Talk 7: Steenrod Squares

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The standard reference is [SE]. The reference I was given is [Bre]. Other good textbook accounts include [Hau], [FF], [MT] and [Swi].

1 Motivation

Recall that singular cohomology with coefficients in some abelian group A is representable on nice spaces. Specifically, we have so-called Eilenberg-MacLane spaces $K(A,n)$ which are CW complexes whose only nontrivial homotopy group is $\pi_n(K(A,n)) \cong A$. We saw that for CW-complexes X, there is a natural isomorphism

$$[X, K(A,n)] \cong H^n(X; A), \quad [f] \mapsto f^*(\iota)$$

where $\iota \in H^n(K(A,n); A)$ is to be thought of as the "universal n-cocycle". So if we want to study cohomology, it makes sense to start by studying the cohomology of these special representing spaces. But by the Yoneda Lemma we then have a natural bijection

$$\begin{aligned} H^m(K(A,n); B) &\cong [K(A,n), K(B,m)] \\ &\cong \mathrm{Nat}([-, K(A,n)], [-, K(B,m)]) \\ &\cong \mathrm{Nat}(H^n(-; A), H^m(-; B)), \end{aligned}$$

so this leads us to studying natural transformation $H^n(-; A) \Rightarrow H^m(-; B)$.

Definition 1.1. A cohomology operation Θ of type (n, m, A, B) is a natural transformation of singular cohomology functors

$$\begin{tikzcd} \mathsf{Top}^2 \arrow[r, bend left=50, "H^n(-;A)"{name=U}] \arrow[r, bend right=50, "H^m(-;B)"'{name=D}] & \mathsf{Ab} \arrow[Rightarrow, from=U, to=D, "\Theta"'] \end{tikzcd}$$

We say Θ has degree $m - n$.

Note that since $\pi_k(K(A,n)) = 0$ for $k < n$ the Hurewicz and universal coefficient theorems tell us that also $H^k(K(A,n); B) = 0$ for $k < n$. This means that cohomology operations with negative degree are trivial.

Example 1.2. We have a SES $0 \to \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$, which for any space-pair (X, A) induces a LES in cohomology

$$\cdots \to H^n(X, A; \mathbb{Z}/4) \to H^n(X, A; \mathbb{Z}/2) \xrightarrow{\beta} H^{n+1}(X, A; \mathbb{Z}/2) \to H^{n+1}(X, A; \mathbb{Z}/4) \to \cdots$$

where β is the connecting homomorphism, often called Bockstein-Homomorphism. This gives a cohomology operation

$$\beta : H^n(X, A; \mathbb{Z}/2) \Rightarrow H^{n+1}(X, A; \mathbb{Z}/2)$$

Example 1.3. For $n \geq 0$ the cup-square in $\mathbb{Z}/2$-coefficients is a cohomology operation of degree n:

$$H^n(X, A; \mathbb{Z}/2) \Rightarrow H^{2n}(X, A; \mathbb{Z}/2), \quad x \mapsto x^2 = x \cup x.$$

2 Steenrod Squares

Steenrod squares are a particular family of Cohomology operations of singular cohomology with $\mathbb{Z}/2$ coefficients. They can be seen as refinements of the cup-squaring we saw above. Although we will not give many proofs here, we will see that Steenrod squares are interesting as they are very useful for computations, and also since they generate all the "nice" cohomology operations of singular cohomology with $\mathbb{Z}/2$-coefficients (see Theorem 2.15). There also exist a so-called Steenrod powers, an analogue of the Steenrod squares for $\mathbb{Z}/p$ coefficients (p prime), although these will not play a role here. From here on, unless stated otherwise, all (co)homology will be with $\mathbb{Z}/2$ coefficients.

2.1 Definitions

If $x \in H^*(X, A)$ we denote by $|x| \in \mathbb{N}_0$ its degree, meaning $x \in H^{|x|}(X, A)$. When clear from context we leave out the cup product symbol and just write $xy := x \cup y$.

Definition 2.1 (Steenrod Squares)**.** For $i \geq 0$, the i-th Steenrod square Sq^i denotes a collection of cohomology operations

$$\operatorname{Sq}^i_n : H^n(X, A) \Rightarrow H^{n+i}(X, A), \qquad n \geq 0,$$

often just written as

$$\operatorname{Sq}^i : H^*(X, A) \Rightarrow H^{*+i}(X, A),$$

such that the following axioms hold:

1. $\operatorname{Sq}^0 = \operatorname{id}$.
2. If $|x| = i$, then $\operatorname{Sq}^i(x) = x^2 = x \cup x$.
3. If $|x| < i$, then $\operatorname{Sq}^i(x) = 0$.
4. The Cartan formula holds:

$$\operatorname{Sq}^k(xy) = \sum_{i+j=k} \operatorname{Sq}^i(x) \operatorname{Sq}^j(y).$$

Theorem 2.2 ([SE, Section VIII.3])**.** The Steenrod squares are characterized uniquely by these axioms.

To get a feel for what these cohomology operations look like, let us try to compute them on 1-cocycles. So let (X, A) be any space-pair and $x \in H^1(X, A)$. The axioms tell us that

$$\operatorname{Sq}^0 x = x, \qquad \operatorname{Sq}^1 x = x^2 = x \cup x \qquad \operatorname{Sq}^i x = 0, \ i > 1. \tag{1}$$

Using this, we can use the Cartan formula to compute $\operatorname{Sq}^i(x^n)$ for all $i, n \geq 0$. To make things easier, we define

$$\operatorname{Sq} := \sum_{i=0}^{\infty} \operatorname{Sq}^i.$$

This is well-defined by the axioms, since for any specific cohomology class $y \in H^*(X, A)$, we have $\operatorname{Sq}^i(y) = 0$ for $|y| < i$. Hence Sq is pointwise finite and can be viewed as a natural morphism $\operatorname{Sq} : H^*(X, A) \to H^*(X, A)$. In fact, the Cartan-formula is now equivalent to Sq being a morphism of *rings*: For $y, z \in H^*(X, A)$, we have

$$\operatorname{Sq}(yz) = \sum_{i=0}^{\infty} \operatorname{Sq}^i(yz) = \sum_{i=0}^{\infty} \sum_{j+k=i} \operatorname{Sq}^j(y) \operatorname{Sq}^k(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \operatorname{Sq}^j(y) \operatorname{Sq}^k(z) = \operatorname{Sq}(y) \operatorname{Sq}(z),$$

where in the third step, we can simply reindex the sum because all occuring sums are actually finite. Now our deduced formulas (1) yield

$$\mathrm{Sq}(x^n) = \mathrm{Sq}(x)^n = (x + x^2)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n+i}.$$

By gradedness, we obtain the following proposition.

Proposition 2.3. For any 1-cocycle $x \in H^1(X, A)$, we have

$$\mathrm{Sq}^i(x^n) = \binom{n}{i} x^{n+i} = \begin{cases} x^{n+i}, & \binom{n}{i} \equiv 1 \mod 2 \\ 0, & \binom{n}{i} \equiv 0 \mod 2 \end{cases} \in H^{n+i}(X, A).$$

Definition 2.4. We call a collection of cohomology operations $\Theta = (\Theta_n : H^n(X, A) \Rightarrow H^{n+i}(X, A))_n$ of degree i stable if they commute with the suspension isomorphisms:

$$\begin{tikzcd} H^n(X,*) \arrow[r, "\Sigma"] \arrow[d, "\Theta"'] & H^{n+1}(\Sigma X,*) \arrow[d, "\Theta"] \\ H^{n+i}(X,*) \arrow[r, "\Sigma"'] & H^{n+i+1}(\Sigma X,*) \end{tikzcd}$$

To give a nice application, we will assume for now that the Steenrod squares are stable cohomology operations and give the proof later.

Proposition 2.5. The Steenrod squares are stable cohomology operations.

Note also that it is basically impossible to get a nice interaction of just the cup-squares with suspension, since the cup product on a suspension space is always trivial.[1]

2.2 Application: The first stable stem

We now want to use Steenrod squares to prove that the first stable stem $\pi_1^s = \mathrm{colim}_n\, \pi_{n+1}(S^n)$ is $\mathbb{Z}/2$ (here the colimit is taken over the suspension maps $\Sigma : [S^{n+1}, S^n] \to [S^{n+2}, S^{n+1}]$). Recall the Freudenthal suspension theorem: If X is well-pointed (inclusion of the point is a cofibration) and $\pi_k(X) = 0$ for $k \leq n$, then the suspension morphism $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism for $k \leq 2n$ and surjective for $k = 2n + 1$. Applying this to spheres, this tells us that we are taking the colimit over the sequence

$$\begin{tikzcd} \pi_2(S^1) \arrow[r] \arrow[d, equal] & \pi_3(S^2) \arrow[r, two heads, "\Sigma"] \arrow[d, equal] & \pi_4(S^3) \arrow[r, "\Sigma", "\cong"'] & \pi_5(S^4) \arrow[r, "\Sigma", "\cong"'] & \cdots \\ 0 & \mathbb{Z}\eta \end{tikzcd}$$

So it suffices to determine $\pi_4(S^3)$. We know the suspension $\Sigma : \pi_3(S^2) \to \pi_4(S^3)$ is surjective, and we saw in the talk on fibrations that $\pi_3(S^2) = \mathbb{Z}\eta$ is freely generated by (the class of) the Hopf fibration $\eta : S^3 \to S^2$. So we already know that $\pi_4(S^3)$ is generated by (the class of) $\Sigma\eta$.

Proposition 2.6. $0 \neq [\Sigma\eta] \in \pi_4(S^3)$.

[1] More generally, if X can be covered by n contractible open subsets, then the cup product of any n elements vanishes in $H^*(X)$.

Proof. Note that the Hopf fibration is the attaching map used to build $\mathbb{CP}^2$ out of $\mathbb{CP}^1 = S^2$, as shown on the left below. Since suspension is a left adjoint, it commutes with colimits and we get the pushout square on the right:

$$\begin{tikzcd} S^3 \arrow[r, "\eta"] \arrow[d] & S^2 \arrow[d] \\ D^4 \arrow[r] & \mathbb{CP}^2 \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} S^4 \arrow[r, "\Sigma\eta"] \arrow[d] & S^3 \arrow[d] \\ D^6 \arrow[r] & \Sigma\mathbb{CP}^2 \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Now suppose that $\Sigma\eta$ is nullhomotopic. Since pushouts along homotopic maps yield homotopy-equivalent results (cf. [Hat, Proposition 0.18]), we obtain $\Sigma\mathbb{CP}^2 \simeq S^3 \vee S^5$. But then naturality and Corollary 2.12 give the commutative diagram

$$\begin{tikzcd} H^3(S^3) \arrow[r, "\mathrm{pr}^*", "\cong"'] \arrow[d, "\mathrm{Sq}^2"'] & H^3(S^3 \vee S^5) \arrow[r, "\cong"'] \arrow[d, "\mathrm{Sq}^2"] & H^3(\Sigma\mathbb{CP}^2) \arrow[d, "\mathrm{Sq}^2"] & H^2(\mathbb{CP}^2) \arrow[l, "\Sigma"', "\cong"] \arrow[d, "\mathrm{Sq}^2"] \\ H^5(S^3) \arrow[r, "\mathrm{pr}^*"'] & H^5(S^3 \vee S^5) \arrow[r, "\cong"] & H^5(\Sigma\mathbb{CP}^2) & H^4(\mathbb{CP}^2) \arrow[l, "\cong"', "\Sigma"] \end{tikzcd}$$

The UCT yields $H^*(\mathbb{CP}^2) \cong \mathbb{Z}/2[x]/(x^3)$ with $|x| = 2$, hence the right vertical Sq^2 is nontrivial, as it coincides with the cup-square on $H^2(\mathbb{CP}^2)$. But the diagram shows that it factors through $H^5(S^3) = 0$. Contradiction. $\square$

The rest of the proof of showing that $\pi_1^s \cong \mathbb{Z}/2$ has nothing to do with Steenrod squares, but we give it for completeness.

Proposition 2.7. $0 = 2[\Sigma\eta] \in \pi_4(S^3)$.

Proof. Viewing S^3 as the unit sphere in $\mathbb{C}^2$, we can explicitly describe the Hopf fibration as

$$\eta : S^3 \to \mathbb{CP}^1, (x, y) \mapsto [x : y].$$

This leads to the commutative diagram

$$\begin{tikzcd} S^3 \arrow[r, "(x,y)\mapsto(\overline{x},\overline{y})", "\sigma"'] \arrow[d, "\eta"'] & S^3 \arrow[d, "\eta"] \\ \mathbb{CP}^2 \arrow[r, "\tau", "{[x:y]\mapsto[\overline{x},\overline{y}]}"'] & \mathbb{CP}^2 \end{tikzcd}$$

It is easy to deduce that σ has degree 1 / is homotopic to the identity, whereas τ has degree -1 / is a reflection on the sphere $S^2 \cong \mathbb{CP}^1$, since $[1 : 0]$ is fixed, and $[x : 1]$ is sent to $[\overline{x} : 1]$. Applying suspension and passing to homotopy classes we obtain

$$[\Sigma\eta] = [\Sigma\eta \circ \Sigma\sigma] = [\Sigma\tau \circ \Sigma\eta] = -[\Sigma\eta]$$

where in the last step we use that $\Sigma\tau$ is of degree -1, and we can write $\Sigma = S^1 \wedge -$, so that

$$[\Sigma\tau \circ \Sigma\eta] = [S^1 \wedge (-1) \circ S^1 \wedge \eta] = [(-1) \wedge S^2 \circ S^1 \wedge \eta] = [S^1 \wedge \eta \circ (-1) \wedge S^2] = -[S^1 \wedge \eta].$$

Here $(-1) : S^1 \to S^1$ denotes any map of degree -1. Thus $[\Sigma\eta] = -[\Sigma\eta]$, i.e. $2[\Sigma\eta] = 0 \in \pi_4(S^3)$. $\square$

Corollary 2.8. The first stable stem is $\pi_1^s = \mathrm{colim}_n\, \pi_{n+1}(S^n) \cong \pi_4(S^3) \cong \mathbb{Z}/2$.

2.3 Properties of Steenrod Squares

Proposition 2.9. The first Steenrod square $\mathrm{Sq}^1 : H^n(X,A) \Rightarrow H^{n+1}(X,A)$ agrees with the Bockstein Homomorphism β from Example 1.2.

Proof. By the Yoneda argument in the first section, we have

$$\mathrm{Nat}(H^n(-;\mathbb{Z}/2), H^{n+1}(-;\mathbb{Z}/2)) \cong H^{n+1}(K(\mathbb{Z}/2,n);\mathbb{Z}/2).$$

As both the Bockstein and Sq^1 are nontrivial, it suffices to show that the latter cohomology group is just $\mathbb{Z}/2$. Recall from a previous talk that we can construct $K(\mathbb{Z}/2,n)$ by first considering the free resolution $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ and constructing

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & S^n \arrow[d] \\ D^{n+1} \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where f is some map of degree 2. This already yields $\pi_k(X) = 0$ for $k < n$ and $\pi_n(X) \cong \mathbb{Z}/2$. Now we kill all homotopy groups $\pi_k(X)$ for $k \geq n+1$ by attaching cells of dimension $\geq n+2$, which results in a model of $K(\mathbb{Z}/2,n)$ that has only a single $n+1$-cell. Thus $H^{n+1}(K(\mathbb{Z}/2,n+1))$ is either $\mathbb{Z}/2$ or 0, and the existence of the Bockstein homomorphism implies it must be $\mathbb{Z}/2$. $\square$

Proposition 2.10. The Cartan-formula also works for the cross-product. Specifically, for $x \in H^*(X,A)$ and $y \in H^*(Y,B)$

$$\mathrm{Sq}^k(x \times y) = \sum_{i+j=k} \mathrm{Sq}^i(x) \times \mathrm{Sq}^j(y) \in H^*(X \times Y,\ X \times B \cup A \times Y)$$

Proof. Recall that $x \times y = p_X^*(x) \cup p_Y^*(y)$ where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. Using naturality and the Cartan-formula we get

$$\begin{aligned} \mathrm{Sq}^k(x \times y) &= \sum_{i+j=k} \mathrm{Sq}^i(p_X^*(x)) \cup \mathrm{Sq}^j(p_Y^*(y)) \\ &= \sum_{i+j=k} p_X^*(\mathrm{Sq}^i(x)) \cup p_Y^*(\mathrm{Sq}^j(y)) \\ &= \sum_{i+j=k} \mathrm{Sq}^i(x) \times \mathrm{Sq}^j(y). \end{aligned}$$

$\square$

Proposition 2.11 ([Bre, Proposition VI.15.2])**.** The Steenrod squares commute with the boundary operators:

$$\begin{tikzcd} H^n(A) \arrow[r, "\delta"] \arrow[d, "\mathrm{Sq}^i"'] & H^{n+1}(X,A) \arrow[d, "\mathrm{Sq}^i"] \\ H^{n+i}(A) \arrow[r, "\delta"'] & H^{n+i+1}(X,A) \end{tikzcd}$$

Proof. One can show that we can reduce to the case of the pair $(A \times I, A \times \partial I)$. This involves standard arguments of naturality, excision and homotopy invariance, so we will just assume this here and refer the interested reader to [Bre, Proposition VI.15.2]. Now since we work with $\mathbb{Z}/2$-coefficients, the Künneth formula gives us an isomorphism

$$H^n(A) \otimes H^0(\partial I) \xrightarrow[\cong]{\times} H^n(A \times \partial I)$$

which means that every cohomology class in $H^n(A \times \partial I)$ can be written as $x \times y$ for $x \in H^n(A)$ and $y \in H^0(\partial I)$. Recall how the cross product interacts with the connecting homomorphism δ:

$$\begin{tikzcd}
H^n(A) \otimes H^0(\partial I) \arrow[r, "\times"] \arrow[d, "\mathrm{id} \otimes \delta^0"'] & H^n(A \times \partial I) \arrow[d, "\delta^n"] \\
H^n(A) \otimes H^1(I, \partial I) \arrow[r, "\times"] & H^{n+1}(A \times I, A \times \partial I)
\end{tikzcd}$$

So in our case we get $\delta^n(x \times y) = x \times \delta^0 y \in H^{n+1}(A \times I, A \times \partial I)$. With this formula, Proposition 2.10, and the axiom that $\mathrm{Sq}^i(x) = 0$ for $|x| < i$ we obtain

$$\begin{aligned}
\mathrm{Sq}^i(\delta^n(x \times y)) &= \mathrm{Sq}^i(x \times \delta^0 y) \\
&= \sum_{j+k=i} \mathrm{Sq}^j(x) \times \mathrm{Sq}^k(\delta^0 y) \\
&= \mathrm{Sq}^i(x) \times \mathrm{Sq}^0(\delta^0 y) + \mathrm{Sq}^{i-1}(x) \times \underbrace{\mathrm{Sq}^1(\delta^0 y)}_{\in H^2(I, \partial I) = 0} \\
&= \mathrm{Sq}^i(x) \times \delta^0 y \\
&= \delta^{n+i}(\mathrm{Sq}^i(x) \times y) \\
&= \delta^{n+i}(\mathrm{Sq}^i(x \times y))
\end{aligned}$$

where in the last step we used Proposition 2.10 again. ☐

Corollary 2.12. The Steenrod squares are stable cohomology operations, so that we have commutative diagrams

$$\begin{tikzcd}
H^n(X, *) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{Sq}^i"'] & H^{n+1}(\Sigma X, *) \arrow[d, "\mathrm{Sq}^i"] \\
H^{n+i}(X, *) \arrow[r, "\Sigma"'] & H^{n+i+1}(\Sigma X, *)
\end{tikzcd}$$

Proof. Writing CX for the cone, recall that ΣX is (homotopy equivalent to) the cofiber CX/X and the map $q : (CX, X) \to (\Sigma X, *)$ induces an isomorphism $q^* : H^n(\Sigma X, *) \cong H^n(CX, X)$. Now Proposition 2.11 and naturality give the commuting diagram

$$\begin{tikzcd}
H^n(X, *) \arrow[r, "\delta^n"] \arrow[rr, bend left, "\Sigma"] \arrow[d, "\mathrm{Sq}^i"'] & H^{n+1}(CX, X) \arrow[d, "\mathrm{Sq}^i"'] & H^{n+1}(\Sigma X, *) \arrow[l, "q^*"'] \arrow[d, "\mathrm{Sq}^i"'] \\
H^{n+i}(X, *) \arrow[r, "\delta^{n+i}"] \arrow[rr, bend right, "\Sigma"'] & H^{n+i+1}(CX, X) & H^{n+i+1}(\Sigma X, *) \arrow[l, "q^*"']
\end{tikzcd}$$

If we want to be very pedantic, then the above connecting homomorphism is actually the one in the triple sequence for $(CX, X, *)$, which is defined as $H^n(X, *) \xrightarrow{\mathrm{inc}^*} H^n(X) \xrightarrow{\delta^n} H^{n+1}(CX, X)$, and hence also commutes with the Steenrod squares by naturality. ☐

Definition 2.13. The Steenrod Algebra $\mathcal{A}$ is the graded $\mathbb{Z}/2$-algebra generated by all stable cohomology operations under composition. Composition is defined in the obvious way: if $\alpha, \beta \in \mathcal{A}$ with $|\alpha| = p$ and $|\beta| = q$, then

$$(\alpha\beta)_n \coloneqq \alpha_{n+q}\beta_n : H^n(X, A) \Rightarrow H^{n+p+q}(X, A), \qquad n \geq 0.$$

The operations of degree p are exactly those that raise the cohomology degree by p.

Remark 2.14. For any space-pair (X, A), the graded cohomology ring $H^*(X, A) = \bigoplus_{n=0}^{\infty} H^n(X, A)$ is canonically endowed with the structure of a graded $\mathcal{A}$-module,

$$\mathcal{A} \times H^*(X, A) \to H^*(X, A)$$

where the action is just application of the cohomology operations.

Theorem 2.15 ([FF, Sections 30])**.** The Steenrod Algebra $\mathcal{A}$ is generated by the Steenrod squares, subject to the Adem relations:

$$\operatorname{Sq}^a \operatorname{Sq}^b = \sum_{i=0}^{\lfloor a/2 \rfloor} \binom{b-i-1}{a-2i} \operatorname{Sq}^{a+b-i} \operatorname{Sq}^i, \qquad 0 < a < 2b.$$

Here the binomial coefficients are taken mod 2. Call a sequence of natural numbers $I = (i_1, \ldots, i_n)$ admissible if $i_j \geq 2i_{j+1}$. Then

$$(\operatorname{Sq}^I = \operatorname{Sq}^{i_1} \cdots \operatorname{Sq}^{i_n} \mid I \text{ admissible})$$

forms a $\mathbb{Z}/2$-Basis of the Steenrod Algebra $\mathcal{A}$, called the Serre-Cartan basis.

Proposition 2.16. If i is not a power of 2, then Sq^i is decomposable, meaning that we can write it as a sum of compositions of Steenrod squares of smaller degree than i.

Proof. We can rewrite the Adem relations as

$$\binom{b-1}{a} \operatorname{Sq}^{a+b} = \operatorname{Sq}^a \operatorname{Sq}^b + \sum_{i=1}^{\lfloor a/2 \rfloor} \binom{b-i-1}{a-2i} \operatorname{Sq}^{a+b-i} \operatorname{Sq}^i, \qquad 0 < a < 2b.$$

So if $\binom{b-1}{a} \equiv 1 \mod 2$ then Sq^{a+b} is decomposable. Now if $i = a+b$ with $b = 2^k$ and $a = 0 < a < 2^k$, then a calculation shows that indeed $\binom{b-1}{a} \equiv 1 \mod 2$, see [Bre, Proposition VI.15.6]. □

Example 2.17. A few of the obtained Adem relations are:

$$\operatorname{Sq}^1 \operatorname{Sq}^{2n} = \operatorname{Sq}^{2n+1}, \quad \operatorname{Sq}^1 \operatorname{Sq}^{2n+1} = 0, \quad \operatorname{Sq}^2 \operatorname{Sq}^2 = \operatorname{Sq}^3 \operatorname{Sq}^1, \qquad n \geq 0.$$

Corollary 2.18. We have the following immediate consequences of Proposition 2.16:

1. If i is not a power of 2 and if X is a space such that $H^k(X) = 0$ for $n < k < n+i$, then $0 = \operatorname{Sq}^i : H^n(X) \to H^{n+i}(X)$.
2. If $x \in H^n(X)$ and $x^2 \neq 0$, then $\operatorname{Sq}^{2^i}(x) \neq 0$ for some i with $0 < 2^i \leq n$.
3. If $H^*(X) \cong (\mathbb{Z}/2)[x]$ or $(\mathbb{Z}/2)[x]/(x^q)$ for some $q > 2$, then $|x|$ is a power of 2.
4. If M^{2n} is a closed $2n$-manifold with $H_i(M) = 0$ for $0 < i < n$ and $H_n(M) \cong \mathbb{Z}/2$, then n is a power of 2.

Corollary 2.19. If there exists a fiber bundle $S^{n-1} \to S^{2n-1} \xrightarrow{f} S^n$, then n is a power of 2.

Proof. In this case M_f is a $2n$-manifold with boundary S^{2n-1} and hence C_f is a closed $2n$-manifold with homology as in Corollary 2.18(4.). □

Adams [Ada] showed that in Corollary 2.18(3.) and hence also in the next two corollaries the only possible powers of 2 are in fact 1,2,4 or 8. This is connected via the Hopf-invariant to the classical problem of when $\mathbb{R}^n$ can be given the structure of a division algebra, where the only possibilities are $\mathbb{R}, \mathbb{C}$, the 4-dimensional quaternions $\mathbb{H}$, and the 8-dimensional octonions $\mathbb{O}$.

3 Construction of the Steenrod Squares

For some space X, let $CX = C(X;\mathbb{Z}/2)$ denote the singular chain complex with $\mathbb{Z}/2$-coefficients, and $C^*X = \mathrm{Hom}(CX, \mathbb{Z}/2)$ the singular cochain complex with $\mathbb{Z}/2$-coefficients. We follow [Swi, Chapter 18].

Definition 3.1. A diagonal approximation is a natural chain map

$$\Delta : C(X) \to C(X) \otimes C(X)$$

such that $\Delta_0(x) = x \otimes x$ on 0-simplices $x \in C_0(X)$.

Remark 3.2.

1. Any two diagonal approximations are naturally chain-homotopic by a routine application of the method of acyclic models.

2. Any diagonal approximation Δ can be used to compute the cup-product via

$$H^p(X) \otimes H^q(X) \xrightarrow{[\varphi]\otimes[\psi]\mapsto[\varphi\psi]} H^{p+q}(\mathrm{Hom}(CX \otimes CX, \mathbb{Z}/2)) \xrightarrow{\Delta^*} H^{p+q}(X).$$

   Here $\varphi : C_p \to \mathbb{Z}/2$ and $\psi : C_q \to \mathbb{Z}/2$ and $\varphi\psi : C_p \otimes C_q \to \mathbb{Z}/2$ is their pointwise product.

3. The Alexander-Whitney map is a diagonal approximation giving the usual formula for the singular cup-product.

The existence of Steenrod squares rests on the subtle fact that while the cup product is (graded) commutative on cohomology, on the chain-level such diagonal approximations need not be commutative (and in fact, cannot be[2]) in the following sense. Consider the twist

$$T : CX \otimes CX \to CX \otimes CX, \quad a \otimes b \mapsto b \otimes a.$$

This is a chain map (usually we would need some signs, but recall all chain complexes are over $\mathbb{Z}/2$). Now if $\Delta^0 : CX \to CX \otimes CX$ is any diagonal approximation, then $T\Delta^0$ is one as well. The method of acyclic models provides a natural chain homotopy $\Delta^1 = (\Delta^1_n : C_nX \to (CX \otimes CX)_{n+1})_n$ with

$$\partial\Delta^1 + \Delta^1\partial = T\Delta^0 - \Delta^0 = (T-1)\Delta^0.$$

Similarly $T\Delta^1$ need not agree with Δ^1 (and in fact cannot, just as for Δ^0), but the method of acyclic models provides a natural chain homotopy $\Delta^2 = (\Delta^2_n : C_nX \to (CX \otimes CX)_{n+2})_n$ with

$$\partial\Delta^2 + \Delta^2\partial = T\Delta^1 - \Delta^1.$$

Continuing this process by induction would be one way to prove the following proposition

Proposition 3.3 ([Swi, Proposition 18.1])**.** There are natural homomorphisms $\Delta^k = (\Delta^k_n : C_nX \to (CX \otimes CX)_{n+k})_n$ of degree $+k$ for each $k \geq 0$ such that

1. Δ^0 is a diagonal approximation.
2. $\partial\Delta^{k+1} + \Delta^{k+1}\partial = T\Delta^k - \Delta^k$.

We can now define refined versions of the cup product as follows:

$$\cup_i : C^pX \otimes C^qX \xrightarrow{\varphi\otimes\psi\mapsto\varphi\cdot\psi} \mathrm{Hom}(C_pX \otimes C_qX,\ \mathbb{Z}/2) \xrightarrow{(\Delta^i)^*} C^{p+q-i}(X).$$

and in fact since Δ^0 is a diagonal approximation $\cup_0$ induces the usual cup product on cohomology. The following two lemmata are just straightforward calculations.

[2] Were the diagonal approximations commutative, then in the construction of the Steenrod squares we could take $\Delta^1 = 0$ and the following development would be trivial, implying that $\mathrm{Sq}^i(x) = 0$ for $|x| \neq i$. As this is not the case, diagonal approximations cannot be commutative on the nose.

Lemma 3.4. In the above setting, we have

1. $\cup_i$ is natural, meaning $f^*(c \cup_i d) = f^*c \cup_i f^*d$ for any map $f : X \to Y$ and $c, d \in C^*Y$.
2. $\cup_i$ is bilinear, meaning $(c_1 + c_2) \cup_i (d_1 + d_2) = c_1 \cup_i d_1 + c_1 \cup_i d_2 + c_2 \cup_i d_1 + c_2 \cup_i d_2$.
3. For $c \in C^n X$ and $d \in C^m X$ we have
$$\delta(c \cup_i d) = \delta c \cup_i d + c \cup_i \delta d + c \cup_{i-1} d + d \cup_{i-1} c$$

Proof. The first two statements are clear. For (3.), if $a \in CX$, then

$$\begin{aligned}
\delta(c \cup_i d)(a) &= (c \cdot d)(\Delta^i \partial a) \\
&= (c \cdot d)([(T-1)\Delta^{i-1} - \partial \Delta^i]a) \\
&= (d \cdot c)(\Delta^{i-1} a) - (c \cdot d)(\Delta^{i-1} a) - \delta(c \cdot d)(\Delta^i a) \\
&= (d \cup_i c)(a) - (c \cup_i d)(a) - \delta(c \cdot d)(\Delta^i a) \\
&= (d \cup_i c - c \cup_i d)(a) - (\delta c \cdot d + c \cdot \delta d)(\Delta^i a) \\
&= (d \cup_i c - c \cup_i d - \delta c \cup_i d - c \cup_i \delta d)(a) \\
&= (d \cup_i c + c \cup_i d + \delta c \cup_i d + c \cup_i \delta d)(a).
\end{aligned}$$

where in the last line we use that we are working in $\mathbb{Z}/2$ coefficients. □

We can now define

$$\mathrm{Sq}^i_n : C^n X \Rightarrow C^{n+i} X, \quad x \mapsto \begin{cases} x \cup_{n-i} x, & i \le n \\ 0, & i > n. \end{cases}$$

Lemma 3.5. In the above setting, we have that

1. Sq^i is natural in that $f^* \mathrm{Sq}^i x = \mathrm{Sq}^i f^* x$. In particular, if $x \in C^* X$ vanishes on $C^* A$ for $A \subseteq X$, then so does $\mathrm{Sq}^i x$, and hence $\mathrm{Sq}^i : C^*(X, A) \Rightarrow C^{*+i}(X, A)$ is well-defined.
2. Sq^i sends cocycles to cocycles, and coboundaries to coboundaries.
3. $\mathrm{Sq}^i(x + y) = \mathrm{Sq}^i x + \mathrm{Sq}^i y + \delta(x \cup_{n-i+1} y)$ for cocycles $x, y \in C^* X$.

Proof. Again the first statement is clear from the definition. Suppose $\delta c = 0$. Then

$$\delta \mathrm{Sq}^i c = \delta(c \cup_{n-i} c) = \delta c \cup_{n-i} c + c \cup_{n-i} \delta c + 2(c \cup_{n-i-1} c) = 0.$$

If $c = \delta d$, then

$$\begin{aligned}
\delta[d \cup_{n-i} + d \cup_{n-i-1} d] &= \delta d \cup_{n-i} c + d \cup_{n-i} \delta c + d \cup_{n-i-1} c + c \cup_{n-i-1} d \\
&\quad + \delta d \cup_{n-i-1} d + d \cup_{n-i-1} \delta d + 2(d \cup_{n-i-2} d) \\
&= \delta d \cup_{n-i} c + d \cup_{n-i} \delta\delta c + 2(d \cup_{n-i-1} c + c \cup_{n-i-1} d + d \cup_{n-i-2} d) \\
&= c \cup_{n-i} c \\
&= \mathrm{Sq}^i c.
\end{aligned}$$

Lastly, let $x, y \in C^n(X)$ by cocycles. The additivity is obvious for $n < i$. Otherwise

$$\begin{aligned}
\delta(x \cup_{n-i+1} y) &= \delta x \cup_{n-i+1} y + x \cup_{n-i+1} \delta y + x \cup_{n-i} y + y \cup_{n-i} x \\
&= x \cup_{n-i} y + y \cup_{n-i} x.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{Sq}^i(x + y) &= (x + y) \cup_{n-i} (x + y) \\
&= x \cup_{n-i} x + x \cup_{n-i} y + y \cup_{n-i} x + y \cup_{n-i} y \\
&= \mathrm{Sq}^i x + \mathrm{Sq}^i y + \delta(x \cup_{n-i+1} y).
\end{aligned}$$

□

Overall, the above Lemma shows that Sq^i induces natural homomorphisms on cohomology

$$\mathrm{Sq}^i : H^n(X, A) \Rightarrow H^{n+i}(X, A)$$

for all $n \geq 0$ and (X, A). Note that by construction we already have $\mathrm{Sq}^i(x) = x^2$ for $|x| = i$ and $\mathrm{Sq}^i(x) = 0$ for $|x| < i$. For the other two axioms, we refer to [Swi] or [Bre].

Proposition 3.6 ([Swi, Proposition 18.12] or [Bre, p.418-420])**.** The operations Sq^i satisfy the axioms from Definition 2.1.

References

[Ada] John Frank Adams, *On the non-existence of elements of hopf invariant one.*

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