Suppose that β and β' intersect. Note that we cannot have $\{\alpha_1, \alpha_2\} = \{\alpha'_1, \alpha'_2\}$, since otherwise either (α_1, β), (α_2, β') or (α_1, β'), (α_2, β) would be two intersecting pairs of arcs in $\mathcal{R}$ whose cyclic order around r is not alternating, contradicting assumption (ii).
Now suppose $\{\alpha_1, \alpha_2\} \cap \{\alpha'_1, \alpha'_2\} \neq \emptyset$. Then there are three distinct arcs among α_1, α_2, α'_1, α'_2 that are s-homotopic. Assume without loss of generality that these arcs are α_1, α_2, α'_1. Observe that α'_1 cannot intersect α_i for $i = 1, 2$, since otherwise (α_i, β) and (α'_1, β') would intersect more than once, contradicting assumption (i). Thus α_1 intersects α_2, but then α'_1 lies between α_1 and α_2 at r, and so there is $i \in \{1, 2\}$ such that (α_i, β) and (α'_1, β') contradict assumption (ii).

We conclude that α_1, α_2, α'_1, α'_2 are distinct; in particular, since at most three distinct arcs in $\mathcal{P}$ can be s-homotopic, we must have $\bar{\alpha} \neq \bar{\alpha'}$. Thus by Step 3 the order of $\alpha_1, \alpha_2, \alpha'_1, \alpha'_2$ at r is neither alternating nor nested. By the latter we mean that α_1 and α_2 do not both lie between α'_1 and α'_2 in the linear order on $\mathcal{P}$, and vice versa. Since each of the pairs of arcs α_1, α_2 and α'_1, α'_2 bound a strip or half-bigon containing s, we must have that α_i and α'_j intersect for some $i, j \in \{1, 2\}$, but then (α_i, β) and (α'_j, β') intersect more than once, contradicting assumption (i).

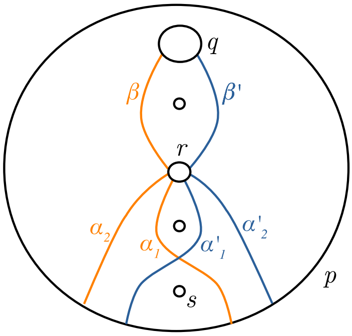

Figure 23 An illustration of Step 6. Here the pairs (α_1, β) and (α'_1, β') contradict assumption (ii).

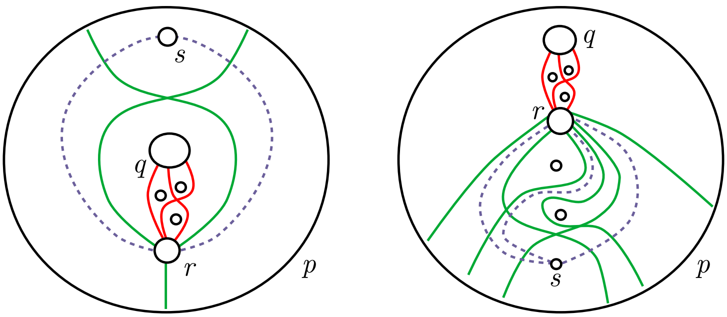

Figure 24 The collection $\mathcal{H}_s$ (the arcs of $\mathcal{H}_s$ are dashed). The case where $\mathcal{P}$ contains an intersecting pair of s-homotopic arcs is depicted on the left, whereas a generic case is depicted on the right. Note that in either case, $|\mathcal{H}_s| = |\mathcal{P}| - |\bar{\mathcal{P}}|$.

Step 6: If $(\bar{\alpha}, \beta), (\bar{\alpha'}, \beta') \in \mathcal{I}$ with $\bar{\alpha} \neq \bar{\alpha'}$, $\beta \neq \beta'$, then the cyclic order of $(\bar{\alpha}, \beta)$ and $(\bar{\alpha'}, \beta')$ at r is alternating. Indeed, suppose otherwise, and let $(\alpha_1, \beta), (\alpha_2, \beta)$ be preimages of $(\bar{\alpha}, \beta)$, and let (α'_1, β') and (α'_2, β') be preimages of $(\bar{\alpha'}, \beta')$ with $\alpha_1 \neq \alpha_2$, $\alpha'_1 \neq \alpha'_2$. Then by Step 3 the order of α_1, α_2, α'_1, α'_2 at r is neither alternating nor nested. Thus, as in Step 5, α_i and α'_j must intersect for some $i, j \in \{1, 2\}$, but then (α_i, β) and (α'_j, β') are two intersecting pairs of arcs in $\mathcal{R}$ whose cyclic order around r is not alternating, contradicting assumption (ii) (see Figure 23).

Let $\mathcal{H}_q$ be the image of $\mathcal{I}$ under the projection map $\bar{\mathcal{P}} \times \mathcal{Q} \to \mathcal{Q}$. Let $\mathcal{H}_s$ be the collection of all geodesic arcs a joining r and s such that a is contained in a strip bounded by a pair of distinct, disjoint s-homotopic arcs in $\mathcal{P}$ (see Figure 24).

Let $\mathcal{H} = \mathcal{H}_s \cup \mathcal{H}_q$, and let $\mathcal{I}' \subset \mathcal{H}_s \times \mathcal{H}_q$ be the set of all $(a, \beta) \in \mathcal{H}_s \times \mathcal{H}_q$ such that $(\alpha, \beta) \in \mathcal{I}$ for an arc α bounding a strip corresponding to a. We extended $\mathcal{R}$ (immediately after Step 3) so that the map $\mathcal{R} \to \bar{\mathcal{R}}$ is injective outside a set of cardinality $|\mathcal{I}'|$. Indeed, each element of $\mathcal{I}$ has either 2 or 3 preimages in $\mathcal{R}$; in the first case, we get 1 element of $\mathcal{I}'$, and in the second, we get 2 elements of $\mathcal{I}'$. Thus, to complete the proof, it suffices to show that $|\mathcal{I}'| \le n-2$.

Step 7: The complement of $\bigcup \mathcal{H}$ consists of punctured strips and a single square, possibly with no punctures. Indeed, the arcs of $\mathcal{H}_s$ are disjoint by construction [Prz15, proof of Theorem 1.7], and the arcs of $\mathcal{H}_q$ are disjoint by Step 5, so that the complements of $\bigcup \mathcal{H}_s$ and $\bigcup \mathcal{H}_q$ consist of punctured strips. By Step 0 the arcs in $\mathcal{H}_q$ (and hence the arcs in $\mathcal{H}_s$) are consecutive at r. Thus $\mathcal{H}_s$ is contained in a single strip of $S - \bigcup \mathcal{H}_q$ and vice versa. Let β, β' be the arcs in $\mathcal{H}_q$ bounding the unique strip of $S - \bigcup \mathcal{H}_q$ containing $\mathcal{H}_s$, and let γ, γ' be the arcs in $\mathcal{H}_s$ bounding the unique strip of $S - \bigcup \mathcal{H}_s$ containing $\mathcal{H}_q$ (note that we do not exclude the possibility that $\beta = \beta'$ or $\gamma = \gamma'$). Then the complement of $\bigcup \mathcal{H}$ consists of the remaining strips of $S - \bigcup \mathcal{H}_q$ and $S - \bigcup \mathcal{H}_s$ and a square bounded by β, β', γ, γ'.

Step 8: $|\mathcal{H}| \le n-1$. Indeed, $|\mathcal{H}|$ is by 2 larger than the number of strips of $S - \bigcup \mathcal{H}$, so it suffices to show that there are at most $n-3$ of these strips. This is true by Step 7 since S has area $2\pi(n-2)$, and a punctured strip and a square each have area at least 2π.

Step 9: $|\mathcal{I}'| \le n-2$. To show this, we intersect $\mathcal{H}$ with a small circle C centered at r. Each element of $\mathcal{I}'$ is determined by a pair of points of this intersection, and we connect them by a straight line segment. We also draw a line segment between the outermost points on C corresponding to elements of $\mathcal{H}_q$. By Step 6 these line segments are pairwise intersecting, so by Lemma 6.1, $|\mathcal{I}'| + 1 \le |\mathcal{H}| \le n-1$. □

REMARK 6.2. Note that we already used planarity in Step 0 of the proof of Lemma 1.4 (see Figure 25).

REMARK 6.3. Note that if $|\mathcal{H}| = n-1$ in Step 8, then $S - \bigcup \mathcal{H}$ necessarily consists of a single square without punctures, $|\mathcal{H}_q| - 1$ once-punctured strips from r to q, and $|\mathcal{H}_s| - 1$ once-punctured strips from r to s, one of which contains p (see Figure 26).

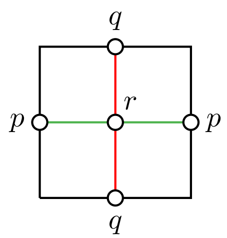

Figure 25 If we allow S to have positive genus then, under the assumptions of Lemma 1.4, the arcs of $\mathcal{P}$ need not be consecutive at r.

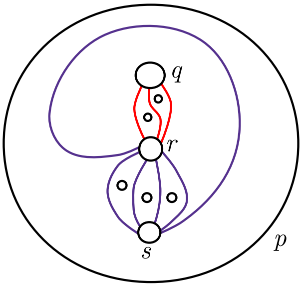

Figure 26 The 0-system $\mathcal{H}$ must have the above configuration if it is of maximum size.

7. Proof of Theorem 1.3

Proof of Theorem 1.3. We proceed by induction on n. The case $n = 3$ is trivial. Now let $n \geq 4$, and assume that the theorem holds if S has fewer punctures. Let r be a puncture of S distinct from p and q. Let $\bar{S}$ be the $(n-1)$-punctured sphere obtained from S by forgetting r. For each arc $\alpha \in \mathcal{A}$, let $\bar{\alpha}$ be the corresponding arc on $\bar{S}$, and let $\bar{\mathcal{A}} = \{\bar{\alpha} : \alpha \in \mathcal{A}\}$. We tighten the arcs of $\bar{\mathcal{A}}$ to geodesics. Note that $\bar{\mathcal{A}}$ is a 2-system on $\bar{S}$, and so $|\bar{\mathcal{A}}| \leq \binom{n-1}{3}$ by the induction hypothesis. Thus it suffices to show that $|\mathcal{A}| - |\bar{\mathcal{A}}| \leq \binom{n-1}{2}$. To that end, we examine the extent to which the map $\pi : \mathcal{A} \to \bar{\mathcal{A}}$, $\alpha \mapsto \bar{\alpha}$ fails to be injective.

By Lemmas 4.4 and 4.7 we may add arcs to $\mathcal{A}$ so that for each $\alpha \in \mathcal{A}$,

$$|\pi^{-1}(\bar{\alpha})| - 1 = |\{\{\alpha_1, \alpha_2\} \in \pi^{-1}(\bar{\alpha}) : \alpha_1, \alpha_2 \text{ distinct and disjoint}\}|.$$

Let $\mathcal{P}$ (resp., $\mathcal{Q}$) be the collection of all geodesic arcs α on S starting at r and ending at p (resp., ending at q) such that α is contained entirely in a strip bounded by a pair of distinct, disjoint r-homotopic arcs in $\mathcal{A}$. Let $\mathcal{R} \subset \mathcal{P} \times \mathcal{Q}$ be the relation consisting of all pairs (α, β) such that both α and β lie in a single such strip. We claim that $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ satisfy the conditions of Lemma 1.4, so that $|\mathcal{R}| \leq \binom{n-1}{2}$. Since $|\mathcal{R}| = |\mathcal{A}| - |\bar{\mathcal{A}}|$, this completes the proof.

We first note that for $(\alpha, \beta), (\alpha', \beta') \in \mathcal{R}$ corresponding to pairs of disjoint r-homotopic arcs $\gamma_1, \gamma_2 \in \mathcal{A}$ and $\gamma_1', \gamma_2' \in \mathcal{A}$, respectively, we have for some $i, j \in \{1, 2\}$ that r produces at least one point of intersection between γ_i and γ_j'. Each point of intersection between the arcs α, β, α', β' produces an additional point of intersection between γ_i and γ_j'. It follows that there is at most one point of intersection between any two pairs of arcs in $\mathcal{R}$.

We now show that no arc in $\mathcal{P}$ intersects an arc in $\mathcal{Q}$. Indeed, suppose we have $(\alpha, \beta), (\alpha', \beta') \in \mathcal{R}$ such that α intersects β'. By the above, α intersects β' exactly once and that is the only point of intersection between the pairs of arcs (α, β) and (α', β'). But then we can find two arcs in $\mathcal{A}$ that intersect thrice, as shown in Figure 27 (left).

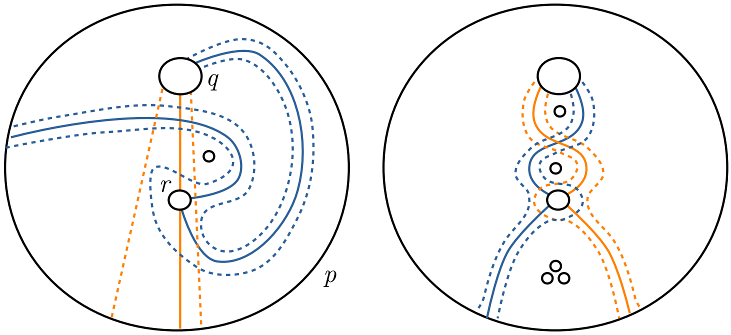

Figure 27 The systems $\mathcal{P}$, $\mathcal{Q}$ and the relation $\mathcal{R}$ satisfy the conditions of Lemma 1.4.

Finally, if there are two intersecting pairs of arcs in $\mathcal{R}$ whose cyclic order around r is not alternating, then we can also find two arcs in $\mathcal{A}$ that intersect thrice, as shown in Figure 27 (right). $\square$

ACKNOWLEDGMENTS. I thank my supervisor Piotr Przytycki for his crucial guidance and remarkable patience throughout the course of preparation of this paper. I also thank Daniel Wise for providing helpful resources. Finally, I thank the referee for penetrating comments and suggestions and in particular for the elegant direct proof of Corollary 5.3.

References

[ABG17] T. Aougab, I. Biringer, and J. Gaster, *Packing curves on surfaces with few intersections,* Int. Math. Res. Not. 2017 (2017), no. 16, 5205–5217.

[Bar17] A. Bar-Natan, *Arcs on punctured disks intersecting at most twice with endpoints on the boundary,* preprint, 2017, arXiv:1708.06402.

[Erd46] P. Erdös, *On sets of distances of n points,* Amer. Math. Monthly 53 (1946), no. 5, 248–250.

[FM12] B. Farb and D. Margalit, *A primer on mapping class groups,* Princeton University Press, Princeton, 2012.

[Gre18a] J. E. Greene, *On curves intersecting at most once,* preprint, 2018a, arXiv:1807.05658.

[Gre18b] ———, *On curves intersecting at most once, II,* preprint, 2018b, arXiv:1811.01413.

[JMM96] M. Juvan, A. Malnič, and B. Mohar, *Systems of curves on surfaces,* J. Combin. Theory Ser. B 68 (1996), no. 1, 7–22.

[MRT14] J. Malestein, I. Rivin, and L. Theran, *Topological designs,* Geom. Dedicata 168 (2014), no. 1, 221–233.

[Prz15] P. Przytycki, *Arcs intersecting at most once,* Geom. Funct. Anal. 25 (2015), no. 2, 658–670.

[Wis12] D. T. Wise, *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry,* CBMS Reg. Conf. Ser. Math., 117, American Mathematical Soc., Providence, 2012.

Department of Mathematics and
Statistics
McGill University
Montreal, QC H3A 0B9
Canada

sami.douba@mail.mcgill.ca