Exercises for Topology I – Sheet 4

University of Bonn, WS 2018/19

Exercise 13. Let X be a topological space. Construct an explicite R-isomorphism

$$\bigoplus_{C \in \pi_0(X)} R \xrightarrow{\cong} H_0^{\mathrm{sing}}(X;R)$$

by inspecting the first differential of the singular R-chain complex of X.

Exercise 14. Let $f\colon X \to X$ be a self-map of a topological space. Define the mapping torus T_f by the pushout

$$\begin{tikzcd}
X \amalg X \arrow[r, "\mathrm{id}_X \amalg f"] \arrow[d, "j_0 \amalg j_1"'] & X \arrow[d] \\
X \times [0,1] \arrow[r] & T_f
\end{tikzcd}$$

where $j_k\colon X \to X \times [0,1]$ sends x to (x,k) for $k = 0,1$. Let $\mathcal{H}_*$ be a homology theory with values in R-modules. Denote by $k\colon X \to T_f$ the obvious inclusion.

Construct a long exact sequence of R-modules, the so called *Wang sequence*,

$$\cdots \xrightarrow{\delta_{n+1}} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(f)-\mathrm{id}_{\mathcal{H}_n(X)}} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(k)} \mathcal{H}_n(T_f) \xrightarrow{\delta_n} \mathcal{H}_{n-1}(X)$$
$$\xrightarrow{\mathcal{H}_{n-1}(f)-\mathrm{id}_{\mathcal{H}_{n-1}(X)}} \mathcal{H}_{n-1}(X) \xrightarrow{\mathcal{H}_{n-1}(k)} \cdots .$$

Exercise 15. Let $\mathcal{H}_*$ be a homology theory with values in $\mathbb{Z}$-modules satisfying the dimension axiom. Compute $\mathcal{H}_n(\mathbb{RP}^2)$ for $n \in \mathbb{Z}$.

Exercise 16. Let n be a natural number. Let C_* be a projective R-chain complex such that $C_i = 0$ holds for $i \leq -1$ and $i \geq n+1$ and $H_i(C_*)$ is projective for $0 \leq i \leq n-1$. Prove or disprove that $H_n(C_*)$ is projective.

---

to be handed in on 05.11. during the lecture