Exercises for Topology I – Sheet 12

University of Bonn, WS 2018/19

Exercise 45. Let G be a group and let X be a contractible CW-complex with a cellular G-action. Assume that for every $g \in G$ and every open cell e the implication $ge \cap e \neq \emptyset \implies g = 1$ holds, where $1 \in G$ is the neutral element. Let $\mathbb{Z}G$ be the group ring of G, i.e., the free abelian group $\mathbb{Z}G$ with G as basis and the multiplication coming from the group structure of G.

Show that the cellular $\mathbb{Z}$-chain complex $C_*^c(X)$ inherits from the G-action the structure of a free $\mathbb{Z}G$-chain complex which is a $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.

Exercise 46. Let $\mathbb{Z}$ be the trivial $\mathbb{Z}[\mathbb{Z}/2]$-module, i.e., its underlying abelian group is $\mathbb{Z}$ and $\mathbb{Z}/2$ acts trivially. Let t be a generator of $\mathbb{Z}/2$.

(a) Show that we obtain a projective $\mathbb{Z}[\mathbb{Z}/2]$-resolution of $\mathbb{Z}$ by the $\mathbb{Z}[\mathbb{Z}/2]$-chain complex

$$\cdots \xrightarrow{t+1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t+1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t+1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/2];$$

where $t \pm 1$ means multiplication with the element $t \pm 1 \in \mathbb{Z}G$;

(b) Show that $\mathrm{Tor}_n^{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to $H_n(\mathbb{RP}^\infty, \mathbb{Z})$ for $n \geq 0$;

(c) Compute $\mathrm{Tor}_n^{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}, \mathbb{Z})$ for $n \geq 0$.

Exercise 47. Let F a field and let $F[x]$ be the ring of polynomials in one variable x with coefficients in F. Consider F as a $F[x]$-module by letting x act trivially on F. Compute $\mathrm{Ext}^n_{F[x]}(F, F)$ for $n \geq 0$.

Exercise 48. Let G be a finite group. Let M be any finitely generated $\mathbb{Z}G$-module. Show that there exists a projective $\mathbb{Z}[G]$-resolution P_* of M such each P_n is a finitely generated free $\mathbb{Z}[G]$-module.

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to be handed in on 21.01. during the lecture

This is the last sheet of this course