Exercises for Algebraic Topology I – Sheet 10

Uni Bonn, WS 2018/19

Aufgabe 37. Let (X,A) be a relative CW-complex and Y be a $(n-1)$-connected space for $n \geq 2$. Suppose that $H^{q+1}(X,A;\pi_q(Y)) = 0$ for $n < q < \dim(X,A)$.

Prove or disprove that a map $f\colon A \to Y$ can be extended to X if and only if the primary obstruction $\gamma^{n+1}(f) \in H^{n+1}(X,A,\pi_n(Y))$ vanishes.

Aufgabe 38. Let G be an abelian group.

(a) Show that there exists up to homotopy precisely one map $\mu\colon K(G,n) \times K(G,n) \to K(G,n)$ which induces on π_n the map $G \times G \to G$ sending (g_1,g_2) to $g_1 + g_2$, and up to homotopy precisely one map $i\colon K(G,n) \to K(G,n)$ which induces on π_n the map $G \to G$ sending g to $-g$.

(b) Let X be a CW-complex. Construct using μ and i the structure of an abelian group on $[X,K(G,n)]$.

(c) Show that the bijection $[X,K(G,n)] \xrightarrow{\cong} H^n(X,G)$ sending $[f]$ to $f^*\iota_n$ for the preferred element $\iota_n \in H^n(K(G,n);G)$ is an isomorphism of abelian groups.

Aufgabe 39. Let X be a CW-complex. We have defined for a 1-dimensional complex vector bundle ξ over a CW-complex X its first Chern class $c_1(\xi) \in H^2(X;\mathbb{Z})$. Prove or disprove:

(a) Two 1-dimensional complex vector bundles over X are isomorphic if and only if they have the same first Chern class.

(b) Every element in $H^2(X;\mathbb{Z})$ occurs as the first Chern class of a 1-dimensional complex vector over X.

Aufgabe 40. Classify up to homotopy all compact n-dimensional manifolds N (possibly with boundary and possibly non-connected) such that for every compact n-dimensional manifold M the map

$$[M,N] \to \hom_{\mathbb{Z}}(H_n(M;\mathbb{Z}), H_n(N;\mathbb{Z})), \quad [f] \mapsto H_n(f;\mathbb{Z})$$

is bijective.

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handover on Wednesday, 18.12 in the lecture