# THE CHERN CLASSES AND THE EULER CHARACTERISTIC OF THE MODULI SPACES OF ABELIAN DIFFERENTIALS

MATTEO COSTANTINI, MARTIN MÖLLER, AND JONATHAN ZACHHUBER

Abstract. For the moduli spaces of Abelian differentials, the Euler characteristic is one of the most basic intrinsic topological invariants. We give a formula for the Euler characteristic that relies on intersection theory on the smooth compactification by multi-scale differentials. It is a consequence of a formula for the full Chern polynomial of the cotangent bundle of the compactification.

The main new technical tools are an Euler sequence for the cotangent bundle of the moduli space of Abelian differentials and computational tools in the Chow ring, such as normal bundles to boundary divisors.

Contents



## 1. Introduction

Only few aspects of the topology of the moduli spaces of holomorphic or meromorphic Abelian differentials $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ with singularities of type $\mu = (m_1, \dots, m_n)$ are currently known, such as the connected components ([KZ03], [Boi15]), and partial information about (quotients of) the fundamental group. This paper provides an expression for the Chern classes of the cotangent bundle of the compactified moduli spaces of abelian differentials and a formula to compute the Euler characteristic of these moduli spaces.

The moduli spaces of Abelian differentials can be thought of as relatives of the moduli space of curves $\mathcal{M}_{g,n}$, for which the Euler characteristic was computed in [HZ86] using a cellular decomposition (given by the arc complex) and counting of cells. Our strategy here is quite different. While the Euler characteristic is an

Research of the second and third author is supported by the DFG-project MO 1884/2-1 and by the LOEWE-Schwerpunkt "Uniformisierte Strukturen in Arithmetik und Geometrie".

intrinsic quantity associated to $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$, our strategy heavily uses the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ constructed in [BCGGM3] and all its properties that make it quite similar to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. Moreover, our strategy is not available to compute the Euler characteristic $\overline{\mathcal{M}}_{g,n}$, as it rather mimics the case of the projective space $\mathbb{P}^d$: The unprojectivized moduli spaces $\Omega\mathcal{M}_{g,n}(\mu)$ are linear manifolds and thus the cotangent bundle of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is governed by the Euler sequence, as in the case of $\mathbb{P}^d$.

Using this strategy we obtain the complete information about the Chern classes of the (logarithmic) canonical bundle of the compactified moduli spaces of Abelian differentials, and thus e.g. the $\chi_y$-genus. A special case, the formula for the canonical class, is particularly easy to state. We recall that the boundary divisors in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are the divisor $D_\mathrm{h}$ of irreducible curves with one self-node and the divisors $D_\Gamma$ parameterized by level graphs $\Gamma \in \mathrm{LG}_1(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ that have one level below the zero level and no horizontal edges (joining vertices of the same level). As for the moduli space of curves, the boundary divisors are nearly (in a sense that we elucidate further down) a product of two lower-dimensional moduli spaces, corresponding to top and bottom level. Those boundary divisors $D_\Gamma$ come with the integer $\ell_\Gamma$, the least common multiple of the prongs $\kappa_e$ along the edges, see Section 3.3 for a review of these notions. We let $\xi = c_1(\mathcal{O}(-1))$ be the first Chern class of the tautological bundle on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ (see Section 3.1).

**Theorem 1.1.** *The first Chern class of the logarithmic cotangent bundle of the projectivized compactified moduli space $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is*

$$\mathrm{c}_1(\Omega^1_{\overline{B}}(\log D)) \;=\; N\cdot\xi + \sum_{\Gamma\in\mathrm{LG}_1(\mathrm{B})} (N - N_\Gamma^\top)\ell_\Gamma[D_\Gamma] \qquad \in \mathrm{CH}^1(\overline{B})\,, \tag{1}$$

*where $N = \dim(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ and where $N_\Gamma^\top := \dim(B_\Gamma^\top)$ is the dimension of the unprojectivized top level stratum in $D_\Gamma$.*

To compute the Euler characteristic we need to understand the top Chern class as we recall in Section 2 along with standard terminology from intersection theory. To state a formula for the full Chern character we need to recall a procedure that also determines adjacency of boundary strata. It is given by undegeneration maps $\delta_i$ that contract all the edges except those that cross from level $-i+1$ to level $-i$, see Section 3.3 and Figure 4 in Section 10. This construction can obviously be generalized so that a larger subset of levels remains, for example the complement of $i$, denoted by the undegeneration map $\delta_i^{\complement}$. We can now define for any graph $\Gamma \in \mathrm{LG}_L(\overline{B})$ with $L$ levels below zero and without horizontal edges the quantity $\ell_\Gamma = \prod_{i=1}^L \ell_{\delta_i(\Gamma)}$.

**Theorem 1.2.** *The Chern character of the logarithmic cotangent bundle is*

$$\mathrm{ch}(\Omega^1_{\overline{B}}(\log D)) \;=\; e^\xi \cdot \sum_{L=0}^{N-1} \sum_{\Gamma\in\mathrm{LG}_L(\overline{B})} \ell_\Gamma\left(N - N^T_{\delta_L(\Gamma)}\right) \mathfrak{i}_{\Gamma*}\Big(\prod_{i=1}^{L} \mathrm{td}\left(\mathcal{N}_{\Gamma/\delta_i^{\complement}(\Gamma)}^{\otimes -\ell_{\delta_i(\Gamma)}}\right)^{-1}\Big)\,,$$

*where $\mathcal{N}_{\Gamma/\delta_i^{\complement}(\Gamma)}$ denotes the normal bundle of $D_\Gamma$ in $D_{\delta_i^{\complement}(\Gamma)}$, where* td *is the Todd class and $\mathfrak{i}_\Gamma : D_\Gamma \hookrightarrow \overline{B}$ is the inclusion map.*

We also give closed expressions for the Chern polynomial in Theorem 9.10, both fully factored and as a sum over level graphs.

To compute the Euler characteristics, we can simplify this expression significantly. Moduli spaces of Abelian differentials are not homogeneous spaces and we should not expect a proportionality between the top Chern class and the Masur-Veech volume form ([Mas82], [Vee82]). For comparison we note however that Masur-Veech volumes of holomorphic minimal strata (where $\mu = (2g-2)$) in each genus are essentially given by the top $\xi$-power ([Sau18]). For non-minimal holomorphic strata (that is, if all $m_i \geq 0$) this top $\xi$-power is zero and the Masur-Veech volume is computed by a product of $\xi^{2g-1}$ and $\psi$-classes ([CMSZ19]). The top $\xi$-powers of all levels of all strata – and only these – are combined to give the Euler characteristic of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. One thus needs the top $\xi$-powers for meromorphic moduli spaces, even if one might be only interested in the holomorphic case. Let $K_\Gamma = \prod_e \kappa_e$ be the product of the prongs over all edges of $\Gamma$.

**Theorem 1.3.** *The orbifold Euler characteristic of the moduli space $\Omega\mathcal{M}_{g,n}(\mu)$ is the dimension-weighted sum over all level graphs $\Gamma \in \mathrm{LG}_L(\overline{B})$ without horizontal nodes*

$$\chi(\Omega\mathcal{M}_{g,n}(\mu)) \;=\; (-1)^d \sum_{L=0}^{d} \sum_{\Gamma \in \mathrm{LG}_L(\overline{B})} \frac{K_\Gamma \cdot N_\Gamma^\top}{|\operatorname{Aut}(\Gamma)|} \cdot \prod_{i=0}^{-L} \int_{B_\Gamma^{[i]}} \xi_{B_\Gamma^{[i]}}^{d_\Gamma^{[i]}} \tag{2}$$

*of the product of the top power of the first Chern class $\xi_{B_\Gamma^{[i]}}$ of the tautological bundle at each level, where $d_\Gamma^{[i]} = \dim(B_\Gamma^{[i]})$ and $d = \dim(\overline{B}) = N - 1$.*

The stratum $B_\Gamma^{[i]}$ at the level $i$ of a graph $\Gamma$ is defined in Section 4.1.

| $\mu$ | $(0)$ | $(2)$ | $(1,1)$ | $(4)$ | $(3,1)$ | $(2,2)$ | $(2,1,1)$ | $(1,1,1,1)$ |
|---|---|---|---|---|---|---|---|---|
| $\chi(B)$ | $-\frac{1}{12}$ | $-\frac{1}{40}$ | $\frac{1}{30}$ | $-\frac{55}{504}$ | $\frac{16}{63}$ | $\frac{15}{56}$ | $-\frac{6}{7}$ | $\frac{11}{3}$ |
| $\mu$ | $(6)$ | $(5,1)$ | $(4,2)$ | $(3,3)$ | $(4,1,1)$ | $(3,2,1)$ | $(2,2,2)$ | $(8)$ |
| $\chi(B)$ | $-\frac{1169}{720}$ | $\frac{27}{5}$ | $\frac{76}{15}$ | $\frac{188}{45}$ | $-\frac{200}{9}$ | $-\frac{96}{5}$ | $-\frac{187}{10}$ | $-\frac{4671}{88}$ |

TABLE 1. Euler characteristics of some holomorphic strata

Table 1 gives the Euler characteristics of some strata of holomorphic differentials. A table of values of top $\xi$-powers and more examples are provided in Section 10. The evaluation of these formulas is performed by a sage package `diffstrata` that builds on the package `admcycles` for computation in the moduli space of curves ([DSZ20]). Specifically, the evaluation of tautological classes below is performed using the formula for fundamental classes of strata conjectured in [FP18] and [Sch18] and proven recently in [BHPSS20] based on results from [HS19]. The algorithms in this package are explained in [CMZ20].

**The Euler sequence.** Next we outline the ingredients needed to prove these theorems. Recall that for projective space the Euler sequence is the exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}(V)} \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(-1)^{\oplus \dim(V)} \xrightarrow{\mathrm{ev}} \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0\,. \tag{3}$$

Over the moduli space $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ this admits the following generalization, that combines Theorem 6.1 and Theorem 9.2. It states roughly that, using the local projective structure induced by period coordinates, in the interior of the stratum we indeed have an Euler sequence, if we replace the direct sum in the middle of the sequence by a local system. This local system naturally extends across the boundary, but the Euler sequence needs a correction term that we determine explicitly via a local computation using perturbed period coordinates.

**Theorem 1.4.** *The logarithmic cotangent bundle sits in an exact sequence*

$$0 \longrightarrow \Omega^1_{\overline{B}}(\log D)\Big(-\sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \ell_\Gamma D_\Gamma\Big) \to \mathcal{K} \to \mathcal{C} \longrightarrow 0\,, \tag{4}$$

*where $\mathcal{C}$ is an explicitly computable sheaf (see Lemma 9.4) supported on the boundary and where the vector bundle $\mathcal{K}$ on $\overline{B}$ fits into the Euler exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow (\overline{\mathcal{H}}^1_{rel})^\vee \otimes \mathcal{O}_{\overline{B}}(-1) \xrightarrow{\mathrm{ev}} \mathcal{O}_{\overline{B}} \longrightarrow 0\,. \tag{5}$$

*Here $\overline{\mathcal{H}}^1_{rel}$ is the Deligne extension of the local system of relative cohomology.*

This theorem directly implies Theorem 1.1. To deduce the other two theorems, we need to exploit further information on the Chow ring of the compactification.

**The tautological rings.** In Section 8 we define a notion of a system of *tautological rings* $R^\bullet(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ inside the Chow rings of the compactifications $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of the projectivized strata $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ that have been constructed in [BCGGM3]. This is the smallest system of $\mathbb{Q}$-subalgebras $R^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subset \mathrm{CH}^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ which

- contains the $\psi$-classes attached to the marked points,
- is closed under the pushfoward of the map forgetting a regular marked point (a zero of order zero), and
- is closed under the clutching homomorphisms $\zeta_{\Gamma,*}p^{[i],*}$, defined in Section 4.

For the moduli space of curves $\overline{\mathcal{M}}_g$ the clutching homomorphisms build a boundary divisor from a product of two smaller moduli spaces, or from just one for the irreducible boundary divisor that plays the role of our $D_\mathrm{h}$. For multi-scale differentials the situation is more involved. First, to relate $D_\Gamma$ to a product of moduli spaces, we need to allow spaces of disconnected curves and allow to impose residue conditions since the levels of $\Gamma$ have that property. We define such *generalized strata* and their modular compactification in Section 4. Second, the boundary divisors $D_\Gamma$ do not admit maps to such generalized strata, since the levels are tied to one another by a datum of the multi-scale differential, the prong-matchings. We need to construct a covering space $c_\Gamma : D^s_\Gamma \to D_\Gamma$ that removes the stacky structure of $D_\Gamma$, which has two properties. First, there are projection maps $p^{[i]}$ from $D^s_\Gamma$ to generalized strata and second, there are clutching maps $\zeta_\Gamma : D^s_\Gamma \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, that factor as $\zeta_\Gamma = \mathfrak{i}_\Gamma \circ c_\Gamma$ into the finite map $c_\Gamma$ and a closed embedding $\mathfrak{i}_\Gamma$. (The upper index of $D^s_\Gamma$ refers to the use of the simple twist group as in [BCGGM3] in the construction of this covering.)

**Theorem 1.5.** *For each $\mu$, a finite set of additive generators of $R^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ is given by the classes*

$$\zeta_{\Gamma_*}\Big(\prod_{i=0}^{-L} p^{[i],*}\alpha_i\Big) \tag{6}$$

*where* $\Gamma$ *runs over all level graphs for all boundary strata of* $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ *including the trivial graph and where* $\alpha_i$ *is a monomial in the* $\psi$*-classes supported on level i of the graph* $\Gamma$*.*

*The tautological ring contains the* $\kappa$*-classes and all level-wise tautological line bundle classes* $\zeta_{\Gamma *} p^{[i],*}\xi_{B_\Gamma^{[i]}}$ *of all level graphs* $\Gamma$*.*

An algorithm to perform the multiplication of these generators is given along with the proof of Theorem 1.5 in Section 8. An important technical tool in the proof is the excess intersection formula (see Proposition 8.1) which, like the above formulation of the tautological ring, has large structural similarities with the case of the Deligne-Mumford compactification. It is useful only if the normal bundles to the boundary divisors are known. Contrary to the Deligne-Mumford compactification the normal bundles to the boundary divisors defined by two-level graphs are indeed bundles, i.e. those boundary divisors do not self-intersect (see Section 5). Along with the clutching morphisms we define in Section 4.3 the tautological bundles on the top and bottom level strata of divisors and their first Chern classes $\xi^\top$ and $\xi^\perp$. In Section 7 we show:

**Theorem 1.6.** *The normal bundle* $\mathcal{N}_\Gamma$ *of a divisor* $D_\Gamma \in \mathrm{LG}_1(\overline{B})$ *has first Chern class*

$$c_1(\mathcal{N}_\Gamma) \;=\; \frac{1}{\ell_\Gamma}\big(-\xi_\Gamma^\top - c_1(\mathcal{L}_\Gamma^\top) + \xi_\Gamma^\perp\big) \quad \textit{in} \quad \mathrm{CH}^1(D_\Gamma)\,, \tag{7}$$

*where* $\mathcal{L}_\Gamma^\top$ *defined in* (49) *is a line bundle supported on the boundary of* $D_\Gamma$ *where the top-level stratum degenerates further.*

We define tautological rings $R^\bullet(D_\Gamma)$ of strata using the analogs of the additive generators (6) and as a consequence of the preceding theorem the normal bundle of each $D_\Gamma$ belongs to the tautological ring $R^\bullet(D_\Gamma)$.

**Organization and strategy of proof.** After recalling some background on intersection theory in Section 2, we provide the necessary details on the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ in Section 3. Each of the levels of a level graph gives rise to the notion of generalized strata, that are defined in Section 4. There we also introduce the covering of boundary strata that allows a decomposition into a product of levels. Section 5 provides a dimension count argument that implies the smoothness of all non-horizontal boundary strata and that is at the heart of a formula for exponentials of sums of over boundary graphs. This formula allows, together with Theorem 1.6, the passage from Theorem 1.4 to Theorem 1.3. Section 6 proves the restriction of Theorem 1.4 to the interior of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and Section 7 proves Theorem 1.6. In Section 8 we prove the properties of the tautological ring announced above. In Section 9 a local calculation at the boundary completes the proof of Theorem 1.4 and computations in the tautological ring allows the passage from Theorem 1.2 to Theorem 1.3.

The strategy used here applies to other linear manifolds for which a compactification similar to that in [BCGGM3] has been constructed. It is already available for meromorphic $k$-differentials for $k > 0$ (see [CMZ19]) and expected to work for any affine invariant manifold. The proof of the main theorems should carry over with very few adaptations. We hope to address these cases in a sequel.

**Acknowledgments.** We thank Martin Lüdtke, Matt Bainbridge, Dawei Chen, Vincent Delecroix, Quentin Gendron, Sam Grushevsky, and Johannes Schmitt for inspiring discussions and help with implementation of the algorithmic part of the project. We are also grateful to the Mathematical Sciences Research Institute (MSRI, Berkeley) and the Hausdorff Institute for Mathematics (HIM, Bonn), where significant progress on this paper was made during their programs and workshops. The authors thank the MPIM, Bonn, for hospitality and support for [SageMath] computations.

## 2. Euler characteristics via logarithmic differential forms

This section connects Euler characteristic to integrals of characteristic classes of the sheaf of logarithmic differential forms. The following proposition is certainly known, but not easy to locate in the literature. We use the occasion to give a self-contained proof, see also [Fio17], and recall some standard exact sequences.

**Proposition 2.1.** *Let $\overline{B}$ be a compact smooth $k$-dimensional manifold, let $D$ be a normal crossing divisor and $B = \overline{B} \smallsetminus D$. Then the Euler characteristic of $B$ can be computed as integral*

$$\chi(B) \;=\; (-1)^k \int_{\overline{B}} c_k(\Omega^1_{\overline{B}}(\log D)) \tag{8}$$

*over the top Chern class of the logarithmic cotangent bundle.*

In all our applications, $\overline{B}$ will be a compact orbifold or proper smooth Deligne-Mumford stack. We work throughout with orbifold Euler characteristics, and since then both sides of (8) are multiplicative in the degree of a covering, we can apply Proposition 2.1 verbatim.

2.1. **The compact case and the Riemann-Roch theorem.** We start with the proof of the special case of the main theorem.

**Proposition 2.2.** *If $B = \overline{B}$ is smooth, compact and $k$-dimensional, then*

$$\chi(B) \;=\; \int_{\overline{B}} c_k(T_B)\,. \tag{9}$$

We start by recalling some intersection theory. Let $\mathcal{E}$ be a holomorphic vector bundle on $B$. Denote by $c_i \coloneqq c_i(\mathcal{E}) \in \mathrm{CH}^i(B)$ the $i$th Chern class of $E$. Recall that $c_0 = 1$ and $c_i = 0$ for $i > \operatorname{rk} E =: r$. The *total Chern class* of $\mathcal{E}$ is the formal sum $\mathrm{c}(\mathcal{E}) = 1 + c_1 + \ldots + c_r$. in $\mathrm{CH}(B)$. Splitting formally $\mathrm{c}(\mathcal{E}) = \prod_{i=1}^r (1+\alpha_i)$ into the Chern roots, the *Chern character* is defined as the formal power series

$$\mathrm{ch}(E) = \sum_{i=1}^r \exp(\alpha_i) \;=\; \sum_{s\geq 0} \frac{1}{s!} \sum_{i=1}^r \alpha_i^s \;=\; \mathrm{rk}(E) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots .$$

Furthermore, the Todd class is defined as

$$\mathrm{td}(E) = \prod_{i=1}^r \frac{\alpha_i}{1-\exp(-\alpha_i)} \;=\; 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots .$$

The Grothendieck-Riemann-Roch theorem in the case of a map $f : X \to Y$ and for the special case of that the higher direct images $R^i f_* \mathcal{E}$ vanish, states that

$$\mathrm{ch}(f_*\mathcal{E}) \cdot \mathrm{td}(T_Y) \;=\; f_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(T_X))\,. \tag{10}$$

*Proof of Proposition 2.2.* For a topological proof, see e.g. [BT82, Proposition 11.24]. Using the notations already set up, we give a quick proof if moreover $B$ is Kähler. From the Borel-Serre identity ([Ful98, Example 3.2.5]) on a $k$-dimensional manifold

$$c_k(T_B) \;=\; \mathrm{ch}\Big(\sum_{j=1}^{k}(-1)^j\Omega_B^j\Big)\cdot \mathrm{td}(T_B)$$

and the application

$$\int_{\overline{B}} \mathrm{ch}((-1)^j\Omega_B^j)\cdot \mathrm{td}(T_B) \;=\; \sum_{\ell\geq 0}(-1)^{\ell+j}h^\ell(B,\Omega_B^j)$$

of Grothendieck-Riemann-Roch theorem for the map from $B$ to a point we get

$$\int_{\overline{B}} c_k(T_B) \;=\; \sum_{\ell,j\geq 0}(-1)^{\ell+j}h^\ell(B,\Omega_B^j) \;=\; \chi(B)$$

by the Hodge decomposition. $\square$

2.2. **The non-compact case and log differential forms.** We suppose throughout that $D = \cup_{j=1}^s D_j$ is a reduced normal crossing divisor, i.e., with distinct irreducible components $D_i$ intersecting each other transversally. In this situation $\Omega^1_{\overline{B}}(\log D)$ is defined to be the vector bundle of rank $n$ with the following local generators. In a neighborhood $U$ of a point where (say) the first $r \leq s$ divisors meet and where $x_1,\ldots,x_k$ is a local coordinate system with $D_j = \{x_j = 0\}$, then logarithmic cotangent bundle is defined by

$$\Omega^1_{\overline{B}}(\log D)(U) \;=\; \Big\langle \frac{dx_1}{x_1},\ldots,\frac{dx_r}{x_r},x_{r+1},\ldots,x_k\Big\rangle \tag{11}$$

as an $\mathcal{O}_{\overline{B}}(U)$-module. There is a fundamental exact sequence for log differential forms, namely

$$0 \to \Omega^1_{\overline{B}} \to \Omega^1_{\overline{B}}(\log D) \to \oplus_{j=1}^s(\mathfrak{i}_j)_*\mathcal{O}_{D_j} \to 0\,, \tag{12}$$

where $\mathfrak{i}_j : D_j \to \overline{B}$ is the inclusion map. More details can be found e.g. in [EV92, Proposition 2.3].

*Proof of Proposition 2.1.* We first reduce to the case that $D$ has simple normal crossings, i.e., to the case the the $D_j$ are all smooth. This can always be achieved by an étale covering. Since both sides of (8) are multiplied by the degree under such a covering, we can assume simple normal crossings. Our goal is to prove

$$\int_{\overline{B}} c_k\Big(\Omega^1_{\overline{B}}\Big(\log\sum_{i\geq 2}D_i\Big)\Big) \;=\; \int_{\overline{B}} c_k\Big(\Omega^1_{\overline{B}}(\log D)\Big) - \int_{D_1} c_{k-1}\Big(\Omega^1_{D_1}\Big(\log(\sum_{i\geq 2}D_i\cap D_1\Big)\Big)\,,$$

The claim follows then from the additivity $\chi(B)+\chi(D) = \chi(\overline{B})$ of the Euler characteristic, Proposition 2.2 and an application to the preceding identity to $B_j = \overline{B}\smallsetminus\cup_{i=j}^s D_j$.

We consider the inclusion of the boundary divisor $D_1$ and deduce from the ideal sheaf sequence that $c((\mathfrak{i}_1)_*\mathcal{O}_{D_1}) = (1-[D_1])^{-1}$ and that $c(\mathcal{N}_{D_1}) = 1+\mathfrak{i}_1^*[D_1]$. Moreover the normal bundle sequence $0 \to \mathcal{T}_{D_1} \to \mathfrak{i}_1^*\mathcal{T}_{\overline{B}} \to \mathcal{N}_{D_1} \to 0$ implies

$$c(\Omega^1_{D_1}) \;=\; \mathfrak{i}_1^*\Big(c(\Omega^1_{\overline{B}})\cdot\frac{1}{1-[D_1]}\Big)\,. \tag{13}$$

On the other hand, the sequence (12) gives

$$c(\Omega^1_{\overline{B}}(\log D)) \;=\; c(\Omega^1_{\overline{B}})\cdot \frac{1}{1-[D_1]}\cdot \prod_{j=2}^{s}\frac{1}{1-[D_j]} \tag{14}$$

and also

$$c(\Omega^1_{D_1}(\log\Big(\sum_{i\geq 2} D_i\cap D_1\Big))) \;=\; c(\Omega^1_{D_1})\cdot \prod_{j=2}^{s}\frac{1}{1-[D_1\cap D_j]}. \tag{15}$$

Hence comparing with (13) we get

$$c\Big(\Omega^1_{D_1}\big(\log\Big(\sum_{i\geq 2} D_i\cap D_1\Big)\big)\Big) \;=\; \mathfrak{i}_1^* c\big(\Omega^1_{\overline{B}}(\log D)\big)\,. \tag{16}$$

Finally from (14) and from the appropriate version of the sequence (12) we also get

$$c(\Omega^1_{\overline{B}}(\log D)) \;=\; \frac{1}{1-[D_1]}c\Big(\Omega^1_{\overline{B}}\Big(\log\sum_{i\geq 2} D_i\Big)\Big). \tag{17}$$

The claim now follows by multiplying this last expression with $1-[D_1]$, integrating and taking the $k$-th coefficient, using that $\int_{\overline{B}}[D]\cdot c_{k-1}(\Omega^1_{\overline{B}}(\log D)) = \int_D \mathfrak{i}_1^* c_{k-1}(\Omega^1_{\overline{B}}(\log D))$. □

## 3. The moduli space of multi-scale differentials

We recall here from [BCGGM3] basic properties of the moduli space of multi-scale differentials $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and its projectivization $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ that compactifies the moduli space $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ of projectivized meromorphic differentials. Throughout we suppose that $\mu=(m_1,\dots,m_n)\in\mathbb{Z}^n$ is the type of a differential, i.e., that $\sum_{j=1}^n m_j = 2g-2$. We usually abbreviate $B=\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ and $\overline{B}=\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$.

3.1. **Enhanced level graphs.** To define strata and the ambient space in the meromorphic case, we assume that there are $r$ positive $m$'s, $s$ zeroes, and $l$ negative $m$'s, with $r+s+l=n$, i.e., that we have $m_1\geq\cdots\geq m_r > m_{r+1}=\cdots=m_{r+s}=0 > m_{r+s+1}\geq\cdots\geq m_n$. Note that $m_j=0$ is allowed, representing an ordinary marked point. A pointed flat surface is usually denoted by $(X,\omega,\mathbf{z})$ where $\mathbf{z}=(z_1,\dots,z_n)$ are the marked points corresponding to the zeros, ordinary marked points, and poles of $\omega$. The sections over $\Omega\overline{\mathcal{M}}_{g,n}(\mu)$ corresponding to those marked points are denoted by $\mathcal{Z}_i$. We denote the polar part of $\mu$ by $\tilde{\mu}=(m_{r+s+1},\dots,m_n)$. The strata of meromorphic differentials are then naturally defined inside the twisted Hodge bundle

$$K\overline{\mathcal{M}}_{g,n}(\tilde{\mu}) \;=\; f_*\Big(\omega_{\mathcal{X}/\overline{\mathcal{M}}_{g,n}}\Big(-\sum_{j=r+s+1}^{n} m_j\mathcal{Z}_j\Big)\Big)$$

The strata are smooth complex substacks $\Omega\mathcal{M}_{g,n}(\mu)$ of dimension $N=2g-1+n$ in the holomorphic case $r=n$ and $N=2g-2+n$ in the meromorphic case.

To each boundary point in $D=\Xi\overline{\mathcal{M}}_{g,n}(\mu)\setminus\Omega\mathcal{M}_{g,n}(\mu)$ there is an associated *enhanced level graph* and $D$ is stratified by the type of this associated graph. Here a *level graph* is defined to be a stable graph $\Gamma=(V,E,H)$, with half-edges in $H$ that are either paired to form edges $E$ or correspond to the $n$ marked points, together with a total order on the vertices (with equality permitted). The graph $\Gamma$ is supposed to be connected here, from Section 8 on its components are in bijection with the components of the flat surfaces the generalized stratum parameterizes. For convenience we usually define the total order using a *level function* $\ell : V(\Gamma) \to \mathbb{Z}$, usually normalized to take values in $\{0, -1, \dots, -L\}$. We usually write $H_m = H \smallsetminus E$ for the half-edges corresponding to the marked points. Moreover, an *enhancement* (in [FP18] or [CMSZ19] this number is called a *twist*) is an assignment of a number $\kappa_e \geq 0$ to each edge $e$, so that $\kappa_e = 0$ if and only if the edge is horizontal. The triple $(\Gamma, \ell, \{\kappa_e\}_{e \in E(\Gamma)})$ is called and *enhanced level graph.* We denote the closure of the boundary stratum parametrizing multi-scaled differentials (as defined below) compatible with $(\Gamma, \ell, \{\kappa_e\})$ by $D_{(\Gamma, \ell, \{\kappa_e\})}$ or usually simply by $D_\Gamma$.

**Theorem 3.1** ([BCGGM3]). *There is a proper smooth Deligne-Mumford stack*[1] $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ *that contains the projectivized stratum* $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ *as open dense substack with the following properties.*

(i) *The boundary* $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \smallsetminus \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ *is a normal crossing divisor.*

(ii) *The codimension of a boundary stratum* $D_\Gamma$ *in* $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ *is equal to the number of horizontal edges plus the number* $L$ *of levels below zero.*

In particular, the *boundary divisors* consist of the divisor $D_{\mathrm{h}}$ (if $g \geq 1$) with just one non-separating horizontal edge and the ('vertical') boundary divisors indexed by two-level graphs without horizontal edges. (Separating horizontal edges are impossible because of the absence of a residue.) We give local coordinates near the boundary divisors in Section 6.2.

Note that the boundary strata $D_\Gamma$ may be empty for some enhanced level graphs. Deciding non-emptyness is the same as the realizability question that was addressed in [MUW17] purely in terms of graphs. The general version taking into account the residue conditions is stated in the algorithmic part of [CMZ20]. Note that these boundary strata may also be non-connected, see the discussion of prong-matching equivalence classes below.

Recall that the construction of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ in [BCGGM3] gives a morphism $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) = \overline{B} \to \mathbb{P}\left(f_*\omega_{\overline{\mathcal{X}}/\overline{\mathcal{M}}_{g,n}}(-\sum_{j=r+s+1}^{n} \mu_j \mathcal{Z}_j)\right)$ to the projectivised twisted Hodge bundle over the Deligne-Mumford compactification. The line bundle $\mathcal{O}_{\overline{B}}(-1)$ is the pullback of the tautological bundle from there.

3.2. **Twisted differentials and multi-scale differentials.** The space $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is a moduli stack for families of a certain collection of differentials, called multi-scale differentials, and this modular interpretation will be used e.g. in the Section 4 to define clutching maps and projection maps at the boundary. We will however refer to $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ as a moduli space to stick to the commonly used terminology. We recall the definition of a single multi-scale differentials, referring for full details of the definition for families to [BCGGM3]. We will recall further details where needed.

When referring to prongs we fix a direction in $S^1$ throughout, say the horizontal direction. Suppose that a differential $\omega$ has a zero of order $m \geq 0$ at $q \in X$. The differential $\omega$ selects inside the real projectivized tangent space $P_q = T_pX/\mathbb{R}_{>0}$

[1]This is not exactly the statement of the current version of [BCGGM3]. There, the space is introduced as a compact orbifold or proper Deligne-Mumford stack with finite quotient singularities at some boundary points. We anticipate here the forthcoming version that improves the structure by changing the definition of families of multi-scale differentials that locally represents the structure of a quotient stack instead of the underlying quotient space.

a collection of $\kappa = m+1$ horizontal (outgoing) *prongs* at $q$, the tangent vectors $\mathbb{R}_{>0} \cdot \zeta_\kappa^i \partial/\partial z$ in a chart where $\omega = z^m dz$ is in standard form and where $\zeta_\kappa$ is a primitive $\kappa$-th root of unity. We denote them by $P_q^{\mathrm{out}} \subset P_q$. The prongs are equivalently the tangent vectors to the outgoing horizontal rays. Dually, if $\omega$ has a pole of order $m \leq -2$, then $\omega$ has $\kappa = -m-1$ horizontal (incoming) *prongs* at $q$, denoted by $P_q^{\mathrm{out}} \subset P_q$, the tangent vectors $\mathbb{R}_{>0} \cdot -\zeta_\kappa^i \partial/\partial z$ in a chart where $\omega = z^m dz$.

We start with an auxiliary notion of differentials from [BCGGM1]. Given a pointed stable curve $(X, \mathbf{z})$, a *twisted differential* is a collection of differentials $\eta_v$ on each component $X_v$ of $X$, that is *compatible with a level structure* on the dual graph $\Gamma$ of $X$, i.e. vanishes as prescribed by $\mu$ at the marked points $z$, satisfies the matching order condition at vertical nodes, the matching residue condition at horizontal nodes and global residue condition of [BCGGM1]. We usually group the differentials on the components of level $i$ of $X$ to form the collection $\eta_{(i)}$ and refer to a twisted differential by $\boldsymbol{\eta} = (\eta_{(i)})$.

A *multi-scale differential of type* $\mu$ on a stable curve $X$ consists of an enhanced level structure $(\Gamma, \ell, \{\kappa_e\})$ on the dual graph $\Gamma$ of $X$, a twisted differential of type $\mu$ compatible with the enhanced level structure, and a prong-matching for each node of $X$ joining components of non-equal level. Here the *compatibility with the enhanced level structure* requires that at each of the two points $q^\pm$ glued to form the node corresponding to the edge $e \in E(\Gamma)$ the number of prongs of the differential $\boldsymbol{\eta}$ is equal to $\kappa_e$. Moreover, a *prong-matching* is an order-reversing isometry $\sigma_q : P_{q^-} \to P_{q^+}$ that induces a cyclic order-reversing bijection $\sigma_q : P_{q^-}^{\mathrm{in}} \to P_{q^+}^{\mathrm{out}}$ between the incoming prongs at $q^-$ and the outgoing prongs at $q^+$.

Finally we state the equivalence relation on multi-scale differentials used to construct $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. The space of isomorphism classes of twisted differentials compactible with $(\Gamma, \ell, \{\kappa_e\})$ and a prong-matching is a finite cover $\mathfrak{W}_{\mathrm{pm}}(\Gamma)$ of a product of strata (set $k=1$ in [CMZ19, Section 3.3] or see [BCGGM3, Section 5] for the viewpoint with an additional Teichmüller marking). Multi-scale differentials only retain the information on lower level up to projectivization. This rescaling of the lower levels is roughly given by a multiplicative torus $T^{L(\Gamma)}$. More precisely, the universal cover $\mathbb{C}^{L(\Gamma)} \to T^{L(\Gamma)}$ acts by rescaling the differentials on each level and simultaneously by fractional Dehn twists on the prong-matching. In fact a subgroup acts trivially, the twist group $\mathrm{Tw}_\Gamma$ that we describe in detail in Section 3.4. So the action factors through the action of the quotient $T_\Gamma = \mathbb{C}^L/\mathrm{Tw}_\Gamma$, called the *level rotation torus*, and two multi-scale differentials are defined to be equivalent, if they differ by the action of $T_\Gamma$.

The projectivized space $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ parametrizes projectivized multi-scale differentials, where $\mathbb{C}^*$ acts by simultaneously rescaling the differentials on all levels and leaving the prong-matchings untouched.

3.3. **Divisors, degeneration, undegeneration.** We let $\mathrm{LG}_L(B)$ be the set of all enhanced $(L+1)$-level graphs without horizontal edges. Recall that boundary divisors of $\overline{B}$ are $D_{\mathrm{h}}$ and $D_\Gamma$ for $\Gamma \in \mathrm{LG}_1(\mathrm{B})$. For later use we define

$$D \;=\; D_{\mathrm{h}} + \sum_{\Gamma \in \mathrm{LG}_1(\mathrm{B})} D_\Gamma \tag{18}$$

to be the total boundary divisor. The structure of the normal crossing boundary of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is encoded by *undegenerations*. Given a non-horizontal level graph $\Gamma$ with

$L+1$ levels, the associated boundary stratum $D_\Gamma$ is contained in the intersection of $L$ boundary divisors $D_{\Gamma_i}$ for $i = 1, \dots, L$ and we can describe this inclusion as follows. View the $i$-th level passage as a horizontal line just above level $-i$. Contract in $\Gamma$ all edges that do not cross this horizontal line to obtain a contraction map $\delta_i : \Gamma \to \Gamma_i$ of enhanced level graphs, where $\Gamma_i$ obtains a two-level structure with the top level corresponding to the components above the horizontal line and the bottom level those below that line. We call this the $i$-th undegeneration of $\Gamma$. This can be generalized for any subset $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, L\}$ and results in the undegeneration map

$$\delta_{i_1,\dots,i_n} \colon \mathrm{LG}_L(B) \to \mathrm{LG}_n(B)\,,$$

which contracts all the passage levels of a non-horizontal level graph $D_\Gamma$ except for the passages between levels $-i_k+1$ and $-i_k$, for those $i_k \in I$. For notational convenience we define $\delta_I^{\complement} = \delta_{I^{\complement}}$.

A *degeneration* of level graphs is simply the inverse of an undegeneration. It is convenient to have a symbol to express this dual process and we write

$$\Gamma \rightsquigarrow \widehat{\Delta} \qquad \text{or} \qquad \Gamma \overset{[i]}{\rightsquigarrow} \widehat{\Delta} \tag{19}$$

for a general undegeneration resp. specifically for an undegeneration where the $i$-th level is split into two levels.

**Remark 3.2.** *With the convention used here and in all of the rest, the levels of a level graph with $L+1$ levels are indexed by negative integers $\{0, -1, \dots, -L\}$, while the level passages are indexed by positive integers $\{1, \dots, L\}$. This implies for examples that $\Gamma \overset{[i]}{\rightsquigarrow} \widehat{\Delta}$ is equivalent to $\Gamma = \delta^{\complement}_{(-i+1)}(\widehat{\Delta})$.*

Note the map of graphs $\delta_I$ is only well-defined up to post-composition by automorphism of the enhanced level graph $\Gamma_i$. Taking this into account will be important for intersection theory, see Proposition 8.1.

3.4. **Prong-matchings and their equivalence classes.** In this section we illustrate the amount of combinatorial information encoded in the notion of a prong-matching, given that we also have to take into account the action of the level rotation torus. We start with a recurrent example.

*Case of a level graph $\Gamma \in \mathrm{LG}_1(\mathrm{B})$, i.e. a divisor $D_\Gamma$ different from $D_h$.* Such an enhanced level graph has $|E(\Gamma)|$ edges each of which carries the information of the prongs, and consequently then there are $K_\Gamma = \prod_{e \in E(\Gamma)} \kappa_e$ prong-matchings. However, this does not imply that locally $D_\Gamma$ is a degree $K_\Gamma$-cover of the product of the moduli spaces corresponding to the upper and lower level. Instead the effect of projectivization of the lower level on prong-matchings has to be taken into account. This effect is given by the action of the level rotation group $R_\Gamma \cong \mathbb{Z}^L \subset \mathbb{C}^L$ in the universal cover of the level rotation torus. This group $R_\Gamma$ acts diagonally turning the prong-matching at each edge by one (in a fixed direction). The stabilizer of a prong-matching is the *twist group* $\mathrm{Tw}_\Gamma$ referred to above. It is isomorphic to $\ell_\Gamma\mathbb{Z}$ as subgroup of $R_\Gamma$ where

$$\ell_\Gamma \;=\; \mathrm{lcm}(\kappa_e \colon e \in E(\Gamma))\,. \tag{20}$$

Orbits of $R_\Gamma$ are also called *equivalence classes of prong-matchings.* For divisors there are $g_\Gamma := K_\Gamma/\ell_\Gamma$ such equivalence classes.

*For a general level graph* $\Delta$ the situation is more complicated and the compactification $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ acquires a non-trivial quotient stack structure that can be computed as follows. As above, there are $K_\Delta = \prod_{e\in E(\Delta)} \kappa_e$ prong matchings. Now the level rotation group is $R_\Delta \cong \mathbb{Z}^L$, where the $i$-th factor twists by one all prong-matchings that cross the horizontal line above level $-i$. The stabilizer of a prong-matching is still called the twist group $\mathrm{Tw}_\Delta$. However, this group is no longer a product of the level-wise factors. In fact, for each $i \in \mathbb{N}$ the twist group of the level-undegeneration $D_{\delta_i(\Delta)}$ is a subgroup of $\mathrm{Tw}_\Delta$ and we call the sum of these subgroups the *simple Twist group* $\mathrm{Tw}_\Delta^s$. The generic stack structure of $D_\Delta$ is given by the product of the action of the group $\mathrm{Aut}(\Delta)$ of enhanced level graphs automorphisms and a cyclic group of order

$$e_\Delta = [\mathrm{Tw}_\Delta : \mathrm{Tw}_\Delta^s].$$

The number of prong-matching equivalence classes is

$$g_\Delta := |R_\Delta - \text{orbits on the set } K_\Delta| \;=\; K_\Delta/[R_\Delta : \mathrm{Tw}_\Delta]\,. \tag{21}$$

These indices can easily be computed using the elementar divisor theorem.

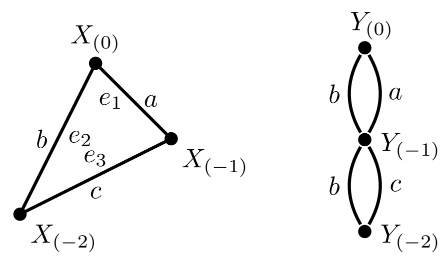



FIGURE 1. The triangle level graph and a graph with the same undegenerations

We discuss a simple case below where $\mathrm{Tw}_\Delta \neq \mathrm{Tw}_\Delta^s$ in preparation for the examples in Section 10. Finally, we generalize for later use the lcm defined above. We define $\ell_\Delta = \prod_{i=1}^L \ell_{\Delta,i}$ and where we use from now on the notation

$$\ell_{\Delta,i} \;=\; \mathrm{lcm}\Big(\kappa_e \colon e \in E(\Gamma)^{\geq -i}_{\leq -i}\Big) \;=\; \ell_{\delta_i(\Delta)} \tag{22}$$

as abbreviation of the one defined in the introduction, where $E(\Gamma)^{\geq -i}_{\leq -i}$ are the edges starting at level $-i+1$ or above and ending at level $-i$ or below.

**Example 3.3.** In the graph $\Delta$ in Figure 1 (left) there are three edges $e_1$,$e_2$ and $e_3$ with enhancements $a$, $b$ and $c$. The group $R_\Delta \cong \mathbb{Z}^2$ acts on $\coprod_{i=1}^{k_1+k_2+k_3} \mathbb{Z}/\kappa_i$ by mapping

$$(1,0) \mapsto (\underbrace{1,\ldots\ldots,1}_{k_1+k_2},\underbrace{0,\ldots,0}_{k_3}) \quad \text{and} \quad (1,0) \mapsto (\underbrace{0,\ldots,0}_{k_1},\underbrace{1,\ldots\ldots,1}_{k_2+k_3})\,.$$

Consequently, there are $\gcd(a,b,c)$ orbits, i.e. that many equivalence classes of prong-matchings near such a boundary point. The index of the twist group $\mathrm{Tw}_\Delta$ in $R_\Delta$ is thus $abc/\gcd(a,b,c)$. On the other hand, as a consequence of the discussion in the divisor case, the index of the simple twist group $\mathrm{Tw}_\Delta^s$ in $R_\Delta$ is $ab/\gcd(a,b)\cdot$

$bc/\gcd(b,c)$. Since $\Delta$ has no level graphs automorphisms, i.e., since $\mathrm{Aut}(\Delta)$ is trivial, we conclude that in this case $D_\Delta$ is a quotient stack by a group of order

$$e_\Delta \;=\; \frac{\gcd(a,b,c)\ \mathrm{lcm}(a,b)\ \mathrm{lcm}(b,c)}{abc}\,. \tag{23}$$

## 4. Clutching and projection to generalized strata

In this section we define *generalized strata* where we allow disconnected surfaces and residues constrained to a residue space $\mathfrak{R}$. This is similar to a discussion in [Sau19]. More precisely, we show in Section 4.1 how the construction of [BCGGM3] carries over to this generalized context to give a compactification $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ of generalized strata.

The reason for dealing with generalized strata is to be able to work with objects (like line bundles and Chow rings) on the individual levels of a boundary stratum, and those might be disconnected and have with residue conditions imposed by the GRC. We construct in Section 4.2 for each boundary stratum $D_\Gamma$ a finite covering $D^s_\Gamma \to D_\Gamma$ that admits projections $p^{[i]}_\Gamma : D^s_\Gamma \to B^{[i]}_\Gamma$ where $B^{[i]}_\Gamma$ are the generalized strata at level $i$ of $D_\Gamma$.

4.1. **The compactification of generalized strata.** We start with the definition of strata in the generality that we need. First, we allow for disconnected surfaces. Throughout $\mu_i = (m_{i,1},\dots,m_{i,n_i}) \in \mathbb{Z}^{n_i}$ is the type of a differential, i.e., we require that $\sum_{j=1}^{n_i} m_{i,j} = 2g_i - 2$ for some $g_i \in \mathbb{Z}$ for $i = 1,\dots,k$. For a tuple $\mathbf{g} = (g_1,\dots,g_k)$ of genera and a tuple $\mathbf{n} = (n_1,\dots,n_k)$ together with $\boldsymbol{\mu} = (\mu_1,\dots,\mu_k)$ we define the disconnected stratum

$$\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu}) \;=\; \prod_{i=1}^{k} \Omega\mathcal{M}_{g_i,n_i}(\mu_i)\,. \tag{24}$$

The projectivized stratum $\mathbb{P}\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ is the quotient by the diagonal action of $\mathbb{C}^*$, not the quotient by the action of $(\mathbb{C}^*)^k$.

Next, we prepare for global residue conditions. Let $H_p \subseteq \cup_{i=1}^k \{(i,1),\cdots(i,n_i)\}$ be the set of marked points such that $m_{i,j} < -1$. Now consider vector spaces $\mathfrak{R}$ of the following special shape, modeled on the global residue condition from [BCGGM1]. Let $\lambda$ be a partition of $H_p$ with parts denoted by $\lambda^{(k)}$ and a subset $\lambda_{\mathfrak{R}}$ of the parts of $\lambda$ such that

$$\mathfrak{R} := \left\{ r = (r_{i,j})_{(i,j)\in H_p} \in \mathbb{C}^{H_p} \quad \text{and} \sum_{(i,j)\in\lambda^{(k)}} r_{i,j} = 0 \quad \text{for all} \quad \lambda^{(k)} \in \lambda_{\mathfrak{R}} \right\}.$$

The subspace of surfaces with residues in $\mathfrak{R}$ will be denoted by $\Omega\mathcal{M}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ and we will refer to them as *generalized strata*, too.

To compute e.g. dimensions it is convenient to define the *residue subspace*

$$R \;=\; \prod_{i=1}^{k} R_i \;\subseteq\; \prod_{i=1}^{k} \mathbb{C}^{l_i} \tag{25}$$

of differentials of the generalized stratum $\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$, where $l_i$ is the number of negative entries in $\mu_i$. Here $R_i$ is the vector subspace cut out by the residue theorem in the $i$-th component in the space generated by the vectors $r_{i,j}$ for each $(i,j)$ with $m_{i,j} \le -1$. When writing $\mathfrak{R} \cap R$ we consider the intersection inside $\prod_{i=1}^k \mathbb{C}^{l_i}$.

**Remark 4.1.** *The dimension of the generalized stratum $\Omega\mathcal{M}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ is*

$$\dim(\Omega\mathcal{M}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})) = \left(\sum_{i=1}^{k} 2g_i + n_i - 1\right) - (l - \dim(\mathfrak{R}\cap R))\,, \tag{26}$$

*where $l = \sum l_i$ is the total number of poles, i.e., marked points with $m_{i,j} < 0$.*

We claim that the construction in [BCGGM3] can be carried out for disconnected surfaces and for surfaces with an assigned residue subspace. We only have to replace in the definition of the twisted differentials $(X = (X_v)_{v\in V(G)}, \eta = (\eta_v)_{v\in V(G)})$ compatible with an enhanced level graph $\Gamma$ the global residue condition by the following condition. We construct a new *auxiliary level graph* $\widetilde{\Gamma}$ by adding a new vertex $v_{\lambda^{(k)}}$ to $\Gamma$ at level $\infty$ for each element $\lambda^{(k)} \in \lambda_{\mathfrak{R}}$ and converting a tuple $(i,j) \in \lambda^{(k)}$ into an edge from the marked point $(i,j)$ to the vertex $v_{\lambda^{(k)}}$.

- **$\mathfrak{R}$-global residue condition ($\mathfrak{R}$-GRC).** The tuple of residues at the poles in $H_p$ belongs to $\mathfrak{R}$ and for every level $L < \infty$ of $\widetilde{\Gamma}$ and every connected component $Y$ of the subgraph $\widetilde{\Gamma}_{>L}$ one of the following conditions holds.
  i) The component $Y$ contains a marked point with a prescribed pole that is *not* in $\lambda_{\mathfrak{R}}$.
  ii) The component $Y$ contains a marked point with a prescribed pole $(i,j) \in H_p$ and there is an $r \in \mathfrak{R}$ with $r_{(i,j)} \neq 0$.
  iii) Let $q_1, \dots, q_b$ denote the set of edges where $Y$ intersects $\widetilde{\Gamma}_{=L}$. Then
  $$\sum_{j=1}^{b} \operatorname{Res}_{q_j^-} \eta_{v^-(q_j)} \;=\; 0\,,$$
  where $v^-(q_j) \in \widetilde{\Gamma}_{=L}$.

This differs from the global residue condition in [BCGGM1] only in the subdivision of cases in i) and ii). As for the normal GRC (see [MUW17]),the $\mathfrak{R}$-GRC also has an algorithmic graph theoretic description, see [CMZ20].

**Proposition 4.2.** *There is a proper smooth Deligne-Mumford stack $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ containing $\mathbb{P}\Omega\mathcal{M}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ as an open dense substack with the following properties:*

(i) *The boundary $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu}) \smallsetminus \Omega\mathcal{M}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ is a normal crossing divisor.*

(ii) *A multi-scale differential defines a point in $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ if and only if it is compatible with an enhanced level graph $\Gamma$ that satisfies the $\mathfrak{R}$-GRC.*

(iii) *The codimension of a boundary stratum $D_\Gamma$ in $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ is equal to the number of horizontal edges plus the number $L$ of levels below zero.*

*Proof.* The residue spaces matters only for the existence of modification differentials needed for gluing. In [BCGGM1, Lemma 4.6] their existence for each component $Y$ as in the global residue condition was shown in case iii). This lemma also covers case i), since we can impose a residue at that marked point to ensure that the total sum equals zero. Since $\mathfrak{R}$ is a vector space, this can still be done in case ii).

The smoothness and the normal crossing divisor properties follow from the same reasoning as in [BCGGM3]. We leave the straightforward verification of those many hidden claims of the proposition to the reader. □

Again, as in the usual situation, also the divisors $D_\Gamma$ of generalized strata may be disconnected or empty.

**Example 4.3.** To illustrate the $\mathfrak{R}$-global residue condition we consider the generalized stratum $\overline{B} \;=\; \mathbb{P}\left(\Omega\mathcal{M}_{0,3}(-2,-2,2)\times\Omega\mathcal{M}_{0,4}(-2,-2,1,1)\right)^{\mathfrak{R}}$ where the special legs are given by the first two marked points of the first component and the first two marked points of the second components, i.e.,

$$H_s \;=\; \{(1,1),(2,1),(1,2),(2,2)\}$$

and the residue space is given by the the partition

$$\lambda_{\mathfrak{R}} \;=\; \{\{(1,1),(1,2)\},\{(2,1),(2,2)\}\}.$$

This means that

$$\mathfrak{R} \;=\; \{r_{(1,1)}+r_{(1,2)} \;=\; 0,\quad r_{(2,1)}+r_{(2,2)} \;=\; 0\}\subset\mathbb{C}^4,$$

and $R$ is the subspace defined by the residue theorem on each of the two components, namely

$$R \;=\; \{r_{(1,1)}+r_{(2,1)} \;=\; 0,\; r_{(1,2)}+r_{(2,2)} \;=\; 0\}.$$

By Remark 4.1, the above generalized stratum has dimension 1. We want to show that the $\mathfrak{R}$-GRC implies that there is only one 2-level boundary divisor in the compactification defined in Proposition 4.2. This divisor is given by the 2-level graph with the 4-marked component on level 0 and the other component on level $-1$.

The only two possible level graphs that could occur are the 2-level graph $\Gamma_1$ described above and the 2-level graph $\Gamma_2$ where the two components are inverted. Consider the auxiliary level graphs $\widetilde{\Gamma_1}$ and $\widetilde{\Gamma_2}$ needed in order to check the $\mathfrak{R}$-GRC given in Figure 2. It is easy to see that condition (iii) of the $\mathfrak{R}$-GRC implies that

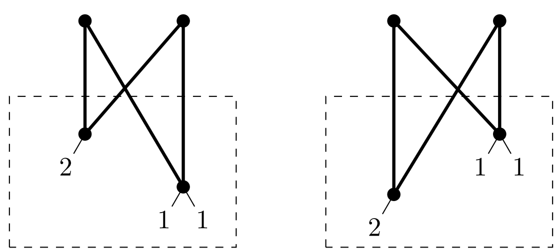



FIGURE 2. Auxiliary level graphs $\widetilde{\Gamma_1}$ (left) and $\widetilde{\Gamma_2}$ (right) for the boundary strata $\Gamma_1$ and $\Gamma_2$ (in the dashed boxes)

the graph $\Gamma_1$ is illegal since both residues on the genus 0 component with the single zero of order 2 on the top level are zero, and this cannot happen.

4.2. **Level projections and clutching.** Consider a boundary stratum $D_\Gamma$ given by an enhanced level graph $\Gamma$. It parameterizes multi-scale differentials, a differential on each level together with a prong-matching. However, there are no well-defined projection morphisms to the generalized strata on each level. E.g. $D_\Gamma$ might have generically trivial quotient stack structure and the generalized strata on its levels might have everywhere trivial stack structure, and yet special points of $D_\Gamma$ have non-trivial quotient structure. A graph $\Gamma$ with two edges and two levels degenerating to a triangle (Figure 1, left) provides an example. This is due to the fact that the equivalence relation in the notion of multi-scale differentials involves the twist group, which in the presence of edges across multiple levels intertwines what happens at the levels. Our goal here is to define a cover of $D_\Gamma$ that has such projection maps.

To define the generalized strata at the levels of $D_\Gamma$ we let $(\mathbf{g}^{[i]}, \mathbf{n}^{[i]}, \boldsymbol{\mu}^{[i]})$ for $i = 0, \ldots, -L$ be the discrete parameters genus, number of points and type at level $i$ and let $\mathfrak{R}^{[i]}$ be the residue condition imposed at level $i$. These residue conditions are constructed via the $\mathfrak{R}$-GRC described before. Our goal is:

**Proposition 4.4.** *There exists a stack $D_\Gamma^s$, called the* simple boundary stratum of type $\Gamma$ *that admits a finite map $c_\Gamma : D_\Gamma^s \to D_\Gamma$ and finite forgetful maps*

$$p_\Gamma^{[i]} : D_\Gamma^s \to B_\Gamma^{[i]} := \mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}^{[i]}}_{\mathbf{g}^{[i]},\mathbf{n}^{[i]}}(\boldsymbol{\mu}^{[i]}) \tag{27}$$

*for each $i = 0, \ldots, -L$.*

We denote by $p_\Gamma = \prod_{i=0}^{-L} p_\Gamma^{[i]}$ the product of all level projections. In the case that $D_\Gamma$ is a divisor we will also denote the two projections by

$$p_\Gamma^\top : D_\Gamma^s \to B_\Gamma^\top \;=\; \mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}^\top}_{\mathbf{g}^\top,\mathbf{n}^\top}(\boldsymbol{\mu}^\top) \quad \text{and} \quad p_\Gamma^\perp : D_\Gamma^s \to B_\Gamma^\perp \;=\; \mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}^\perp}_{\mathbf{g}^\perp,\mathbf{n}^\perp}(\boldsymbol{\mu}^\perp)\,.$$

With the help of the finite coverings $c_\Gamma$ and the inclusion of the boundary strata $\mathfrak{i}_\Gamma : D_\Gamma \to \overline{B}$, we have now the *clutching maps* maps $\zeta_\Gamma = \mathfrak{i}_\Gamma \circ c_\Gamma$ at our disposal in order to define the generators of what we will be defined as tautological ring, appearing in Theorem 1.5.

The strategy of proof of the proposition is to construct $D_\Gamma^s$ as a cover dominating the local covers of neighborhoods of more degenerate boundary strata, following the strategy already used in [Mum83]. We do not attempt to analyze whether $D_\Gamma^s$ is smooth, but the covering we construct is branched at worst over the boundary divisors (hence locally over the coordinate axis, since the boundary is normal crossing), so $D_\Gamma^s$ has at worst Cohen-Macaulay singularities (Proposition 2.2 in loc. cit.), which allows us to perform intersection theory as in loc. cit. The objects of the construction are summarized in the following diagram that we now explain.

$$\begin{tikzcd}
 & \widetilde{U}_\Delta \arrow[r, "\subset"] \arrow[dr, "q_\Delta"] & D_\Gamma^s \arrow[ddll, "p_\Gamma"'] \arrow[ddrr] \arrow[dddrrr, bend left, "c_\Gamma"] & & & \\
 & & U_\Delta^s \arrow[dl, "p_\Gamma^\Delta"] \arrow[dr] \arrow[ddrr, bend right, "c_\Gamma^\Delta"'] & & & \\
B_\Gamma = \prod_i B_\Gamma^{[i]} & B_{\Gamma,\Delta} \arrow[l, hook', "\supset"] & & U_\Delta^{\mathrm{int}} \arrow[r, "\subset"] \arrow[dr] & D_\Gamma^{\mathrm{gs}} \arrow[dr] & \\
 & & & & U_\Delta \arrow[r, "\subset"] & D_\Gamma
\end{tikzcd}$$

For any level graph $\Delta$ which is a degeneration of $\Gamma$ we let $U_\Delta \subset D_\Gamma$ be the open subset parametrizing multi-scale differentials compatible with an undegeneration of $\Delta$. In particular $U_\Gamma \subset D_\Gamma$ is the complement of all boundary strata where $\Gamma$ degenerates further. In symbols (with notation as in Section 3.2)

$$U_\Delta \;=\; \Big( \coprod_{\Gamma \rightsquigarrow \Pi \rightsquigarrow \Delta} \mathfrak{W}_{\mathrm{pm}}(\Pi)/T_\Delta \Big)/\mathbb{C}^*\,, \tag{28}$$

with the complex structure and modular interpretation as in [BCGGM3, Section 12 and Section 7], forgetting the Teichmüller marking there and intersecting with $D_\Gamma$.

Next, we define $U^s_\Delta$. It will be the moduli stack of *simple multi-scale differential compatible with an undegeneration of* $\Delta$ that we define now. (This notion with additional Teichmüller marking is essentially also given in [BCGGM3, Section 8].) Recall that we defined multi-scale differentials compatible with $\Delta$ as the quotient by the action of the level rotation torus $T_\Delta = \mathbb{C}^{L(\Delta)}/\mathrm{Tw}_\Delta$. Simple multi-scale differentials compatible (precisely) with $\Delta$ are defined as the quotient by the simple level rotation torus $T^s_\Delta = \mathbb{C}^{L(\Delta)}/\mathrm{Tw}^s_\Delta$. (To define the notion in families, one should use the simple rescaling ensemble instead of the rescaling ensemble, see [BCGGM3, Section 7].) Multi-scale differentials compatible with an undegeneration $\Pi$ of $\Delta$, i.e. for level graphs with $\Gamma\rightsquigarrow\Pi\rightsquigarrow\Delta$, are also included in the stack we are about to define. We have to specify the correct notion of equivalence $\mathfrak{W}_{\mathrm{pm}}(\Pi)$ so that these objects fit together in families. Similar to the definition of the simple Dehn space in [BCGGM3], this works if we declare two differentials in $\mathfrak{W}_{\mathrm{pm}}(\Pi)$ to be declared equivalent if they differ by the action of the image of $T^s_\Delta \to T_\Delta \to T_\Pi$. To be able to define projection maps, we moreover mark all the half-edges of $\Gamma$ (i.e. the edges of $\Gamma$, keeping the labels at the marked points) in our notion of simple multi-scale differentials. The moduli stack of simple multi-scale differential compatible with an undegeneration of $\Delta$ is denoted by $U^s_\Delta$ and comes with a covering map $c^\Delta_\Gamma : U^s_\Delta \to U_\Delta$. In symbols

$$U^s_\Delta \;=\; \Big(\coprod_{\Gamma\rightsquigarrow\Pi\rightsquigarrow\Delta} \mathfrak{W}^*_{\mathrm{pm}}(\Pi)/T^s_\Delta\Big)/\mathbb{C}^*\,, \tag{29}$$

with topology and complex structure again as in [BCGGM3], and where the star should remind of the extra edge marking.

Since the edges of $\Gamma$ are labelled for points in $U^s_\Delta$ and since the equivalence relation is defined by $T^s_\Delta$, hence level by level, we may decompose the simple multi-scale differentials parameterized by $U^s_\Delta$ according to the levels of $\Gamma$. In this we we obtain maps $p^{\Delta,[i]}_\Gamma : U^s_\Delta \to B^{[i]}_\Gamma$ such that the product map $p^\Delta_\Gamma = \prod_i p^{\Delta,[i]}_\Gamma$ is a finite cover of an open subset $B_{\Gamma,\Delta}$ of the product of level strata $B_\Gamma = \prod_i B^{[i]}_\Gamma$.

The last step is to define a covering dominating all the $c^\Delta_\Gamma$. For technical reasons we first define the 'generically simple' intermediate space $D^{\mathrm{gs}}_\Gamma$, that removes the stack structure over the open subset $U_\Gamma$ (if there is). The space $D^{\mathrm{ps}}_\Gamma$ contains $U^s_\Gamma$ as open dense subset. The covering $D^{\mathrm{gs}}_\Gamma \to D_\Gamma$ is defined by marking all the edges of $\Gamma$ and by using the covering of level rotation tori $T^s_\Gamma \to T_\Gamma$ over $U_\Gamma$ as well as over the boundary strata $D_\Gamma \setminus U_\Gamma$. This is to say that for a degeneration $\Gamma\rightsquigarrow\Delta$ the points in the intermediate space $U^{\mathrm{int}}_\Delta$ are multi-scale differentials up to the equivalence relation given by the hybrid torus $T_\Delta \times_{T_\Gamma} T^s_\Gamma$. Finally, we take $D^s_\Gamma$ to be the normalization of $D^{\mathrm{gs}}_\Gamma$ in a Galois field extension of the function field of $D^{\mathrm{gs}}_\Gamma$ that contains all the extensions defined by $U^s_\Delta \to D^{\mathrm{gs}}_\Gamma$. (If $D_\Gamma$ happens to be reducible, we perform the construction on each connected component. Actually, the $U^s_\Delta$ still have a stack structure due to automorphism of the underlying stable curves. The details how to construct the covering with this caveat are in [Mum83, Section 2b].) This space comes with a forgetful map $c_\Gamma : D^s_\Gamma \to D_\Gamma$ that factors as $c_\Gamma = c^\Delta_\Gamma \circ q_\Delta : \widetilde{U}_\Delta \to U_\Delta$ over the preimages of $U_\Delta$. We may now define $p^{[i]}_\Gamma = p^{\Delta,[i]}_\Gamma \circ q_\Delta$, since the $\widetilde{U}_\Delta$ for all degenerations $\Gamma\rightsquigarrow\Delta$ cover $D^s_\Gamma$. This completes the *proof of Proposition 4.4.*

4.3. **Push-pull comparison.** Let $\Gamma \in \mathrm{LG}_L(\overline{B})$ be a level graph. Several recursive computations in the sequel are performed on the level strata $B_\Gamma^{[i]}$ and we want to transfer the result via $p^{[i]}$-pullback and $c_\Gamma$-pushforward to $D_\Gamma$. This section provides the basic relations in this push-pull procedure. The degree of $c_\Gamma$ seems difficult to compute. In applications we only the following relative statement.

**Lemma 4.5.** *The ratios of the degrees of the projections in Proposition 4.4 is*

$$\frac{\deg(p_\Gamma)}{\deg(c_\Gamma)} \;=\; \frac{K_\Gamma}{|\operatorname{Aut}(\Gamma)|\,\ell_\Gamma}\,. \tag{30}$$

*Proof.* The degrees can be computed at the generic point, where both maps factor through $q_\Gamma$. The degree of $p_\Gamma^\Gamma$ is the number of equivalence classes of prong-matchings, which is $K_\Gamma/[R_\Gamma : \mathrm{Tw}_\Gamma]$. The degree of $c_\Gamma^\Gamma$ is the index $[\mathrm{Tw}_\Gamma : \mathrm{Tw}_\Gamma^s] \cdot |\operatorname{Aut}(\Gamma)|$. The claimed equality

$$\frac{\deg(p_\Gamma)}{\deg(c_\Gamma)} \;=\; \frac{\deg(p_\Gamma^\Gamma)}{\deg(c_\Gamma^\Gamma)} \;=\; \frac{1}{|\operatorname{Aut}(\Gamma)}\frac{K_\Gamma}{[R_\Gamma : \mathrm{Tw}_\Gamma^s]} \;=\; \frac{K_\Gamma}{|\operatorname{Aut}(\Gamma)|\,\ell_\Gamma} \tag{31}$$

follows from the definition of the simple twist group. □

Next we compare codimension 1 boundary classes on the strata $D_\Gamma \in \mathrm{LG}_L(B)$ and on their level strata $B_\Gamma^{[i]}$ in order to pull back tautological relations. We use the symbol $[D_\Gamma]$ to denote the fundamental class of the substack of $\overline{B}$ parameterizing multi-scale differentials compatible with a degeneration of $\Gamma$. Let $i \in \mathbb{Z}_{\le 0}$.

Consider a graph $\Delta \in \mathrm{LG}_1(B_\Gamma^{[i]})$ defining a divisor in $B_\Gamma^{[i]}$. We aim to compute its pullback to $D_\Gamma^s$ and the push forward to $D_\Gamma$ and to $\overline{B}$. Recall that in $D_\Gamma^s$ the edges of $\Gamma$ have been labeled once and for all (we write $\Gamma^\dagger$ for this labeled graph) and that the level strata $B_\Gamma^{[i]}$ inherit these labels. Consequently, there is unique graph $\widehat{\Delta}^\dagger$ which is a degeneration of $\Gamma^\dagger$ and such that extracting the levels $i$ and $i-1$ of $\widehat{\Delta}^\dagger$ equals $\Delta$. The resulting unlabeled graph will simply be denoted by $\widehat{\Delta}$. (Recall from Remark 3.2 that $\delta^{\complement}_{(-i+1)}(\widehat{\Delta}) = \Gamma$.) On the other hand, the procedure of gluing in and forgetting labels is not injective. For a fixed labeled graph $\Gamma^\dagger$ we denote by $J(\Gamma^\dagger, \widehat{\Delta})$ the set of $\Delta \in \mathrm{LG}_1(B_\Gamma^{[i]})$ such that $\widehat{\Delta}$ is the result of that procedure. Obviously the graphs in $J(\Gamma^\dagger, \widehat{\Delta})$ differ only by the labeling of their half-edges.

**Lemma 4.6.** *The cardinality of $J(\Gamma^\dagger, \widehat{\Delta})$ is determined by*

$$|J(\Gamma^\dagger, \widehat{\Delta})| \cdot |\operatorname{Aut}(\widehat{\Delta})| \;=\; |\operatorname{Aut}(\Delta)| \cdot |\operatorname{Aut}(\Gamma)|\,.$$

*Proof.* Consider the map $\varphi : \operatorname{Aut}(\widehat{\Delta}) \to \operatorname{Aut}(\Gamma)$ induced by the undegeneration $\delta^{\complement}_{(-i+1)}$ of the $(-i+1)$-th level passage of $\widehat{\Delta}$. For an element in the kernel, the graph $\Gamma$ is fixed, so we may as well label it. Thanks to these labels, extraction of the levels $i$ and $i-1$ now defines a graph $\Delta \in \mathrm{LG}_1(B_\Gamma^{[i]})$ and the restriction map $\operatorname{Ker}(\varphi) \to \operatorname{Aut}(\Delta)$ is an isomorphism. To determine the cokernel of $\varphi$ we use the labels given by $\Gamma^\dagger$ and a degeneration $\widehat{\Delta}^\dagger$ labeled except for the edges interior to that pair of levels. After restriction to the levels $i$ and $i-1$ the elements in the image of $\varphi$ act trivially. The resulting bijection of $\operatorname{Coker}(\varphi)$ and $J(\Gamma^\dagger, \widehat{\Delta})$ proves the result. □

We now determine the multiplicities of the push-pull procedure. Recall from (22) the definition of $\ell_{\Gamma,j}$, for $j \in \mathbb{Z}_{\ge 1}$.

**Proposition 4.7.** *For a fixed* $\Delta \in \mathrm{LG}_1(B^{[i]}_\Gamma)$*, the divisor classes of* $D_{\widehat{\Delta}}$ *and the clutching of* $D_\Delta$ *are related by*

$$\frac{|\operatorname{Aut}(\widehat{\Delta})|}{|\operatorname{Aut}(\Delta)||\operatorname{Aut}(\Gamma)|}\cdot c_\Gamma^*[D_{\widehat{\Delta}}] \;=\; \frac{\ell_\Delta}{\ell_{\widehat{\Delta},-i+1}}\cdot p_\Gamma^{[i],*}[D_\Delta]\,. \tag{32}$$

*in* $\mathrm{CH}^1(D^s_\Gamma)$ *and consequently by*

$$\frac{|\operatorname{Aut}(\widehat{\Delta})|}{|\operatorname{Aut}(\Gamma)|}\cdot \ell_{\widehat{\Delta},-i+1}\cdot[D_{\widehat{\Delta}}] \;=\; \frac{|\operatorname{Aut}(\Delta)|}{\deg(c_\Gamma)}\cdot \ell_\Delta\cdot c_{\Gamma,*}\big(p_\Gamma^{[i],*}[D_\Delta]\big) \tag{33}$$

*in* $\mathrm{CH}^1(D_\Gamma)$.

*Proof.* If suffices to show the first equation, the second follows by taking $c_{\Gamma,*}$. Since the two sides are supported on the same set, it suffices to verify the multiplicities. Since near the divisors under consideration both sides are pullback via $q_{\widehat{\Delta}}$ this can be done by computing the ramification orders of the finite maps $c_\Gamma^{\widehat{\Delta}}$ and $p_\Gamma^{\widehat{\Delta}}$ over the divisor $D_{\widehat{\Delta}}$ and over $\tilde{D}_\Delta = D_\Delta \times \prod_{j\neq i} B_\Gamma^{[j]}$ respectively.

We start with $c_\Gamma^{\widehat{\Delta}}$. There, passing to the equivalence relation by the torus $T^s_\Gamma$ gives a covering of degree $[\mathrm{Tw}_\Gamma : \mathrm{Tw}^s_\Gamma]$, both at a generic point and over $D_{\widehat{\Delta}}$. Adding the markings on the edges of $\Gamma$ gives $|\operatorname{Aut}(\Gamma)|$ additional choices at a generic point. Over $D_{\widehat{\Delta}}$ only the automorphism in image of the map $\varphi$ (as in the proof of Lemma 4.6) can be rigidified by adding the marking. This image has cardinality $|\operatorname{Aut}(\widehat{\Delta})|/|\operatorname{Aut}(\Gamma)|$ and thus the ramification order is the reciprocal of the factor on the left hand side of (32).

Next we consider the map $p_\Gamma^{\widehat{\Delta}}$. Since in $\prod_j B_\Gamma^{[j]}$ and thus also on $\tilde{D}_\Delta$ the half-edges that form the edges of $\Gamma$ are labeled, graph automorphism do not contribute to branching. However, after adding the prong matching for $\Gamma$, the orbits of the $-i+1$-st component of the integer subgroup $\mathbb{Z}^{L+1}\subset\mathbb{C}^{L+1}$ of the level rotation torus change. In $\tilde{D}_\Delta$ (and in $D_\Delta$) the orbit has size $\ell_\Delta$, while in $D^s_\Gamma$ the orbit has size $\ell_{\widehat{\Delta},-i+1}$ since the prongs of edges of $\widehat{\Delta}$ are acted on, too. Since this component of the level rotation torus is not present at a generic point and since all other components have the same effect at a generic point and over $\tilde{D}_\Delta$, we conclude that the ramification order is the reciprocal of the factor on the right hand side of (32). □

Next we compare various versions of the $\xi$-class on boundary strata. A first definition is by a local description. Consider a level $i\in\{0,\dots,-L\}$ of a boundary stratum $D_\Gamma$ and recall that it is a moduli space of multiscale differentials compatible with a degeneration of $\Gamma$. We define the line bundle $\mathcal{O}_\Gamma^{[i]}(-1)$ on $D_\Gamma$ as follows. On open sets where $\Gamma$ does not degenerate further, it is generated by the $i$-th component $\eta_{(i)}$ of the multi-scale differential. If $\Gamma$ degenerates to $\Gamma_1$ the level $i$ splits up into an interval $i$ to $i-k$ of levels, then the local generator of $\mathcal{O}_\Gamma^{[i]}(-1)$ is the multi-scale components $\eta_{(i)}$ for the top of these levels. We let $\xi_\Gamma^{[i]} = c_1(\mathcal{O}_\Gamma^{[i]}(-1))$ and write $\xi_\Gamma^\top$ for the top level contribution.

**Remark 4.8.** Since stable differentials on a boundary stratum are zero on all levels apart from the top one, we have $\xi_\Gamma^\top = \xi|_{D_\Gamma}$.

**Proposition 4.9.** *The first Chern classes of the tautological bundles on the levels of a boundary divisor are related by*

$$c_\Gamma^* \, \xi_\Gamma^{[i]} \;=\; p_\Gamma^{[i],*} \xi_{B_\Gamma^{[i]}} \qquad \text{in} \quad \mathrm{CH}^1(D_\Gamma^s)\,. \tag{34}$$

*Proof.* Comparing local generators, we obtain a collection of isomorphisms

$$c_\Gamma^{\Delta,*} \mathcal{O}_{B_\Gamma^{[i]}(-1)} \;\cong\; (p_\Gamma^{\Delta,[i]})^* \, \mathcal{O}_\Gamma^{[i]}(-1)$$

compatible with restrictions to undegenerations. The $q_\Delta$-pullbacks of this collection of maps gives the isomorphsm on $D_\Gamma^s$, and then we take the first Chern class. □

We will continue the study of the tautological ring in Sections 7 and 8, using local descriptions near the boundary introduced along with Section 6.

## 5. The structure of the boundary

In this section we show that the non-horizontal boundary divisors $D_\Gamma$ are smooth. More generally we show that if a collection of non-horizontal divisors intersects, then there is a unique order on this collection such that $i$-th divisors appear as the $i$-th 2-level undegeneration of an intersection point.

In the sequel it will be convenient to assume that the 2-level graphs have been numbered once and for all, say as $\mathrm{LG}_1(\mathrm{B}) = \{\Gamma_1, \dots, \Gamma_M\}$. Note that that the intersection of two divisors, say $D_{\Gamma_1}$ and $D_{\Gamma_2}$, consists a priori of the sublocus $D_{12}$ of unions of $D_\Lambda$, for $\Lambda \in \mathrm{LG}_2(B)$ with $\delta_1(\Lambda) = \Gamma_1$ and $\delta_2(\Lambda) = \Gamma_2$, and the sublocus $D_{21}$, which is the union of $D_\Lambda$ for $\Lambda \in \mathrm{LG}_2(B)$ with $\delta_1(\Lambda) = \Gamma_2$ and $\delta_2(\Lambda) = \Gamma_1$. The notation generalizes to any number of levels. We define the suborbifold

$$D_{i_1,\dots,i_L} \;\subseteq\; \bigcap_{j=1}^{L} D_{i_j} \tag{35}$$

consisting of all $D_\Lambda$, with $\Lambda \in \mathrm{LG}_L(B)$ such that $\delta_j(\Lambda) = \Gamma_{i_j}$ for all $j = 1, \dots, L$ and we refer to this by the ordered set $[i_1, \dots, i_L]$, called the *profile* of the boundary stratum. We denote by $\mathscr{P} = \mathscr{P}(B)$ the set profiles of $B$ and by $\mathscr{P}_L$ those of length $L$. The language of profiles is used mainly in this section and then again in Theorem 9.10, while elsewhere we usually work with set of level graphs. The sage package `diffstrata` makes fully use of the notion of profiles and the following proposition.

**Proposition 5.1.** *If $\cap_{j=1}^{L} D_{\Gamma_{i_j}}$ is not empty, there is a unique ordering $\sigma \in \mathrm{Sym}_L$ on the set $I = \{i_1, \dots, i_L\}$ of indices such that*

$$D_{\sigma(I)} \;=\; \bigcap_{j=1}^{L} D_{\Gamma_{i_j}}\,.$$

*Moreover if $i_k = i_{k'}$ for a pair of indices $k \neq k'$, then $D_{i_1,\dots,i_L} = \emptyset$.*

**Remark 5.2.** In general the intersection of boundary divisors $D_{\sigma(I)}$ is not irreducible, i.e., it consists of boundary strata associated to different enhanced level graphs, see for example the 3-level graphs in Figure 1.

The preceding proposition also gives a useful relation. Suppose two divisors $D_{\Gamma_1}$ and $D_{\Gamma_2}$ meet in a boundary stratum $D_\Delta$. Two situations may occur. Either $\delta_1(\Delta) = \Gamma_1$ and $\delta_2(\Delta) = \Gamma_2$ or vice versa. In the first situation, $\Delta$ arises from degenerating the lower level of $\Gamma_1$. We phrase this by saying that $\Gamma_2$ *goes under* $\Gamma_1$ and write $\Gamma_2 \prec \Gamma_1$. A priori, this notion might depend on the enhanced level graph $\Delta$. But the preceding proposition implies that it does in fact not depend on $\Delta$.

The proof of Proposition 5.1 uses dimension estimates and the following lemma. We define

$$d_\Lambda^{[p]} \;=\; \dim(B_\Lambda^{[p]}) \text{ for all } \Lambda \in \mathrm{LG}_L(B)\,,$$

where $B_\Lambda^{[p]}$ is the projectivized substratum at level $p \in \{0, \dots, -L\}$ of $D_\Lambda$ defined in Proposition 4.4. Note that the sum $\sum_{p=0}^{-L}(d_\Lambda^{[p]} + 1) = N = 1 + \dim(\overline{B})$ is the unprojectivized dimension of the stratum.

**Lemma 5.3.** *The dimensions of the levels of a boundary stratum $D_\Lambda$ and the boundary divisor $D_{\delta_k(\Lambda)}$ given by its $k$-th undegeneration are related by*

$$d_{\delta_k(\Lambda)}^{[0]} \;=\; k - 1 + \sum_{p=0}^{k-1} d_\Lambda^{[-p]}, \quad d_{\delta_k(\Lambda)}^{[-1]} \;=\; L - 1 - k + \sum_{p=k}^{L-1} d_\Lambda^{[-p]}\,.$$

*Proof.* The follows directly from the description of undegeneration, see [BCGGM3]. ∎

*Proof of Proposition 5.1.* Assume that, after reordering, $\cap_{i=1}^L D_i$ is not empty, and that $D_\Lambda$ is a component of $D_{1,\dots,L}$. Assume furthermore that there is a permutation $\sigma \in S_j$, such that also $D_{\sigma(1),\dots,\sigma(L)}$ is non-empty, containing a component $D_{\Lambda^\sigma}$. Now, by definition

$$\delta_1(\Lambda^\sigma) \;=\; \delta_{\sigma(1)}(\Lambda) \;=\; D_{\Gamma_{\sigma(1)}}\,.$$

By Lemma 5.3 we can then write the dimension of the top component of $D_{\Gamma_1}$ and $D_{\Gamma_{\sigma(1)}}$ in two different ways, namely

$$\begin{aligned} d_{\Gamma_1}^{[0]} \;&=\; d_\Lambda^{[0]} \;=\; \sigma^{-1}(1) - 1 + \sum_{p=0}^{\sigma^{-1}(1)-1} d_{\Lambda^\sigma}^{[-p]} \\ d_{\Gamma_{\sigma(1)}}^{[0]} \;&=\; d_{\Lambda^\sigma}^{[0]} \;=\; \sigma(1) - 1 + \sum_{p=0}^{\sigma(1)-1} d_\Lambda^{[-p]}. \end{aligned}$$

By substituting the first expression into the second one we obtain

$$d_{\Lambda^\sigma}^{[0]} = \sigma(1) - 1 + \sigma^{-1}(1) - 1 + \sum_{p=0}^{\sigma^{-1}(1)-1} d_{\Lambda^\sigma}^{[-p]} + \sum_{p=1}^{\sigma(1)-1} d_\Lambda^{[-p]}$$

which simplifies to

$$0 = \sigma(1) - 1 + \sigma^{-1}(1) - 1 + \sum_{p=1}^{\sigma^{-1}(1)-1} d_{\Lambda^\sigma}^{[-p]} + \sum_{p=1}^{\sigma(1)-1} d_\Lambda^{[-p]}.$$

This implies that $\sigma(1) = 1$. By induction we get that $\sigma = \mathrm{id}$.

In order to prove the second statement assume by contradiction that the orbifold $D_{i_1,\dots,i_L}$ is non-empty, with $i_1 = i_k$ for $1 < k \le L$. Let $D_\Lambda$ be a component of

$D_{i_1,\dots,i_L}$. Then by Lemma 5.3 we get

$$d^{[0]}_{\delta_1(\Lambda)} \;=\; d^{[0]}_{\Lambda} = k-1+\sum_{p=0}^{k-1} d^{[-p]}_{\Lambda}.$$

This implies that $k=1$, which is already a contradiction. □

## 6. Euler sequence for strata of abelian differentials

The characteristic classes of the tangent bundle to projective space $\mathbb{P}(V)$ of a vector space $V$ are conveniently computed using the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}(V)} \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(-1)^{\oplus \dim(V)} \xrightarrow{\text{ev}} \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0\,. \tag{36}$$

Our main computational tool uses the affine structure of strata to provide a similar Euler sequence on the compactified strata $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$.

**Theorem 6.1.** *There is a vector bundle $\mathcal{K}$ on $\overline{B}$ that fits into an exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow (\overline{\mathcal{H}}^1_{rel})^\vee \otimes \mathcal{O}_{\overline{B}}(-1) \xrightarrow{\text{ev}} \mathcal{O}_{\overline{B}} \longrightarrow 0\,, \tag{37}$$

*where $\overline{\mathcal{H}}^1_{rel}$ is the Deligne extension of the relative cohomology, such that the restriction of $\mathcal{K}$ to the interior $B$ is the cotangent bundle $\Omega^1_B$.*

An explicit description of local generators of $\mathcal{K}$ is part of the proof in this section. We will have set up the tools to describe $\mathcal{K}$ intrinsically in Theorem 9.2.

We will define the evaluation map ev in the course of the construction. The construction happens first over the open part and then the finite covering charts that exhibit $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ locally as quotient stack.

6.1. **Over the open stratum.** Recall that moduli space of Abelian differential have an affine structure given by period coordinates. Concretely, for a pointed flat surface $(X,\omega,\mathbf{z})$ we denote by $Z=\{z_1,\dots,z_{r+s}\}$ the zeros and by $P=\{z_{r+s+1},\dots,z_n\}$ the poles among the marked points, thus including marked ordinary points in $Z$. By [HM79] or [Vee86] (see also [BCGGM2]) integration of the one-form along relative periods is a local biholomorphism and thus provides local charts of $\Omega\mathcal{M}_{g,n}(\mu)$ in the vector space

$$V \;=\; V_{(X,\omega,\mathbf{z})} \coloneqq \mathrm{H}^1(X\setminus P, Z;\mathbb{C})\,.$$

The changes of charts are linear, in fact with $\mathbb{Z}$-coefficients. This makes the projectivization $B$ into a $(\mathrm{PGL}_N,\mathbb{P}^{N-1})$-manifold.

We denote by $\mathcal{H}^1_{\mathrm{rel}}$ the local system on $B$ with fiber the relative cohomology $V=\mathrm{H}^1(X\setminus P,Z;\mathbb{C})$ and recall that $N=\dim(V)=\dim(B)+1$. Recall that the fiber of $\mathcal{O}_B(-1)$ at the point $(X,\omega,\mathbf{z})$ is the vector space generated by $\omega$. We thus obtain the evaluation map

$$\text{ev}\colon (\mathcal{H}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_B(-1) \to \mathcal{O}_B, \quad \gamma\otimes\omega \mapsto \int_\gamma \omega$$

by integrating the one-form.

**Proposition 6.2.** *There is a short exact sequence of vector bundles on $B$*

$$0 \longrightarrow \Omega^1_B \longrightarrow (\mathcal{H}^1_{rel})^\vee \otimes \mathcal{O}_B(-1) \xrightarrow{\text{ev}} \mathcal{O}_B \longrightarrow 0$$

*that locally on a chart $\mathbb{P}V$ is given by the standard Euler sequence.*

*Proof.* Let $\pi : \tilde{B} \to B$ be the universal cover of $B$. Consider the developing map $\mathrm{dev}\colon \tilde{B} \longrightarrow \mathbb{P}(V)$, which is a $\pi_1(B)$-equivariant local isomorphism. We use the sequence on the standard charts of $\mathbb{P}(V)$ and we claim that its dev-pullback descends to an exact sequence $B$.

To justify this, consider paths $\{\alpha_i\}_{i=1}^N$ that form a local frame of $(\mathcal{H}^1_{\mathrm{rel}})^\vee$. Let $\{a_i\}_{i=1}^N$ be the corresponding local coordinates and $\{\mathrm{d}a_i\}$ the local frame of $\Omega^1_{\mathbb{P}(V)}$. On the open subset $U_k = \{a_k \neq 0\} \subseteq \mathbb{P}(V)$ the monomorphism of the Euler sequence (36) is given by

$$\mathrm{d}a_i \mapsto \Big(\alpha_i - \frac{a_i}{a_k}\alpha_k\Big) \otimes \omega, \qquad i = 1, \dots, \hat{k}, \dots, N\,, \tag{38}$$

where $\omega$ is the representative of the line bundle with $\int_{\alpha_k} \omega = 1$. The pull-back sequence gives rise to an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{dev}^*\Big(\Omega^1_{\mathbb{P}(V)}\Big) \arrow[r] \arrow[d, "\cong"] & \mathrm{dev}^*(V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)) \arrow[r, "\mathrm{ev}"] \arrow[d, "\cong"] & \mathrm{dev}^*(\mathcal{O}_{\mathbb{P}(V)}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi^*(\Omega^1_B) \arrow[r] & \pi^*(\mathcal{H}^1_{\mathrm{rel}})^\vee \otimes \pi^*(\mathcal{O}_B(-1)) \arrow[r, "\mathrm{ev}"] & \pi^*(\mathcal{O}_B) \arrow[r] & 0
\end{tikzcd}$$

Each vector bundle appearing is provided with a canonical $\pi_1(B)$ action and the vertical maps are isomorphisms of $\pi_1(B)$-vector bundles. The first vertical map is an isomorphism since the developing map is a local isomorphism and $\pi^*(\Omega^i_B) \cong \Omega^i_{\tilde{B}}$ for every $i$. Since the evaluation map is $\pi_1(B)$-equivariant, so is the kernel. Hence the short exact sequence passes to the quotient by the action of $\pi_1(B)$ and yields the claim. $\square$

6.2. **Coordinates near the boundary.** Coordinates near the boundary of the moduli space $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are *perturbed period coordinates* ([BCGGM3, Section 11] or [CMZ19, Section 3]) that we now illustrate in typical cases that exhibit all the relevant features. The reader is encouraged to read this subsection in parallel with the subsequent one, where the Euler sequence is extended step by step to these boundary strata.

**Case 1: only horizontal nodes.** Suppose that the level graph $\Gamma$ consists of $k \geq 1$ horizontal edges only, all of them must necessarily be non-separating. At a smooth point near $D_\Gamma$ the relative homology can be grouped into

- the vanishing cycles $\alpha_i$ for $i = 1, \dots, k$ around the nodes,
- loops $\beta_i$ symplectically dual to $\alpha_i$, and
- paths $\gamma_1, \dots, \gamma_{N-2k}$ completing the above to a basis of relative homology.

Coordinates in a chart of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ near $D_\Gamma$ are given by the periods $c_i = \int_{\gamma_i} \omega$, by $a_i = \int_{\alpha_i} \omega$ and by the exponentiated period ratio $q_i = \exp(2\pi i b_i / a_i)$ where $b_i = \int_{\beta_i} \omega$. To provide charts of the projectivization $\overline{B}$ we fix $a_1$ to be identically one.

**Case 2: two levels, only vertical nodes.** For concreteness, we suppose that in the 2-level graph $\Gamma \in \mathrm{LG}_1(B)$ there is only one vertex on each level and for concreteness, say, with three edges $e_1, e_2, e_3$ joining the two vertices. Suppose moreover that there is no marked zero on lower level. (If there is such a marked point on each level, the loops $\beta_i$ below have to be replaced by relative periods across the level, leading to similar constructions.) At a point close to $D_\Gamma$ the relative homology can be grouped into

- loops $\beta_1$ through $e_1$ and $e_3$ and $\beta_2$ through $e_2$ and $e_3$,
- loops $\alpha_1$ and $\alpha_2$, the vanishing cycles corresponding to $e_1$ and $e_2$,
- paths $\gamma_1^{[0]}, \dots, \gamma_{d_0}^{[0]}$ forming a basis of the relative homology on top level,
- loops $\gamma_1^{[-1]}, \dots, \gamma_{d_1}^{[-1]}$ forming a basis of the homology on bottom level,

for some $d_0, d_1 \in \mathbb{Z}$, see also Figure 3.

On the other hand, the surfaces on the boundary stratum $D_\Gamma$ have a basis of relative homology that can be grouped into

- relative periods $\widetilde{\beta}_i$ joining the marked points at the upper ends of the edge $e_i$ to the upper end of the edge $e_3$ for $i = 1, 2$,
- loops $\widetilde{\alpha}_i$ around the poles at lower ends of $e_i$ for $i = 1, 2$,
- paths $\widetilde{\gamma}_1^{[0]}, \dots, \widetilde{\gamma}_{d_0}^{[0]}$ forming a basis of the relative homology on top level,
- loops $\widetilde{\gamma}_1^{[-1]}, \dots, \widetilde{\gamma}_{d_1}^{[-1]}$ forming a basis of the homology on bottom level.

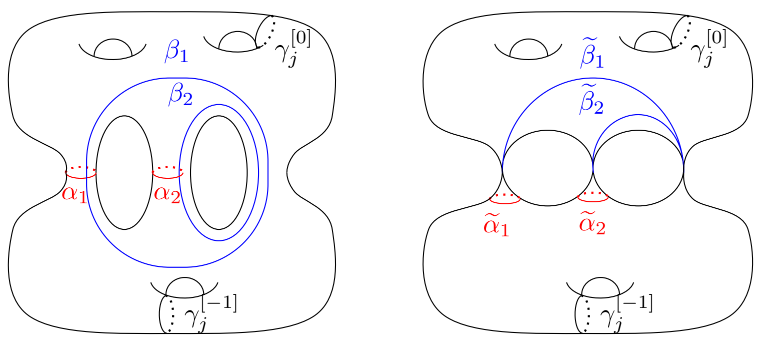



FIGURE 3. Cycles in Case 2, near the boundary stratum and at the boundary stratum

From this description it is apparent that $d_i$ is related to the projectivized and unprojectivized dimensions of the level strata previously introduced by $d_i = N_\Gamma^{[i]} - 2 = d_\Gamma^{[i]} - 1$.

The main statement about perturbed period coordinates [BCGGM3, Section 11] is that on the one hand, coordinates near the boundary are given by the periods on the boundary surfaces and on the other hand, periods with and without tilde are nearly the same after appropriate rescaling. To make this statement concrete, let $\kappa_i$ be the enhancements corresponding the edges $e_i$ and let $\ell = \operatorname{lcm}(\kappa_1, \kappa_2, \kappa_3)$. Near our current boundary divisor $D_\Gamma$ the universal family of curves has a (universal) family of differentials $\omega$ and $\ell$ is chosen so that rescaling $\eta_{(-1)} = t^{-\ell}\omega_{(-1)}$ is holomorphic and generically non-zero for a coordinate with $D_\Gamma = \{t = 0\}$ locally ([BCGGM3, Section 12]). At each point $p \in D_\Gamma$ we find a non-zero $\eta$-period on lower level, say the period along $\widetilde{\gamma}_1^{[-1]}$, and choose $t$ and thus $\eta$ so that $\int_{\widetilde{\gamma}_1^{[-1]}} \eta = 1$.

A chart of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ near $p$ is then nearly the product of a neighborhood of the irreducible components $(X_0, \omega)$ and $(X_1, \eta)$ of the fiber over $p$ in their respective strata of meromorphic differentials. Here, 'nearly' refers to the fact that, because of prong-matchings, it is a $\ell$-fold cover fully ramified over $t = 0$, and moreover, because of enhanced level graph automorphisms, it is a quotient stack by the subgroup $G$ of $S_3$ that exchanges edges with the same enhancement.

Coordinates on this chart are then given by $t$ and the periods

$$\begin{aligned} \widetilde{b}_i &= \int_{\widetilde{\beta}_i} \eta_{(0)} \quad (i=1,2), & r_i &= \int_{\widetilde{\alpha}_i} \eta_{(-1)} \quad (i=1,2), \\ \widetilde{c}_i^{[0]} &= \int_{\widetilde{\gamma}_i^{[0]}} \eta_{(0)} \quad (i=1,\dots,d_0), & \widetilde{c}_i^{[-1]} &= \int_{\widetilde{\gamma}_i^{[-1]}} \eta_{(-1)} \quad (i=2,\dots,d_1)\,. \end{aligned}$$

To provide charts of the projectivization $\overline{B}$ we simply fix one of the periods on top level, say $\widetilde{c}_1^{[0]}$, to be identically one. (If $d_0=0$ we take $\widetilde{b}_1 \equiv 1$ instead.)

In each sector near the boundary, the perturbed period coordinates are related to the $\omega$-periods by

$$\begin{aligned} b_i &:= \int_{\beta_i} \omega \sim \widetilde{b}_i & a_i &:= \int_{\alpha_i} \omega \;=\; t^\ell r_i \\ c_i^{[0]} &:= \int_{\gamma_i^{[0]}} \omega \sim \widetilde{c}_i^{[0]}, & c_i^{[-1]} &:= \int_{\gamma_i^{[-1]}} \omega \;=\; t^\ell \widetilde{c}_i^{[-1]}. \end{aligned} \tag{39}$$

where $\sim$ indicates that the difference is $O(t^\ell)$. The difference stems (for $c_i^{[0]}$) from the fact that the $\omega$ in the universal family is not just the deformation of the twisted differential $(\eta_{(0)}, \eta_{(-1)})$ in the fiber over $p$ in its product moduli space, but blurred by some modification differentials. For the $b_i$ there is an additional error in the same order of magnitude due to a choice of a nearby base point in the plumbing construction.

**Case 3: two levels, additional horizontal nodes.** This is a mixture of the previous two cases. To see the effects, we assume that we are in the situation of Case 2, with one horizontal node and thus additionally a pair of cycles $\alpha^{[j]}$ and $\beta^{[j]}$ with $j=0$ or $j=-1$ depending on the level where the horizontal node is attached.We may then uniformly write the periods $a^{[j]} = \int_{\alpha^{[j]}} \eta_{(j)}$ and $b^{[j]} = \int_{\beta^{[j]}} \eta_{(j)}$. The additional coordinates are $a^{[j]}$ and the exponentiated period ratio $q^{[j]} = \exp(2\pi i b^{[j]}/a^{[j]})$.

**Case 4: three levels, three nodes.** This is the generalization of the triangle case (Figure 1 left), with edges replaced possibly by multiple strands, say $k_i$ strands for the edge $e_i$, including the case $k_i = 0$ for missing edge (as the long edge in Figure 1 right). Let $\ell_1$ be the lcm of the enhancements on the edges starting at level 0 and $\ell_2$ the lcm of the edges ending at level $-2$, as defined in (22).

A point $p \in D_\Gamma$ on the corresponding divisor is given by meromorphic differential forms $(X_{(0)}, \eta_{(0)})$, $(X_{(-1)}, \eta_{(-1)})$, $(X_{(-2)}, \eta_{(-2)})$ together with prong-matchings. We denote by $\widetilde{\gamma}_i^{[j]}$ for $j = 0,-1,-2$ and $i = 1,\dots,N_j$ paths of the relative homology of the surfaces. (There are no global residue conditions in this example.) We may suppose that $\int_{\widetilde{\gamma}_1^{[j]}} \eta_{(j)} = 1$ for $j=0,-1,-2$ to fix the scale of the $\eta_{(j)}$ on lower level and for $j=0$ to fix an open subset of the projectivization.

A chart of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ near $p$ is then nearly the product of a neighborhood of the irreducible components $(X_{(j)}, \eta_{(j)})$ where $j = 0,-1,-2$ of the fiber over $p$ in their respective strata of meromorphic differentials. Slightly abusing notation we call the universal differentials over these neighborhoods also $\eta_{(j)}$. A coordinate system for the neighborhood of $p \in \overline{B}$ is given by functions $t_1$ and $t_2$ that correspond to rescalings of the two levels together with the functions $\widetilde{c}_i^{[j]} = \int_{\widetilde{\gamma}_i^{[j]}} \eta_{(j)}$ for $j = 0, -1, -2$ and $i = 2, \ldots, N_j$. In particular $N_0 + N_{-1} + N_{-2} = N$.

To give the relation of these coordinates to nearby periods, note that the universal differential $\omega$ over $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, has by construction the property that the periods of $\omega$ on bottom level agree with those of $t_1^{\ell_1} t_2^{\ell_2} \eta_{(-2)}$, the periods on level $-1$ of $\omega$ differ from those of $t_1^{\ell_1}\eta_{(-1)}$ by functions that decay like $t_1^{\ell_1} t_2^{\ell_2}$ and periods on top level of $\omega$ differ from those of $\eta_{(0)}$ by functions that decay like $t_1^{\ell_1}$. Here, as we have illustrated in Case 2, the loops around the nodes corresponding to the $k_1 + k_2 + k_3$ edges can be treated as residues and thus as periods on the level at the lower end of the edge, while the loops through those edges (denote previously by $\beta_i$) can be treated as relative periods on the highest level that the loop touches.

6.3. **The Euler sequence on the Deligne extension.** Recall that the Deligne extension of a local system on $B$ is a canonical extension to a vector bundle on $\overline{B}$ admitting an extension of the Gauss-Manin connection to a connection with regular singular points ([Del70]). In this section we want to extend the Euler sequence across the boundary to construct (37). For this purpose we exhibit local generators of the Deligne extension $\overline{\mathcal{H}}^1_{\mathrm{rel}}$ of $\mathcal{H}^1_{\mathrm{rel}}$, extend the map ev and determine its kernel in each of the cases as we discussed perturbed period coordinates in Section 6.2, adopting notation from there.

**Case 1: only horizontal nodes.** A basis of $(\overline{\mathcal{H}}^1_{\mathrm{rel}})^\vee$ consists of the cycles $\alpha_1, \ldots, \alpha_k$ and $\gamma_1, \ldots, \gamma_{N-2k}$ that extend across $D_\Gamma$, together with the linear combinations

$$\widehat{\beta}_i \;=\; \beta_i - \frac{1}{2\pi i}\log(q_i)\alpha_i$$

designed to be monodromy invariant. Since the family one-forms $\omega$ extends across $D_\Gamma$ to a family of stable differentials, the definition

$$\mathrm{ev}(\widehat{\beta}_i \otimes \omega) \;=\; \int_{\beta_i} \omega - \frac{1}{2\pi i}\log(q_i)\int_{\alpha_i}\omega \;=\; b_i - \frac{1}{2\pi i}\log(q_i)a_i \;=\; 0$$

extends the definition of ev in the interior and gives a well-defined holomorphic function. To check the surjectivity of ev we can use any of the periods that extend across $D_\Gamma$. We claim that the kernel of ev is on the chart $U$ with $a_1 \equiv 1$

$$\mathcal{K} \;=\; \big\langle dq_1/q_1, da_2, dq_2/q_2, \ldots, da_k, dq_k/q_k, dc_1, \ldots, dc_{N-2k}\big\rangle \tag{40}$$

as $\mathcal{O}_U$-module. In fact, using the definition (38) in the interior one checks that

$$dq_i/q_i \;=\; d\log(q_i) \;=\; d\left(2\pi i \frac{b_i}{a_i}\right) \;\mapsto\; \frac{2\pi i}{a_i}\Big(\beta_i - \frac{b_i}{a_i}\alpha_i\Big) \otimes \omega \tag{41}$$

is mapped to a local generator of $(\overline{\mathcal{H}}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_{\overline{B}}(-1)$ since the functions $a_i$ do not vanish near such a boundary point. Moreover $dq_i/q_i$ is mapped to the kernel of ev by the preceding calculation. For the other elements these claims follow as in the interior.

**Case 2: two levels, only vertical nodes.** We first work in the special case near a boundary divisor $D_\Gamma$ where $\Gamma$ has three edges as in the case discussed in Section 6.2. A basis of $(\overline{\mathcal{H}}^1_{\mathrm{rel}})^\vee$ consists of the cycles $\alpha_1, \alpha_2, \gamma_1^{[0]}, \dots, \gamma_{d_0}^{[0]}, \gamma_1^{[-1]}, \dots, \gamma_{d_1}^{[-1]}$ that extend across $D_\Gamma$, together with the linear combinations

$$\widehat{\beta}_i \;=\; \beta_i - \log(t)(\alpha_i + (\alpha_1+\alpha_2)) \qquad (i=1,2) \tag{42}$$

that are monodromy invariant since turning once around the divisor acts by simultaneous Dehn-twists around the core curves of the three plumbing cylinders. Sending cycles that extend across $D_\Gamma$ to their $\omega$-integrals and letting

$$\mathrm{ev}(\widehat{\beta}_i \otimes \omega) \;=\; \int_{\beta_i} \omega - \log(t)\Big(\int_{\alpha_i}\omega + \int_{\alpha_1+\alpha_1}\omega\Big) \qquad (i=1,2)$$

extends the definition of ev in the interior and is well-defined since the function $\log(t)\Big(2\int_{\alpha_i}\omega + \int_{\alpha_1+\alpha_1}\omega\Big) = O(t^\ell \log(t))$ is bounded near $D_\Gamma$ and $\int_{\beta_i}\omega \to \int_{\widetilde{\beta}_i}\omega$ is bounded as well.

Obviously on the chart with $c_1^{[0]} = 1$ the kernel of ev is

$$\begin{aligned}\mathrm{Ker}(\mathrm{ev}) \;=\; \Big\langle &\gamma_i^{[0]} - c_i^{[0]}\gamma_1^{[0]} \ (i=2,\dots,d_0);\ \ \alpha_i - a_i\gamma_1^{[0]} \ (i=1,2); \\ &\gamma_i^{[-1]} - c_i^{[-1]}\gamma_1^{[0]} \ (i=1,\dots,d_1);\ \widehat{\beta}_i - \widehat{b}_i\gamma_1^{[0]} \ (i=1,2)\Big\rangle\end{aligned} \tag{43}$$

where $\widehat{b}_i$ is the integral of $\widehat{\beta}_i$. We claim that via the identification of periods in (39) this kernel is precisely the image of

$$\mathcal{K} \;=\; \big\langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{d_0}^{[0]},\, d\widetilde{b}_1, d\widetilde{b}_2,\, t^\ell dt/t,\, t^\ell d\widetilde{c}_2^{[-1]}, \dots, t^\ell d\widetilde{c}_{d_1}^{[-1]},\, t^\ell dr_1,\, t^\ell dr_2, \big\rangle\,.$$

under the map (38). First, since we used the coordinate $\widetilde{c}_1^{[1]}$ to fix the scaling on the bottom level, the differential form $\ell t^\ell dt/t = dc_1^{[-1]}$ is mapped to $\gamma_1^{[-1]} - c_1^{[-1]}\gamma_1^{[0]}$. Then from (39), we see that $t^\ell d\widetilde{c}_i^{[-1]}$ is mapped to a linear combination of $\gamma_i^{[-1]} - c_i^{[-1]}\gamma_1^{[0]}$ and the previous generator for any $i \geq 2$. Similarly $t^\ell dr_i$ maps to $\alpha_i - a_i\gamma_1^{[0]}$ and a linear combination of the previous generators. In the second step we consider the generators that correspond to top level. The form $d\widetilde{c}_i^{[0]}$ does not quite map to $\gamma_i^{[0]} - c_i^{[0]}\gamma_1^{[0]}$ because of the presence of modification differentials, but the difference is a linear combination of the differential of some $\eta$-periods that we have shown already in the first step to belong to Ker(ev). Similarly, the image of $d\widetilde{b}_i$ and $\widehat{\beta}_i - \widehat{b}_i\gamma_1^{[0]}$ is differentials of periods on lower level (from (39), to compare with $db_i$ and from (42)).

We now rename and regroup the generators of $\mathcal{K}$ in a form that generalizes to other level graphs. Since the $\beta$-periods become relative periods and since the $\alpha$-periods for the edges joining the levels are simply residues appearing on lower level, we may name the set of all periods on top level by $\widetilde{c}_i^{[0]}$ for $1 \leq i \leq N_0$ and those on bottom level by $\widetilde{c}_i^{[-1]}$ for $1 \leq i \leq N_1$. Then the above argument gives that

$$\mathcal{K} \;=\; \big\langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]},\, t^\ell dt/t,\, t^\ell d\widetilde{c}_2^{[-1]}, \dots, t^\ell d\widetilde{c}_{N_1}^{[-1]}\big\rangle\,. \tag{44}$$

**Case 3: two levels, additional horizontal nodes.** We mix the conclusion of the two previous cases. If the horizontal node is at the top level, then

$$\mathcal{K} \;=\; \langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]},\, da,\, dq/q,\, t^\ell dt/t,\, t^\ell d\widetilde{c}_2^{[1]}, \dots, t^\ell d\widetilde{c}_{N_1}^{[1]} \rangle\,, \tag{45}$$

while in the case of a horizontal node and the bottom level

$$\mathcal{K} \;=\; \langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]},\, t^\ell dt/t,\, t^\ell d\widetilde{c}_2^{[-1]}, \dots, t^\ell d\widetilde{c}_{N_1}^{[-1]},\, t^\ell da,\, t^\ell dq/q \rangle\,. \tag{46}$$

**Case 4: three levels, three nodes.** We can adopt here from Case 2 the argument the $\alpha_j$-periods corresponding to the graphs become residues and the monodromy-invariant modifications $\widehat{\beta}_j$ of the dual $\beta_j$-periods have ev-images that tend to the $\beta_j$-integrals. We claim that thus $\mathrm{Ker}(\mathrm{ev})$ is the image of

$$\begin{aligned} \mathcal{K} \;=\; \langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]},\, & t_1^{\ell_1} dt_1/t_1,\; t_1^{\ell_1} d\widetilde{c}_2^{[-1]}, \dots,\; t_1^{\ell_1} d\widetilde{c}_{N_1}^{[-1]} \\ & t_1^{\ell_1} t_2^{\ell_2} dt_2/t_2,\, t_1^{\ell_1} t_2^{\ell_2} d\widetilde{c}_2^{[-2]}, \dots, t_1^{\ell_1} t_2^{\ell_2} d\widetilde{c}_{N_2}^{[-2]} \rangle\,. \end{aligned} \tag{47}$$

under the map (38). We justify this, starting at bottom level. The differential form

$$d(\widetilde{c}_1^{[-2]}) \;=\; d(t_1^{\ell_1} t_2^{\ell_2}) \;=\; \ell_2 t_1^{\ell_1} t_2^{\ell_2} dt_2/t_2 + \ell_1 t_1^{\ell_1} t_2^{\ell_2} dt_1/t_1 \quad \in \mathcal{K}\,,$$

since it is mapped to $\gamma_1^{[-2]} - c_1^{[-2]}\gamma_1^{[0]}$, which in analogy with (43) belongs to the natural basis of $\mathrm{Ker}(\mathrm{ev})$. Next, the form $dt_1^{\ell_1} t_2^{\ell_2} d\widetilde{c}_i^{[-2]}$ map to a linear combination of the elements $\gamma_i^{[-2]} - c_i^{[-2]}\gamma_1^{[0]}$ in the natural basis of $\mathrm{Ker}(\mathrm{ev})$ and the previous generator.

We next proceed to the middle level. There, the form $\ell_1 t^{\ell_1} dt_1/t_1$ is not quite equal to $d(c_1^{[-1]})$ because of the presence of modification differentials. It thus does not quite map to the basis element $\gamma_i^{[-1]} - c_i^{[-1]}\gamma_1^{[0]}$ of $\mathrm{Ker}(\mathrm{ev})$. But the difference is a combination of elements that we have already shown to belong to $\mathcal{K}$. As a combination of this form and $d(\widetilde{c}_1^{[-2]})$ we now have $\ell_2 t_1^{\ell_1} t_2^{\ell_2} dt_2/t_2 \in \mathcal{K}$. Considering the remaining form $dc_i^{[-1]}$ from periods on middle level, and then all the form $dc_i^{[0]}$ for $i \geq 2$ on top level identifies the remaining elements listed in $\mathcal{K}$ with elements of $\mathrm{Ker}(\mathrm{ev})$, up to the effect of modification differentials, which produce differentials of periods already shown to belong to $\mathcal{K}$.

The notation

$$t_{\lceil j \rceil} \;=\; \prod_{i=1}^{j} t_i^{\ell_i}, \quad j \in \mathbb{N}. \tag{48}$$

will be convenient here and in the sequel.

*Proof of Theorem 6.1.* Continuing the argument as in the preceding cases, we see that near a point $p \in D_\Gamma$ the elements

- $t_{\lceil j \rceil} dt_j/t_j$, for every level $-j$,
- the $t_{\lceil j \rceil}$-multiples of differential forms associated to periods on level $-j$
- $t_{\lceil j \rceil} dq_k^{[-j]}/q_k^{[-j]}$ for every horizontal node with parameter $q_k$ on level $-j$

freely generate $\mathcal{K}$. ☐

## 7. The normal bundle to boundary strata

In this section we provide formulas to compute the first Chern class of the normal bundle $\mathcal{N}_\Gamma = \mathcal{N}_{D_\Gamma}$ to a boundary divisor $D_\Gamma$. We will encounter here and in the sequel frequently the top level correction line bundle

$$\mathcal{L}_\Gamma^\top \;=\; \mathcal{O}_{D_\Gamma}\Big(\sum_{\substack{\widehat{\Delta}\in \mathrm{LG}_2(\overline{B})\\ \delta_2(\widehat{\Delta})=\Gamma}} \ell_{\widehat{\Delta},1} D_{\widehat{\Delta}}\Big) \tag{49}$$

on $D_\Gamma$ that records all the degenerations of the top level of $\Gamma$.

**Theorem 7.1.** *Suppose that $D_\Gamma$ is a divisor in $\overline{B}$ corresponding to a graph $\Gamma \in \mathrm{LG}_1(\overline{B})$. Then*

$$c_1(\mathcal{N}_\Gamma) \;=\; \frac{1}{\ell_\Gamma}\big(-\xi_\Gamma^\top - c_1(\mathcal{L}_\Gamma^\top) + \xi_\Gamma^\perp\big) \quad \text{in} \quad \mathrm{CH}^1(D_\Gamma)\,. \tag{50}$$

In case the graph $\Gamma$ contains an edge $e$ (which is automatic if the ambient stratum parameterizes connected curves, but often not satisfied in the generalization to higher codimension strata below) there is an alternative expression for the Chern class of the normal bundle, that gives a comparison to the situation in the moduli space of curves. Let $e^\pm$ be the half edges that form the edge $e$.

**Proposition 7.2.** *The first Chern class of the normal bundle $\mathcal{N}_\Gamma$ of a boundary divisor $D_\Gamma$ is*

$$\mathrm{c}_1(\mathcal{N}_\Gamma) \;=\; -\frac{\kappa_e}{\ell_\Gamma}\bigg(\psi_{e^+} + \psi_{e^-}\bigg) - \frac{1}{\ell_\Gamma}\sum_{\widehat{\Delta}\in \mathrm{LG}^\Gamma_{2,e}(B)} \ell_{\widehat{\Delta},a_{\widehat{\Delta},\Gamma}}[D_{\widehat{\Delta}}]. \tag{51}$$

*as an element of $\mathrm{CH}^1(D_\Gamma)$, where $\mathrm{LG}^\Gamma_{2,e}(B)$ is the set of 3-level graphs in $\mathrm{LG}^\Gamma_2(B)$ where the edge $e$ goes from level zero to level $-2$ and where $a_{\widehat{\Delta},\Gamma} \in \{1,2\}$ is the index such that the $a_{\widehat{\Delta},\Gamma}$-th undegeneration of $\widehat{\Delta}$ is not equal to $\Gamma$.*

We say that $\mathrm{LG}^\Gamma_{2,e}(B)$ are the 3-levels graphs where the edge $e$ *becomes long.* We give direct proofs of both expressions for the normal bundle. The equivalence of the statements follows from an application of the relation in Proposition 8.2 below.

*Proof of Theorem 7.1.* We consider over the boundary stratum $D_\Gamma$ the line bundles $\mathcal{L}_1 = \mathcal{O}_\Gamma^{[0]}(-1)\otimes\mathcal{L}_\Gamma^\top$ and $\mathcal{L}_2 = \mathcal{O}_\Gamma^{[-1]}(-1)$ where the tautological bundles on the levels have been introduced in Section 4.3. Roughly the content of the theorem is that the ratio of local sections of these line bundles is the function $t_1^{\ell_\Gamma}$, which is also the $\ell_\Gamma$-th power of a transversal coordinate. For the precise statement we compare the cocycles defining the line bundles $\mathcal{L}_1^{-1}\otimes\mathcal{L}_2$ and $\mathcal{N}_\Gamma^{\ell_\Gamma}$.

We start by considering the open subset of $D_\Gamma$ where $\Gamma$ does not degenerate further. A local section of $\mathcal{L}_1^{-1}\otimes\mathcal{L}_2$ is the ratio of two relative differential forms, thus a function on the base, that we may compute as $u = \int_{\alpha_1}\eta_{(-1)}/\int_{\alpha_0}\eta_{(0)}$ for some paths $\alpha_1$ at level $-1$ and $\alpha_0$ at level 0. Here $\alpha_1$ can be taken as (usual) relative cycle, and for $\alpha_0$ we might have to use a path starting and ending at points *near* the upper ends of connecting nodes, as in the definition of perturbed period coordinates in [BCGGM3, Section 11]. We consider a nearby coordinate patch where now the ratio is $\widetilde{u} = \int_{\widetilde{\alpha_1}}\eta_{(-1)}/\int_{\widetilde{\alpha_0}}\eta_{(0)}$ for some new cycles related to the original ones by a base change $\widetilde{\alpha_1} = \alpha_1 + \gamma_1$ and $\widetilde{\alpha_0} = \alpha_0 + \gamma_0$ in the homology of the upper and lower level subsurfaces respectively. One computes that

$$\widetilde{u} = u \cdot \frac{1+y}{1+x}\,, \quad \text{where} \quad x \;=\; \int_{\gamma_0} \eta_{(0)} \,/ \int_{\alpha_0} \eta_{(0)} \quad \text{and} \quad y \;=\; \int_{\gamma_1} \eta_{(-1)} \,/ \int_{\alpha_1} \eta_{(-1)}$$

In particular these $x, y$ are local functions on the upper and lower level strata.

On the other hand, by construction (of the perturbed period coordinates) the $\ell_\Gamma$-th power of a transversal coordinate is given by

$$t_1^{\ell_\Gamma} \;=\; s \;=\; \int_{\alpha_1} \eta_{(-1)} \,/ \int_{\alpha_0} (\eta_{(0)} + \xi_{(0)})\,,$$

where $\xi_{(0)}$ is the modification differential at level 0 constructed in [BCGGM3, Section 11] and where the $\alpha_i$ are as above. Again a nearby coordinate patch is given by $\widetilde{s} = \int_{\widetilde{\alpha_1}} \eta_{(-1)}/, \int_{\widetilde{\alpha_0}} (\eta_{(0)} + \xi_{(0)})$ with cycles as above. The main point now is that $\xi_0$ is divisible by $s$ by construction, and so its contribution vanishes in after $s$-derivation and setting $s = 0$, so

$$\frac{\partial \widetilde{s}}{\partial s}\Big|_{s=0} \;=\; \frac{1+y}{1+x} \;=\; \frac{\widetilde{u}}{u}\,, \tag{52}$$

showing that the cocycles from $\mathcal{L}_1^{-1} \otimes \mathcal{L}_2$ and $\mathcal{N}_\Gamma^{\ell_\Gamma}$ agree on the subset under consideration.

If the bottom level degenerates or in case of horizontal degenerations of $\Gamma$, the above claims remain valid without modification, if we take $\alpha_1$ to be a period that does not go to lower level. If the top level degenerates into two levels (without loss of generality, higher codimension degenerations do not affect the first Chern classes), the above cocycle comparison is valid verbatim, if all pairs of level indices are shifted from $(0, -1)$ to $(-1, -2)$, that is, if we compare the periods of a form on the middle level with the periods of a form at bottom level. Since the multi-scale differential on the middle level is $t_1^{\ell_{\widehat{\Delta},1}}$ times a top level differential at the intersection with $D_{\widehat{\Delta}}$, the sections of we are locally comparing with are sections of $\mathcal{L}_1 = \mathcal{O}_\Gamma^{[0]}(-1) \otimes \mathcal{L}_\Gamma^\top$ as we claimed. □

*Sketch of proof of Proposition 7.2.* We let $m_e = \ell(\Gamma)/\kappa_e$. In $\overline{\mathcal{M}}_{g,n}$ consider the divisor $D_e$ corresponding to the single edge $e$ and denote by $\mathcal{N}_e$ its normal bundle. With the same symbol we denote also the pullback of this normal bundle under the forgetful map $D_\Gamma \to D_e$. We claim that (at least outside a subvariety of codimension two) there is a short exact sequence of quasi-coherent $\mathcal{O}_{D_\Gamma}$-modules

$$0 \longrightarrow \mathcal{N}_\Gamma^{m_e} \to \mathcal{N}_e \to \mathcal{Q}_\Gamma \to 0 \tag{53}$$

where the coherent sheaf $\mathcal{Q}_\Gamma$ is supported on the set $\mathrm{LG}^\Gamma_{2,e}(B)$ and this sheaf is given by

$$\mathcal{Q}_\Gamma \;=\; \bigoplus_{\Delta \in \mathrm{LG}^\Gamma_{2,e}(B)} \mathcal{O}_{D_\Gamma} / I_{D_{\widehat{\Delta}}}^{\ell_{\widehat{\Delta},a}/\kappa_e}\,, \tag{54}$$

where $a = a_{\widehat{\Delta},\Gamma}$ as above and where $I_{D_{\widehat{\Delta}}}$ is the ideal sheaf of the divisor $D_{\widehat{\Delta}} \subseteq D_\Gamma$. This claim obviously implies the proposition.

To prove it, we use the local description of the universal family over $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ given by the plumbing construction described in Section 12 of [BCGGM3]. At a boundary point that is precisely in the intersection of divisors $D_{\Gamma_i}$ we let $m_{e,i} =$

$\ell_{\Gamma_i}/\kappa_e$. Then the construction states in particular that the universal family is constructed using the plumbing fixture

$$\mathbb{V}_e \;=\; \Big\{(u,v)\in\Delta^2 :\; uv = \prod_{i=L(e^-)}^{L(e^+)} t_i^{m_{e,i}}\Big\}$$

at the node corresponding to the edge $e$, where $u$ and $v$ are coordinates on the surfaces at the upper and lower end of the edge and where $L(e^{\pm})$ denotes the levels at the edges of the edge. A local generator of $\mathcal{N}_e$ is $\partial/\partial f$ if $uv = f$ is a local equation of the node. On the other hand, a local generator of $\mathcal{N}_\Gamma^{m_e}$ is $\partial/\partial(t_i^{m_{e,i}})$ if $\Gamma$ is the undegeneration of the $i$-th level at the point under consideration. (In particular, $m_e = m_{e,i}$ in this situation). This follows from the form of perturbed period coordinates. This implies that at a generic point of $D_\Gamma$ (and more generally whenever the edge does not become long) the natural map $\mathcal{N}_\Gamma^{m_e} \to \mathcal{N}_\Gamma$ is an isomorphism. At the remaining points,

$$\frac{\partial}{\partial f} \;=\; \prod_{\substack{j=L(e^-)\\ j\neq i}}^{L(e^+)} t_j^{m_{e_j}}\, \frac{\partial}{\partial(t_i^{m_{e_i}})} + \cdots$$

where the suppressed tangent vectors vanish when restricted to $D_\Gamma$. Since $t_j$ are the defining equations of divisors $D_\Delta$ where the edge becomes long, this implies (54). $\square$

**Example 7.3.** Consider the stratum $\mathbb{P}\Omega\mathcal{M}_{0,5}(a_1,a_2,a_3,a_4,-b)$ with $a_i \geq 0$ and $b = +2 + \sum a_i \geq 0$. We study the 'cherry' divisor $\Gamma$ (see also [BCGGM3, Section 14.4]) with one vertex on top level, carrying the unique pole, and two vertices on lower level, carrying the first two and the third plus forth point, respectively. The vertices on lower level are each connected to the top level by a single edge, denoted by $e_1$ and $e_2$ respectively. The enhancements are given by $\kappa_1 = a_1+a_2+1$ and $\kappa_2 = a_3+a_4+1$. Hence $\ell_\Gamma = \mathrm{lcm}(\kappa_1,\kappa_2)$.

We compute the degree of the normal bundle using either of the two edges. Note that the boundary divisor $D_\Gamma$ has two intersection points with other boundary strata, where $e_1$ and $e_2$ become long edges. Neighborhoods of these points are quotient stacks by a cyclic group of order $m_i = \ell_\Gamma/\kappa_i$. To see this, say where $e_1$ becomes long, we check that $\mathrm{Tw}_\Lambda^s = \ell\mathbb{Z}\oplus\kappa_1\mathbb{Z}$ and $\mathrm{Tw}_\Lambda = \langle(0,\kappa_1),(\kappa_2,-\kappa_2)\rangle$, hence the index is $m_1$, as claimed.

In this example, the bundle $\mathcal{N}_e$ has degree zero when pulled back to $D_\Gamma$, since $D_\Gamma$ is contracted when mapped to $\overline{\mathcal{M}}_{0,5}$. Applying the theorem, we get

$$\deg(\mathcal{N}^{m_1}) \;=\; 0 - \frac{1}{m_2}\frac{\kappa_2}{\kappa_2}, \quad \text{hence} \quad \deg(\mathcal{N}) = \frac{1}{m_1 m_2}$$

and using $e_2$ we arrive at the same conclusion.

Our next task is to identify the normal bundle as sum of two contributions from the top an bottom level via push-pull through the level projections and clutching maps. For this purpose we define

$$\mathcal{L}_{B_\Gamma^\top} \;=\; \mathcal{O}_{B_\Gamma^\top}\Big(\sum_{\Delta\in\mathrm{LG}_1(B_\Gamma^\top)} \ell_\Delta D_\Delta\Big) \tag{55}$$

**Lemma 7.4.** *We have* $p_\Gamma^{\top,*}\mathcal{L}_{B_\Gamma^\top} \;=\; c_\Gamma^*\mathcal{L}_\Gamma^\top$.

*Proof.* We sum the first equation in Proposition 4.7 over all $\Delta \in \mathrm{LG}_1(B_\Gamma^\top)$. Each $\widehat{\Delta}$ will appear for all graphs in $J(\Gamma^\dagger, \widehat{\Delta})$ as discussed at the beginning of Section 4.3. However thanks to Lemma 4.6 this factor cancels with all the automorphism factors in that proposition to give the statement we claim. $\square$

The lemma obviously implies

$$c_1(\mathcal{L}_\Gamma^\top) \;=\; \frac{1}{\deg(c_\Gamma)} \cdot c_{\Gamma,*}\, p_\Gamma^{\top *} c_1(\mathcal{L}_{B_\Gamma^\top})\,.$$

Since the tautological bundles on top and on bottom level have a pullback description by Proposition 4.9, we have shown that there exist $\nu^\top \in \mathrm{CH}^1(B_\Gamma^\top)$ and $\nu^\perp \in \mathrm{CH}^1(B_\Gamma^\perp)$ such hat

$$\nu_\Gamma \;:=\; \mathrm{c}_1(\mathcal{N}_\Gamma) \;=\; \mathrm{c}_{\Gamma,*}\,(p^\top)^*\nu_\Gamma^\top \;+\; c_{\Gamma,*}\,(p^\perp)^*\nu_\Gamma^\perp\,. \tag{56}$$

The normal bundle computation has a generalization to an inclusion $\mathrm{j}_{\Gamma,\Pi}\colon D_\Gamma \hookrightarrow D_\Pi$ between non-horizontal boundary strata of relative codimension one, say defined by the $L$-level graph $\Pi$ and one of its $(L+1)$-level graph degenerations $\Gamma$. This generalization is needed in Section 8 for recursive evaluations. Such an inclusion is obtained by splitting one of the levels of $\Pi$, say the level $i \in \{0, -1, \dots, -L\}$. We define

$$\mathcal{L}_\Gamma^{[i]} \;=\; \mathcal{O}_{D_\Gamma}\Big(\sum_{\Gamma \overset{[i]}{\rightsquigarrow} \widehat{\Delta}} \ell_{\widehat{\Delta},-i+1} D_{\widehat{\Delta}}\Big) \quad \text{for any} \quad i \in \{0, -1, \dots, -L\}\,, \tag{57}$$

where the sum is over all graphs $\widehat{\Delta} \in \mathrm{LG}_{L+2}(\overline{B})$ that yield divisors in $D_\Gamma$ by splitting the $i$-th level, which in terms of undegenerations means $\delta^{\complement}_{-i+1}(\widehat{\Delta}) = \Gamma$. With the same proof as above, simply shifting attention to level $i$ of $\Pi$, we obtain:

**Proposition 7.5.** *For $\Pi \overset{[i]}{\rightsquigarrow} \Gamma$ (or equivalently $\delta^{\complement}_{-i+1}(\Gamma) = \Pi$) the Chern class of the normal bundle $\mathcal{N}_{\Gamma,\Pi} = \mathcal{N}_{D_\Gamma/D_\Pi}$ is given by*

$$c_1(\mathcal{N}_{\Gamma,\Pi}) \;=\; \frac{1}{\ell_{\Gamma,(-i+1)}}\big(-\xi_\Gamma^{[i]} - c_1(\mathcal{L}_\Gamma^{[i]}) + \xi_\Gamma^{[i-1]}\big) \quad \textit{in} \quad \mathrm{CH}^1(D_\Gamma)\,. \tag{58}$$

With the same proof as in Lemma 7.4 we obtain

$$p_\Gamma^{[i]*}\mathcal{L}_{B_\Gamma^{[i]}} \;=\; c_\Gamma^* \mathcal{L}_\Gamma^{[i]} \quad \text{where} \quad \mathcal{L}_{B_\Gamma^{[i]}} \;=\; \mathcal{O}_{B_\Gamma^{[i]}}\Big(\sum_{\Delta \in \mathrm{LG}_1(B_\Gamma^{[i]})} \ell_\Delta D_\Delta\Big)\,. \tag{59}$$

We can thus write the normal bundle as a sum of bundles that are $c_\Gamma$-pushforwards of pullbacks from $B_\Gamma^{[i]}$ and from $B_\Gamma^{[i-1]}$. We express this by saying that the normal bundle is *supported on the levels $i$ and $i-1$* (for $i \in \mathbb{Z}_{\le 0}$).

We need some compatibility statements for pullbacks of normal bundles to more degenerate graphs. We start with auxiliary bundles, whose pullback we need, too.

**Lemma 7.6.** *Let $\Gamma \in \mathrm{LG}_L(B)$ and let $\Gamma \overset{[i]}{\rightsquigarrow} \widehat{\Delta}$ be a codimension one degeneration of $\Gamma$ obtained by splitting the level $i \in \{0, \dots, -L\}$. Then for every $j \in \{0, \dots, -L\}$*

$$\mathrm{j}^*_{\widehat{\Delta},\Gamma}(\xi_\Gamma^{[j]}) \;=\; \begin{cases} \xi_{\widehat{\Delta}}^{[j]}, & \textit{if } j \ge i \\ \xi_{\widehat{\Delta}}^{[j-1]} & \textit{if } j < i \end{cases}$$

*and*

$$\mathrm{j}^*_{\widehat{\Delta},\Gamma}\left(c_1\left(\mathcal{L}_\Gamma^{[j]}\right)\right) = \begin{cases} c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j]}\right), & \text{if } j > i \\ c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j-1]}\right) & \text{if } j < i \\ c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j-1]}\right) + \xi_{\widehat{\Delta}}^{[j-1]} - \xi_{\widehat{\Delta}}^{[j]} & \text{if } j = i. \end{cases}$$

*Proof.* For the cases $j \neq i$, the claim is obvious since level $i$ is untouched in the degeneration from $\Gamma$ to $\widehat{\Delta}$. If $i = j$ then the second claim follows from

$$\begin{aligned} \mathrm{j}^*_{\widehat{\Delta},\Gamma}\left(c_1\left(\mathcal{L}_\Gamma^{[j]}\right)\right) &= \mathrm{j}^*_{\widehat{\Delta},\Gamma}\left(\sum_{\Gamma \overset{[j]}{\rightsquigarrow} \Lambda,\ \Lambda \neq \widehat{\Delta}} \ell_{\Lambda,-j+1}[D_\Lambda] + \ell_{\widehat{\Delta},-j+1}[D_{\widehat{\Delta}}]\right) \\ &= c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j]}\right) + c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j-1]}\right) + \ell_{\widehat{\Delta},-j+1}\,\mathrm{c}_1(\mathcal{N}_{D_{\widehat{\Delta}}/D_\Gamma}) \\ &= c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j]}\right) + c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j-1]}\right) + \left(-\xi_{\widehat{\Delta}}^{[j]} + \xi_{\widehat{\Delta}}^{[j+1]} - c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j]}\right)\right) \\ &= c_1\left(\mathcal{L}_{\widehat{\Delta}}^{[j-1]}\right) + \xi_{\widehat{\Delta}}^{[j-1]} - \xi_{\widehat{\Delta}}^{[j]}. \end{aligned}$$

The case $j = i$ for the first claim about pulling back $\xi_\Gamma^{[j]}$ follows directly from the definition of $\mathcal{O}_\Gamma^{[j]}(-1)$ by local generators. Alternatively one can compute it by applying the relation (62). If the chosen marked point is supported on the $j$-th level of $\widehat{\Delta}$, the calculation is straightforward. If the marked point $h$ is supported on the $(j-1)$st level of $\widehat{\Delta}$, then $\widehat{\Delta}$ appears among the boundary terms of (62). Pulling back makes the normal bundle appear, and thus $\xi_\Delta^{[i]}$ in the formula from Theorem 7.1. The remaining boundary terms of (62) can be grouped into those where $h$ ends up at level $j-1$ or $j-2$ after pulling back to $\widehat{\Delta}$. These groups cancel with the remaining two terms of the normal bundle. ☐

As a consequence of the preceding lemma and Theorem 7.1 we obtain:

**Corollary 7.7.** *Let $\Gamma \in \mathrm{LG}_L(B)$ and let $\widehat{\Delta}$ be a codimension one degeneration of the $(-i+1)$-level of $\Gamma$, i.e. such that $\Gamma = \delta_i^{\complement}(\widehat{\Delta})$, for some $i \in \{1, \ldots, L+1\}$. Then*

$$\mathrm{j}^*_{\widehat{\Delta},\Gamma}\left(\ell_{\Gamma,j}\,\mathrm{c}_1\big(\mathcal{N}_{\Gamma/\delta_j^{\complement}(\Gamma)}\big)\right) = \begin{cases} \ell_{\widehat{\Delta},j}\ \ \mathrm{c}_1\left(\mathcal{N}_{\widehat{\Delta}/\delta_j^{\complement}(\widehat{\Delta})}\right), & \text{for } j < i \\ \ell_{\widehat{\Delta},j+1}\,\mathrm{c}_1\left(\mathcal{N}_{\widehat{\Delta}/\delta_{(j+1)}^{\complement}(\widehat{\Delta})}\right) & \text{otherwise.} \end{cases}$$

## 8. The tautological ring

In this section we give the precise definition of the tautological ring and prove Theorem 1.5. We define the *tautological rings of strata* as the smallest set of $\mathbb{Q}$-subalgebras $R^\bullet(\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subset \mathrm{CH}^\bullet(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ which

- contains the $\psi$-classes attached to the marked points,
- is closed under the pushfoward of the map forgetting a regular marked point (a zero of order zero), and
- is closed under the maps $\zeta_{\Gamma,*}p^{[i],*}$ defined in Proposition 4.4 for all level graphs $\Gamma$.

Our goal is to provide additive generators of this ring and show that the main players, normal bundles and the logarithmic cotangent bundle have Chern classes in this ring. The main tool is the excess intersection formula that allows to compute the intersection product of boundary strata, possibly decorated with $\psi$-classes.

In fact, there are two definitions of other (refined) tautological rings. One option is the refined ring $R^\bullet_{\mathrm{ref}}(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ that is closed under all the clutching morphisms $\zeta^{\mathrm{ref}}_* p^{[i],*}$ that distinguish the components of boundary strata that are reducible due to inequivalent prong-matchings. Obviously, $R^\bullet(\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subseteq R^\bullet_{\mathrm{ref}}(\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subset \mathrm{CH}^\bullet(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$. There is an analog of Theorem 1.5, replacing in the additive generators the inclusion maps $\mathfrak{i}_\Gamma$ of reducible boundary strata by the inclusion maps of irreducible components. The proofs below can be adapted to that setting.

The second option is to include $D_{\mathrm{h}}$ or equivalently clutching morphism for horizontal nodes into the definition of the tautological ring $R^\bullet_{\mathrm{h}}(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ (and not distinguishing inequivalent prong-matchings, although one could obviously do both). Obviously $R^\bullet(\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subseteq R^\bullet_{\mathrm{h}}(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$.

In order to express $c_1(\Omega_{\overline{B}})$ we need $D_{\mathrm{h}}$, so we need to work in $R^\bullet_{\mathrm{h}}(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$. However, one of the main points of this section is that the Chern polynomial of the logarithmic cotangent bundle belongs to the smallest of the natural candidates for a tautological ring. It seems interesting to decide which of the two inclusions of tautological rings defined above are strict.

8.1. **Excess intersection formula.** Suppose we are given two level graphs $\Lambda_1$ and $\Lambda_2$ without horizontal nodes and the corresponding inclusion maps $\mathfrak{i}_{\Lambda_j}\colon D_{\Lambda_j} \to \Xi\overline{\mathcal{M}}_{g,n}(\mu)$ into a compactified stratum. For a class $\alpha \in \mathrm{CH}^\bullet(D_{\Lambda_2})$, we want to compute $\mathfrak{i}^*_{\Lambda_1}\mathfrak{i}_{\Lambda_2,*}\alpha$ as the push-forward from the maximal-dimensional boundary strata in the support of $D_{\Lambda_1} \cap D_{\Lambda_2}$, in terms of an $\alpha$-pullback and normal bundle classes encoding the excess intersection of $D_{\Lambda_1}$ and $D_{\Lambda_2}$. We say that a level graph $\Pi$ is *a* $(\Lambda_1,\Lambda_2)$*-graph* if there are undegeneration morphisms $\rho_i\colon \Pi \to \Lambda_i$, i.e. edge contraction morphisms with the property that there are subsets $I_{\Lambda_1}$ and $I_{\Lambda_2}$ of level passages of $\Pi$ such that $\delta_{I_{\Lambda_1}}(\Pi) = \Lambda_1$ and $\delta_{I_{\Lambda_2}}(\Pi) = \Lambda_2$. (Automorphisms of $\Lambda_i$, i.e. the stack structure of $D_{\Lambda_i}$ stemming from permuting the edges requires the distinction between $\delta$'s and the $\rho_i$'s.) We call $\Pi$ *a generic* $(\Lambda_1,\Lambda_2)$*-graph*, if $I^{\complement}_{\Lambda_1} \cap I^{\complement}_{\Lambda_2} = \emptyset$. The intersection formula will use the inclusion maps as indicated in the diagram

$$\begin{tikzcd}
D_\Pi \arrow[r, "\mathfrak{j}_{\Pi,\Lambda_2}"] \arrow[d, "\mathfrak{j}_{\Pi,\Lambda_1}"] & D_{\Lambda_2} \arrow[d, "\mathfrak{i}_{\Lambda_2}"] \\
D_{\Lambda_1} \arrow[r, "\mathfrak{i}_{\Lambda_1}"] & \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)
\end{tikzcd}$$

**Proposition 8.1.** *For any $\alpha \in \mathrm{CH}^\bullet(D_{\Lambda_2})$ we can express its push-forward pulled back to $\Lambda_1$ as*

$$\mathfrak{i}^*_{\Lambda_1}\mathfrak{i}_{\Lambda_2,*}\alpha \;=\; \sum_{\Pi} \mathfrak{j}_{\Pi,\Lambda_1,*}\left(\nu^{\Pi}_{\Lambda_1\cap\Lambda_2} \cdot \mathfrak{j}^*_{\Pi,\Lambda_2}\alpha\right), \tag{60}$$

*where the sum is over all generic $(\Lambda_1,\Lambda_2)$-graphs $\Pi$. In this expression*

$$\nu^{\Pi}_{\Lambda_1\cap\Lambda_2} = \prod_{k\in I_{\Lambda_1}\cap I_{\Lambda_2}} \mathfrak{j}^*_{\Pi,\delta_k(\Pi)}\left(\nu_{\delta_k(\Pi)}\right)$$

*is the product of the pull-back to $D_\Pi$ of the first Chern classes of the normal bundles of the divisors containing both $D_{\Lambda_1}$ and $D_{\Lambda_2}$.*

*Proof.* By the excess intersection formula ([Ful98, Proposition 17.4.1]) we have to show that the fiber product $\mathcal{F}_{\Lambda_1,\Lambda_2} = D_{\Lambda_1} \times_{\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)} D_{\Lambda_1}$ is the coproduct

$\mathcal{D} = \coprod D_\Pi$ over all generic $(\Lambda_1, \Lambda_2)$-graphs $\Pi$ and to identify the excess normal bundle.

First we define a map $\varphi : \mathcal{D} \to \mathcal{F}_{\Lambda_1,\Lambda_2}$ via the universal properties of the coproduct and the fiber product. It is the map induced by the inclusions $\mathrm{j}_{\Pi,\lambda_i} : D_\Pi \to D_{\Lambda_i}$, for each generic $(\Lambda_1, \Lambda_2)$-graph $\Pi$.

To give a converse natural transformation on objects we take a family parameterized by $\mathcal{F}_{\Lambda_1,\Lambda_2}$, i.e. a pair of a family $(\mathcal{X}_1, \boldsymbol{\eta}_1)$ of multi-scale differentials compatible with an undegeneration of $\Lambda_1$ and a family $(\mathcal{X}_2, \boldsymbol{\eta}_2)$ compatible with an undegeneration of $\Lambda_2$. If we forget the differentials, we can construct a family of pointed stable curves $(\mathcal{X}, \mathbf{z})$ over some stable graph $\Pi$, which is generic as a $(\Lambda_1, \Lambda_2)$-stable graph (see [GP03] or [SZ18]). We make $\Pi$ into a level graph by declaring a vertex $v_1$ to be on top of $v_2$ if this holds for either of their images in $\Lambda_1$ or in $\Lambda_2$. Compatibility of the fiber product ensures that this definition is consistent. This definition moreover ensures that $\Pi$ is $(\Lambda_1, \Lambda_2)$-generic in our sense of enhanced level graphs. The construction of $\mathcal{X}$ moreover exhibits a bijection of its $f$-relative components (relative to the structure morphism $f$ to the base) with the $f$-relative components of $\mathcal{X}_1$ (and also those of $\mathcal{X}_2$). We can thus pull back the differential $\eta_1$ on each of those components of $\mathcal{X}$ (or we could pull back $\eta_2$) to a collection of differentials $\boldsymbol{\eta}$ on $\mathcal{X}$. To see that this indeed defines a twisted differential compatible with $\Pi$, only the global residue conditions requires a non-trivial verification. By definition of $(\Lambda_1, \Lambda_2)$-genericity and because of the unique ordering of profiles shown in Proposition 5.1, for each level $-i$ of $\Pi$ there is an index $j \in \{1, 2\}$ and a level $-i'$ of $\Lambda_j$ such that the connected components of the subgraph of $\Pi$ above level $-i$ are in natural bijection with the connected components of the subgraph of $\Lambda_j$ above level $-i'$. This implies the global residue condition. The enhancements of the edges $\Pi$ are given by the identification of the edges with those of $\Lambda_1$ and $\Lambda_2$ in the first step of the converse construction. In the same way we provide $(\mathcal{X}, \mathbf{z}, \boldsymbol{\eta})$ with a collection of prong-matchings and pull back the rescaling ensembles as in [BCGGM3, Section 7] to complete the construction of a family of multi-scale differentials compatible with an undegeneration of $\Pi$. The converse natural transformation on morphisms is simply the map constructed for families of pointed stable curves.

The excess normal bundle is in general given by $E = \mathrm{j}^*_{\Pi,\Lambda_1}\mathcal{N}_{\Lambda_1}/\mathcal{N}_{\Pi,\Lambda_2}$, where the normal sheaves appearing are the normal sheaves of the morphisms $\mathrm{i}_{\Lambda_1}$ and $\mathrm{j}_{\Pi,\Lambda_2}$. Since by Proposition 5.1 the non-horizontal boundary strata are smooth and simple normal crossing, the previous normal sheaves are vector bundles and they are given as the direct sum of the pull-back of the normal bundles of appropriate divisors. More specifically $\mathcal{N}_{\Lambda_1} = \oplus_{i=1}^{L(\Lambda_1)} \mathcal{N}_{\delta_i(\Lambda_1)}$ and $\mathcal{N}_{\Pi,\Lambda_2} = \oplus_{i \in I^{\complement}_{\Lambda_2}} \mathcal{N}_{\delta_i(\Pi)}$. This implies that $E$ is the direct sum of the the normal bundles of the levels common to both $\Lambda_1$ and $\Lambda_2$ (pulled back to $D_\Pi$) and thus its top Chern class is as claimed in the proposition. $\square$

At the expense of introducing more notation, the excess intersection formula can be generalized in two ways. First, the ambient space might be a boundary stratum associated to a codimension $L$-level graph $\Gamma$, as summarized in the diagram

$$\begin{array}{ccc} D_\Pi & \xrightarrow{\mathrm{j}_{\Pi,\Lambda_2}} & D_{\Lambda_2} \\ \big\downarrow{\scriptstyle \mathrm{j}_{\Pi,\Lambda_1}} & & \big\downarrow{\scriptstyle \mathrm{j}_{\Lambda_2,\Gamma}} \\ D_{\Lambda_1} & \xrightarrow{\mathrm{j}_{\Lambda_1,\Gamma}} & D_\Gamma \end{array}$$

of inclusions. In this situation we define $\nu^{\Pi}_{(\Lambda_1\cap\Lambda_2)/\Gamma}$ to be the product of the pull-back to $\Pi$ of the Chern classes of the normal bundles $\mathcal{N}_{\Gamma'/\Gamma}$, where $\Gamma'$ ranges over all codimension 1 non-horizontal degenerations $\Gamma'$ of $\Gamma$ that are common to $\Lambda_1$ and $\Lambda_2$. As above, we denote appropriate pullbacks of this product by the same letter. The excess intersection formula then reads

$$\mathrm{j}^*_{\Lambda_1,\Gamma}\mathrm{j}_{\Lambda_2,\Gamma*}\alpha \;=\; \sum_{\Pi} \mathrm{j}_{\Pi,\Lambda_1,*}\Big(\nu^{\Pi}_{(\Lambda_1\cap\Lambda_2)/\Gamma}\cdot \mathrm{j}^*_{\Pi,\Lambda_2}\alpha\Big)\,, \tag{61}$$

where the sum ranges over all $(\Lambda_1,\Lambda_2)$-graphs $\Pi$.

In the more general case that the level graphs $\Lambda_i$ also have horizontal nodes, there is an obvious generalization of this proposition. A general undegeneration of boundary graphs is given by a pair $\delta = (\delta_{\mathrm{ver}},\delta_{\mathrm{hor}})$ consisting of a level undegeneration $\delta_{\mathrm{ver}}$ as in Section 3.3 and an undegeneration of horizontal nodes $\delta_{\mathrm{hor}}$. One defines $\Pi$ to be a $(\Lambda_1,\Lambda_2)$-graph if there are undegenerations $\delta_i$ such that $\delta_i(\Pi)=\Lambda_i$, for $i=1,2$. Such a a graph is generic if the vertical undegenerations are generic as above and, moreover, if the horizontal contractions are generic in the usual sense of $\overline{\mathcal{M}}_g$ (see [GP03] or [ACG11, Chapter XVII]). We leave it to the reader to adapt the previous proposition and the subsequent argument to the general context.

8.2. **Relations in the tautological ring and the proof of Theorem 1.5.** Before concluding the proof of Theorem 1.5 we need some relations in the tautological ring. These relations are essentially known, but we restate them here for convenience and to justify a version for the spaces $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$, i.e. possibly disconnected, with residue conditions, and for multi-scale differentials rather than on the incidence variety compactification. Recall the notation of Section 4.1 for generalized strata, where the $(i,j)$-th marked point is the $j$-th marked point of the $i$-th surface and has order $m_{i,j}\in\mathbb{Z}$.

**Proposition 8.2** ([Sau18, Theorem 6(1)]). *The class $\xi$ on $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ can be expressed using the $\psi$-class at the $(i,j)$-the marked point as*

$$\xi \;=\; (m_{i,j}+1)\psi_{(i,j)} \;-\; \sum_{\Gamma\in\, _{(i,j)}\mathrm{LG}_1(\overline{B})} \ell_\Gamma [D_\Gamma] \tag{62}$$

*where $_{(i,j)}\mathrm{LG}_1(\overline{B})$ are two-level graphs with the leg $(i,j)$ on lower level.*

The fact that our $D_\Gamma$ record prong-matching equivalence classes makes up for the difference between our formula and the one appearing in [Sau18], since the $\kappa_\Gamma$-factor appearing in loc. cit. become $\ell_\Gamma = \kappa_\Gamma/g_\Gamma$ in our formula.

*Proof.* We expand the argument given in [Che19, Proposition 2.1] including the boundary terms. Let $\pi:\mathcal{X}\to\overline{B}$ be the universal family and $S_i$ be the image of the section given by the $i$-th marked point. The evaluation map gives an isomorphism of $\pi^*\mathcal{O}(-1)$ and $\omega_{\mathcal{X}/\overline{B}}$ outside the locus $S_i$ and the lower level components of the boundary divisors. Consider the construction of the universal differential over $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ in [BCGGM3, Section 12], in particular in the plumbing fixture (12.6) of loc. cit. The difference of $t$-powers at the two branches is just $\ell_\Gamma$ in our notation, and all this is unchanged in the presence of a GRC $\mathfrak{R}$. We we deduce that

$$\pi^*\xi \;=\; c_1(\omega_{\mathcal{X}/\overline{B}}) - \sum_{i=1}^{n} m_i S_i - \sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \ell_\Gamma [\mathcal{X}_\Gamma^\perp]\,, \tag{63}$$

where $\mathcal{X}_\Gamma^\perp$ is the lower level component in the universal family over the divisor $D_\Gamma$. We intersect both sides with $S_i$ and apply $\pi_*$. Using $\pi_*(S_i^2) = -\psi_i$ and $\pi_*(\omega_{\mathcal{X}/\overline{B}}\cdot S_i) = \psi_i$, this gives the claim. □

We need a similar generalization of another relation of Sauvaget to our framework that will be needed for the final evaluation of top degree classes (see the end of Section 9 and [CMZ20])). Consider a generalized stratum defined by a residue condition $\mathfrak{R}$ as defined in Section 4.1. Suppose we remove one element from the set $\lambda_\mathfrak{R}$ constraining the residues in the definition of $\mathfrak{R}$. We denote this new set by $\lambda_{\mathfrak{R}_0}$ and $\mathfrak{R}_0$ the new set of residue conditions. Two cases might occur. Either $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu}) = \mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}_0}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ or $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu}) \subsetneq \mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}_0}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ is a divisor. We consider the second case here and note that this condition is equivalent to $S := R\cap\mathfrak{R} \subset S_0 := R\cap\mathfrak{R}_0$ is codimension one (rather than the two being equal), where $R$ is the space of residues defined in (25). Consider now a boundary stratum $D_\Gamma$ in $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}_0}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$. For each level $i$ of $D_\Gamma$ and any GRC $\mathfrak{R}$ containing $\mathfrak{R}_0$, we define the *residue condition* $\mathfrak{R}^{[i]}$ *induced by* $\mathfrak{R}$ to be the residue condition given at level $i$ by the auxiliary level graph $\widetilde{\Gamma}_\mathfrak{R}$ as defined in Section 4.1, created with the help of the auxiliary vertices of $\mathfrak{R}$. For the top level we write $\mathfrak{R}^\top$ for the induced residue condition on top level. It can be simply computed by discarding from the parts $\lambda_\mathfrak{R}$ all indices of edges that go to lower level in $D_\Gamma$.

**Proposition 8.3** ([Sau18, Proposition 7.6])**.** *The class of the stratum* $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ *with residue condition* $\mathfrak{R}$ *compares inside Chow ring of the generalized stratum* $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}_0}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ *to the class* $\xi$ *by the formula*

$$[\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})] \;=\; -\xi \;-\sum_{\Gamma\in {\mathrm{LG}^{\mathfrak{R}}}_1(\overline{B})} \ell_\Gamma[D_\Gamma] \;-\sum_{\Gamma\in \mathrm{LG}_{1,\mathfrak{R}}(\overline{B})} \ell_\Gamma[D_\Gamma]\,, \tag{64}$$

*where* $\mathrm{LG}_1^{\mathfrak{R}}(\overline{B})$ *are two-level graphs with* $R_\Gamma\cap\mathfrak{R}^\top = R_\Gamma\cap\mathfrak{R}_0^\top$*, i.e., where the GRC on top level induced by* $\mathfrak{R}$ *does no longer introduce an extra condition and where* $\mathrm{LG}_{1,\mathfrak{R}}(\overline{B})$ *are two-level graphs where all the legs involved in the condition forming* $\mathfrak{R}\setminus\mathfrak{R}_0$ *go to lower level.*

*Proof.* Consider that map $s : \mathcal{O}_{\overline{B}}(-1) \to S_0/S$ to the constant rank one vector bundle, mapping a points $(X,\omega)$ to the (equivalence class mod $S$ of the) tuple of residues of $\omega$, which defines point in $S_0$. The vanishing locus in the interior of $\overline{B}$ is by definition $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$, and in usual period coordinates we see that the vanishing order is one there. To understand the boundary contribution, consider first boundary divisors neither in $\mathrm{LG}_1^{\mathfrak{R}}(\overline{B})$ nor in $\mathrm{LG}_{1,\mathfrak{R}}(\overline{B})$. For those, being in the vanishing locus of $s$ is a non-trivial (divisorial) condition, and thus this locus is of codimension two and irrelevant for the equation. It remains to justify the vanishing statement and the vanishing order for the other divisors. Any section of $\mathcal{O}_{\overline{B}}(-1)$ decays like $t_1^{\ell_\Gamma}$ near lower level components by construction of the compactification in [BCGGM3, Section 12], where $t_1$ is a transversal coordinate. Consequently, any $D_\Gamma \in \mathrm{LG}_{1,\mathfrak{R}}(\overline{B})$ is in the support of the cokernel of the map $s$, with multiplicity $\ell_\Gamma$.

For $D_\Gamma$ in $\Gamma \in \mathrm{LG}_1^{\mathfrak{R}}(\overline{B})$ the residues at the poles going to level zero are zero (mod $S$) all along $D_\Gamma$ by definition. Transversally, they become non-zero with the growth of the modification differential (see the construction in [BCGGM3, Section 11]), since the modification differential must be generically non-zero on $D_\Gamma$ if $\mathfrak{R}$ imposes a non-trivial condition generically on the stratum, but none along $D_\Gamma$. Since the modification differential scales with $t_1^{\ell_\Gamma}$ this proves the claim on the multiplicity of $D_\Gamma$ in this case, too. □

We are now ready to prove that the tautological ring is finitely generated by the additive generators displayed in Theorem 1.5.

*Proof of Theorem 1.5.* We let $R^\bullet_{fg}(\overline{B})$ be the vector space spanned by the classes $\zeta_{\Gamma *}\big(\prod_{i=0}^{-L(\Gamma)} p_\Gamma^{[i],*}\alpha_i\big)$, where $\alpha_i$ is a monomial in the $\psi$-classes supported on level $i$ of the graph $\Gamma$, where $\Gamma \in \mathrm{LG}(\overline{B})$ ranges among all level graphs without horizontal nodes. Obviously this is a finite dimensional v ector space since for any stratum $\mu$ there are only finitely many level graphs, and for each of them there is a finite number of monomials that give a non-zero class.

By our definition of the tautological ring of the moduli space of multi-scaled differentials, clearly of $R^\bullet_{fg}(\overline{B}) \subseteq R^\bullet(\overline{B})$.

We show now that $R^\bullet_{fg}(\overline{B})$ is actually a subring of the tautological ring, i.e., that it is closed under the intersection product. We prove this by iteratively applying the projection formula and the excess intersection formula (61). In the first step, for any two classes $\alpha_j \in \mathrm{CH}^*(D_{\Lambda_j})$ the projection formula and Proposition 8.1 imply

$$\begin{aligned} \mathrm{i}_{\Lambda_1 *}(\alpha_1)\cdot \mathrm{i}_{\Lambda_2 *}(\alpha_2) &= \sum_{\Pi} \mathrm{i}_{\Lambda_1 *}\Big(\alpha_1 \cdot \mathrm{j}_{\Pi,\Lambda_1,*}\Big(\nu^{\Pi}_{\Lambda_1\cap\Lambda_2}\cdot \mathrm{j}^*_{\Pi,\Lambda_2}\alpha_2\Big)\Big) \\ &= \sum_{\Pi} \mathrm{i}_{\Pi,*}\big(\nu^{\Pi}_{\Lambda_1\cap\Lambda_2}\cdot \mathrm{j}^*_{\Pi\Lambda_1}(\alpha_1)\cdot \mathrm{j}^*_{\Pi,\Lambda_2}(\alpha_2)\big) \end{aligned} \tag{65}$$

where the sums are over all generic $(\Lambda_1, \Lambda_2)$-graphs $\Pi$. The excess intersection class $\nu^{\Pi}_{\Lambda_1\cap\Lambda_2}$ is given by pull-backs of normal bundles of divisors. By repeatedly applying Corollary 7.7, we see that the pull-back of the class of the normal bundle of a divisor is given by the class of the normal bundle of $D_\Pi$ in a codimension one undegeneration. The shape of such a class was computed in (58). By using the compatibility expressed in (4.9) between level-wise tautological line classes and the tautological line classes on the level strata, together with Proposition 8.2, we see that the classes of these normal bundles are given by $\psi$-class contributions and boundary contributions given by codimension one degenerations of $\Pi$. If there are no boundary contributions, then we are done since we obtained an expression in terms of elements of $R^\bullet_{fg}(\overline{B})$ supported on $\Pi$. If this is not the case, we can apply the same projection formula and excess intersection formula argument as before to these boundary contributions. (Now we have to use the more general excess intersection formula (61) with ambient $\Pi$). This process has to terminate since the dimension of the boundary strata appearing in the excess intersection factor is decreasing, so at some point the excess class contribution will be trivial. Hence we have shown that $R^\bullet_{fg}(\overline{B})$ is a subring of the tautological ring.

In order to show that $R^\bullet_{fg}(\overline{B})$ is equal to $R^\bullet(\overline{B})$, we need to show that $R^\bullet_{fg}(\overline{B})$ is closed under push-forward of clutching morphism and under $\pi$-pushforward. The first statement is clear. For the second we argue inductively on the dimension of $\overline{B}$, starting with the obvious case $\dim(\overline{B}) = 0$. We may assume by induction hypothesis that the $\pi$-pushforwards of elements in $R^\bullet_{fg}(\overline{B})$ are in $R^\bullet_{fg}(\pi(\overline{B}))$ for any stratum of dimension less than the dimension of $\overline{B}$.

We first show that $\pi_*(\zeta_{\Gamma,*}\psi_{n+1}^{\ell+1}) \in R^\bullet_{fg}(\overline{B})$ for any graph $\Gamma$ with at least two levels. Let $i$ be the level of $\Gamma$ that contains the $n+1$-st marked point. For $\psi_{n+1}^{\ell+1}$ to be non-zero we need the component containing the $n+1$-st marked point to be positive-dimensional (taking GRC into account). Let $\Gamma'$ be the level graph obtained from $\Gamma$ by forgetting this point. There is thus a well-defined projection map $\pi^{[i]} : B_\Gamma^{[i]} \to B_{\Gamma'}^{[i]}$ of generalized strata. Recalling that $\psi_{n+1} = p_\Gamma^*\psi_{n+1}$ by our general abuse of notation we find $\pi_*(\zeta_{\Gamma,*}\psi_{n+1}^{\ell+1}) = \zeta_{\Gamma',*}p_{\Gamma'}^*\pi_*^{[i]}\psi_{n+1}^{\ell+1}$. By induction we know that $\pi_*^{[i]}\psi_{n+1}^{\ell+1} \in R^\bullet_{fg}(\pi^{[i]}(B_\Gamma^{[i]}))$ and since the collection of rings $R^\bullet_{fg}(\cdot)$ is already known to be stable under $\zeta_{\Gamma',*}p_{\Gamma'}^*$, we conclude that $\pi_*(\zeta_{\Gamma,*}\psi_{n+1}^{\ell+1}) \in R^\bullet_{fg}(\overline{B})$.

Second, in order to treat the case when $\Gamma$ is the trivial graph, we consider $\overline{\mathcal{X}} = \Xi\overline{\mathcal{M}}_{g,n+1}(\mu,0)$, $\overline{B} = \Xi\overline{\mathcal{M}}_{g,n}(\mu,0)$ and the commutative diagram

$$\begin{tikzcd}
\Xi\overline{\mathcal{M}}_{g,n+1}(\mu,0) \arrow[r, "f_{n+1}"] \arrow[d, "\pi"] & \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi_{n+1}"] \\
\Xi\overline{\mathcal{M}}_{g,n}(\mu) \arrow[r, "f_n"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

where $\pi$ and $\pi_{n+1}$ are the maps forgetting the last point and $f_{n+1}$ and $f_n$ are the maps forgetting the twisted differential. These vertical maps are the universal families over their images respectively. Consequently,

$$f_n^*\kappa_\ell \;=\; f_n^*(\pi_{n+1})_*(\psi_{n+1}^{\ell+1}) \;=\; \pi_*(f_{n+1}^*(\psi_{n+1}^{\ell+1})).$$

Recall that we abuse notation and identify $\psi$ and $\kappa$-classes in $\mathrm{CH}^*(\overline{B})$ with their pull-back from $\overline{\mathcal{M}}_{g,n}$. We have thus shown that $\pi_*(\psi_{n+1}^{\ell+1}) = \kappa_\ell$ also holds in $\mathrm{CH}^*(\overline{B})$. We thus only need to show that $\kappa_\ell \in R^\bullet_{fg}(\overline{B})$. As before, the special case of the dilaton equation $\kappa_\ell = \pi_*(\omega_{\mathcal{X}/\overline{B}}^{\ell+1})$ holds also in $\mathrm{CH}^*(\overline{B})$. Recall that $[\mathcal{X}_\Gamma^\perp]$ is the lower level component in the universal family over the divisor $D_\Gamma$. From (63) we deduce that

$$\kappa_\ell \;=\; \pi_*\left(\Big(\pi^*\xi + \sum_{i=1}^n m_i S_i + \sum_{\Gamma\in\mathrm{LG}_1(\mathrm{B})} \ell_\Gamma[\mathcal{X}_\Gamma^\perp]\Big)^{\ell+1}\right)$$

is a linear combination of terms of the form $\xi^a\pi_*(S_i^{b_i}\prod[\mathcal{X}_\Gamma^\perp]^{c_\Gamma})$ with $a + b_i + \sum_\Gamma c_\Gamma = \ell+1$, since the sections $S_i$ are disjoint. The $\xi$-powers are tautological by Proposition 8.2 and so we only need to study the $\pi_*$-term. Let $\mathfrak{i} : D_0 := \bigcap_{\Gamma: c_\Gamma>0, i\in\Gamma^\perp} D_\Gamma \to \overline{B}$ be the inclusion of the intersection of boundary divisors where the $i$-th marked point is on the bottom level, which the image of the support $S_i^{b_i}\prod[\mathcal{X}_\Gamma^\perp]^{c_\Gamma}$ under $\pi$. Let $\widetilde{\mathfrak{i}} : \mathcal{X}_0 := \bigcap_{\Gamma: c_\Gamma>0, i\in\Gamma^\perp} \mathcal{X}_\Gamma^\perp \to \overline{\mathcal{X}}$ be the corresponding inclusion in the total space of the family. Let $\mathfrak{j}_{0,\Gamma} : D_0 \to D_\Gamma$ and $\widetilde{\mathfrak{j}}_{0,\Gamma} : \mathcal{X}_0 \to \mathcal{X}_\Gamma$ the inclusions into codimension one divisors. Finally let $\sigma_i$ be the section of the $i$-th marked point and abusively also its restriction to $D_\Gamma$ and to $D_0$.

Suppose that $b_i > 0$. Then using $\sigma_i^* S_i^k = (-\psi_i)\sigma_i^*(S_i^{k-1})$ we find

$$\begin{aligned}
\pi_*\Big(S_i^{b_i}\prod_\Gamma[\mathcal{X}_\Gamma^\perp]^{c_\Gamma}\Big) &= \pi_*\sigma_{i,*}\sigma_i^*\Big(S_i^{b_i-1}\cdot\tilde{\mathfrak{i}}_*\big(\prod_\Gamma \tilde{\mathfrak{j}}_{0,\Gamma}^*\mathcal{N}_{\mathcal{X}_\Gamma^\perp}^{c_\Gamma-1}\big)\Big) \\
&= (-\psi_i)^{b_i-1}\cdot\sigma_i^*\Big(\tilde{\mathfrak{i}}_*\big(\prod_\Gamma \tilde{\mathfrak{j}}_{0,\Gamma}^*\mathcal{N}_{\mathcal{X}_\Gamma^\perp}^{c_\Gamma-1}\big)\Big) \\
&= (-\psi_i)^{b_i-1}\cdot\mathfrak{i}_*\Big(\prod_\Gamma \sigma_i^*\big(\tilde{\mathfrak{j}}_{0,\Gamma}^*\mathcal{N}_{\mathcal{X}_\Gamma^\perp}^{c_\Gamma-1}\big)\Big) \\
&= (-\psi_i)^{b_i-1}\cdot\mathfrak{i}_*\Big(\prod_\Gamma \mathfrak{j}_{0,\Gamma}^*\mathcal{N}_\Gamma^{c_\Gamma-1}\Big),
\end{aligned}$$

which is in $R^\bullet_{fg}(\overline{B})$ by Theorem 7.1. If $b_i = 0$ the expression $\pi_*(\prod_\Gamma[\mathcal{X}_\Gamma^\perp]^{c_\Gamma})$ is the $\pi_*$-pushforward of a sum of tautological generators supported on non-trivial boundary strata and we have already shown before that they belong to $R^\bullet_{fg}(\overline{B})$.

Since we have shown that $R^\bullet_{fg}(\overline{B})$ is a subring of the tautological ring closed under clutching and $\pi$-pushforward, it has to be the same as the tautological ring by minimality.

We finally show the last statement of the theorem, namely that the $\mathfrak{i}_{\Gamma*}$ of the level-wise tautological classes $\xi_\Gamma^{[i]}$ and the $\kappa$-classes are tautological. For the $\xi$-classes, it is enough to notice that by Proposition 8.2 the class $\xi_{B_\Gamma^{[i]}}$ can be expressed as a linear combination of a $\psi$-class and boundary classes, so it is tautological on $B_\Gamma^{[i]}$ by the main statement of the theorem that we just proved. Since the tautological rings are closed under clutching morphisms, also the class $\zeta_{\Gamma_*}p_\Gamma^{[i],*}\xi_{B_\Gamma^{[i]}}$ is tautological. Notice that this is, up to constant, the same as $\mathfrak{i}_{\Gamma*}(\xi_\Gamma^{[i]})$. Finally, the $\kappa$-classes are tautological since we have previously shown that they belong to $R^\bullet_{fg}(\overline{B})$, which we have proven to be the same as the tautological ring. □

## 9. The Chern classes of the logarithmic cotangent bundle

In this section we relate the logarithmic cotangent bundle to bundles whose Chern classes can be expressed in standard generators. We will first prove in Theorem 9.2, a restatement of Theorem 1.4. We will then complete the proofs of the remaining main theorems of the introduction, Theorem 1.2 and Theorem 1.3.

The first step is a direct consequence of the Euler sequence (37).

**Corollary 9.1.** *The Chern character and the Chern polynomial of the kernel $\mathcal{K}$ of the Euler sequence are given by*

$$\mathrm{ch}(\mathcal{K}) \;=\; Ne^{\xi}-1 \quad \textit{and} \quad \mathrm{c}(\mathcal{K}) \;=\; \sum_{i=0}^{N-1}\binom{N}{i}\xi^i\,.$$

*Proof.* The result follows from the properties of the Chern character and the Chern polynomial, together with the fact that all higher Chern classes of the Deligne extension $\overline{\mathcal{H}}^1_{\mathrm{rel}}$ vanish. Indeed the Chern classes of a logarithmic sheaf are given in terms of symmetric polynomials of residues of the logarithmic connection (see [EV86, B3]) and the Deligne extension is defined such that all these terms are zero, since the residues are given by nilpotent matrices. (See also the discussion around [ACG11, Theorem 17.5.21].) □

The second step relates the kernel of the Euler sequence to the vector bundle we are actually interested in. We will use the abbreviations

$$\mathcal{E}_B \;=\; \Omega^1_{\overline{B}}(\log D) \quad \text{and} \quad \mathcal{L}_B \;=\; \mathcal{O}_{\overline{B}}\Big(\sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \ell_\Gamma D_\Gamma\Big) \tag{66}$$

throughout in the sequel.

**Theorem 9.2.** *There is a short exact sequence of quasi-coherent $\mathcal{O}_{\overline{B}}$-modules*

$$0 \longrightarrow \mathcal{E}_B \otimes \mathcal{L}_B^{-1} \to \mathcal{K} \to \mathcal{C} \longrightarrow 0 \tag{67}$$

*where $\mathcal{C} = \bigoplus_{\Gamma\in\mathrm{LG}_1(\mathrm{B})} \mathcal{C}_\Gamma$ is a coherent sheaf supported on the non-horizontal boundary divisors, whose precise form is given in Lemma 9.4 below.*

*Proof.* We start analyzing the injection claimed in (67). As in Section 6, all local calculations happen on the finite covering charts of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. At a generic point of a divisor $D_\Gamma$ the vector bundle $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}$ is generated (using the notation of Case 2 of Section 6.2) by $\langle t^\ell d\widetilde{c}_2^{[0]}, \dots, t^\ell d\widetilde{c}_{N_0}^{[0]}, t^\ell dt/t, t^\ell d\widetilde{c}_2^{[-1]}, \dots, t^\ell d\widetilde{c}_{N_1}^{[-1]}\rangle$. It is hence obviously a subbundle of the kernel $\mathcal{K}$ as given in (44). Similarly, at the intersection point of $L$ divisors different from $D_{\mathrm{h}}$, the vector bundle $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}$ is generated by the elements $t_{\lceil L\rceil} d\widetilde{c}_j^{[-i]}$ and $t_{\lceil L\rceil} dt_i/t_i$ for $j = 2, \dots, N_i$ and for $i = 0, \cdots, L$, where recall that $t_{\lceil L\rceil} = \prod_{i=1}^L t_i^{\ell_i}$ was introduced in (48). This is obviously a subbundle of $\mathcal{K}$ as given in proof of Theorem 6.1. In the presence of a horizontal edge, this argument still works, see the form of the cokernel in Case 1 and Case 3 above. The precise form of $\mathcal{C}$ is isolated in several lemmas below. $\square$

To start with the computation of $\mathcal{C}$, we will also need an infinitesimal thickening the of the boundary divisor $D_\Gamma$, namely we define $D_{\Gamma,\bullet}$ to be its $\ell_\Gamma$-th thickening, the non-reduced substack of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ defined by the ideal $\mathcal{I}_{D_\Gamma}^{\ell_\Gamma}$. We will factor the above inclusion using the notation

$$\mathfrak{i}_\Gamma = \mathfrak{i}_{\Gamma,\bullet} \circ j_{\Gamma,\bullet} \colon D_\Gamma \overset{j_{\Gamma,\bullet}}{\hookrightarrow} D_{\Gamma,\bullet} \overset{\mathfrak{i}_{\Gamma,\bullet}}{\hookrightarrow} \overline{B}\,.$$

We need three more bundles. First, we recall from (49) the definition of the line bundle $\mathcal{L}_\Gamma^\top$ and we define $\mathcal{L}_{\Gamma,\bullet}^\top = (j_{\Gamma,\bullet})_* \mathcal{L}_\Gamma^\top$. Second we need the analog of $\mathcal{E}_B$, but as a bundle on $D_\Gamma$. Since the projections are defined only on $D_\Gamma^s$ rather than on $D_\Gamma$ we cannot define this bundle as a $p^\top$-pullback, but we need to define it by local generators. That is, we define $\mathcal{E}_\Gamma^\top$ to be the vector bundle of rank $N_\Gamma^\top - 1$ on $D_\Gamma$ with generators $dc_j^{[0]}$ as $\mathcal{O}_{D_\Gamma}$-module at a generic point of $\Gamma$ with the usual coordinates from (39). At a point where the top level degenerates, into say $k$ levels, it is generated as $\mathcal{O}_{D_\Gamma}$-module by the differentials $dc_j^{[-i]}$ of level-wise periods and by $dt_i/t_i$ for $i = 0, \dots, k-1$. Third, we define $\mathcal{E}_{\Gamma,\bullet}^\top = (j_{\Gamma,\bullet})_*(\mathcal{E}_\Gamma^\top)$.

**Lemma 9.3.** *There is an equality of Chern characters*

$$\mathrm{ch}\Big((\mathfrak{i}_{\Gamma,\bullet})_*(\mathcal{E}_{\Gamma,\bullet}^\top \otimes (\mathcal{L}_{\Gamma,\bullet}^\top)^{-1})\Big) \;=\; \mathrm{ch}\Big((\mathfrak{i}_\Gamma)_*\big(\bigoplus_{j=0}^{\ell_\Gamma-1} \mathcal{N}_\Gamma^{\otimes -j} \otimes \mathcal{E}_\Gamma^\top \otimes (\mathcal{L}_\Gamma^\top)^{-1}\big)\Big)\,.$$

*Proof.* If $\mathcal{F}_\Gamma$ is a vector bundle on $D_\Gamma$ and $\mathcal{F}_{\Gamma,\bullet} = (\mathfrak{i}_{\Gamma,\bullet})_*(\mathcal{F}_\Gamma)$ is its push-forward to the $\ell_\Gamma$-thickening, we consider the exact sequences

$$0 \to \mathcal{I}_{D_\Gamma}^{k+1}\mathcal{F}_{\Gamma,\bullet} \to \mathcal{I}_{D_\Gamma}^{k}\mathcal{F}_{\Gamma,\bullet} \to (j_{\Gamma,\bullet})_*\left(\frac{\mathcal{I}_{D_\Gamma}^{k}}{\mathcal{I}_{D_\Gamma}^{k+1}} \otimes_{\mathcal{O}_D} \mathcal{F}_\Gamma\right) \to 0, \quad k = 0, \dots, \ell_\Gamma - 1.$$

Notice that $\mathcal{I}^{\ell_\Gamma}\mathcal{F}_{\Gamma,\bullet} = 0$.

We specialize to $\mathcal{F}_\Gamma = \mathcal{E}_\Gamma^\top \otimes (\mathcal{L}_\Gamma^\top)^{-1}$ and compute the Chern character of its push-forward to the thickening via the previous sequences. The statement then follows from the identification $\mathcal{I}^k_{D_\Gamma}/\mathcal{I}^{k+1}_{D_\Gamma} = \mathcal{N}_\Gamma^{\otimes -k}$ and from the fact that $(\mathrm{i}_{\Gamma,\bullet})_*$ is exact, since $\mathrm{i}_{\Gamma,\bullet}$ a closed embedding. □

The cokernel of (67) can be described using the bundles we just introduced.

**Lemma 9.4.** *The cokernel of* (67) *is given by*

$$\mathcal{C} \;=\; \bigoplus_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \mathcal{C}_\Gamma \quad \textit{where} \quad \mathcal{C}_\Gamma \;=\; (\mathrm{i}_{\Gamma,\bullet})_*(\mathcal{E}^\top_{\Gamma,\bullet}\otimes(\mathcal{L}^\top_{\Gamma,\bullet})^{-1})\,. \tag{68}$$

*Proof.* Recall that local generators of $\mathcal{K}$ had been given in the proof of Theorem 6.1. At a generic point of the boundary divisor $D_\Gamma$, there is a map of coherent sheaves $\mathcal{K} \to (\mathrm{i}_{\Gamma,\bullet})_*(\mathcal{E}^\top_{\Gamma,\bullet}\otimes(\mathcal{L}^\top_{\Gamma,\bullet})^{-1})$ which is given in terms of the generators (44) by $t^\ell dt/t \mapsto 0$, by $t^\ell d\widetilde{c}_j^{[-1]} \mapsto 0$, and by $d\widetilde{c}_j^{[0]} \mapsto d\widetilde{c}_j^{[0]} \bmod t^\ell$ for all $j$. The kernel of this map is obviously $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}$.

In a neighborhood $U$ of the intersection of $L$ boundary divisors $D_{\Gamma_i}$, labeled so that $\Gamma_i$ is the $i$-th undegeneration, we recall the shorthand notation $t_{\lceil s\rceil} = \prod_{i=1}^s t_i^{\ell_i}$ and we assign for every level $-i \in \{0,\dots,L\}$

$$\begin{aligned} t_{\lceil i\rceil} dt_s/t_s &\mapsto t_{\lceil i\rceil} dt_s/t_s && \bmod t_{i+1}^{\ell_{i+1}}, \quad \in \mathcal{C}_{\Gamma_{i+1}}\\ t_{\lceil i\rceil} d\widetilde{c}_j^{[-s]} &\mapsto t_{\lceil i\rceil} d\widetilde{c}_j^{[-s]} && \bmod t_{i+1}^{\ell_{i+1}}, \quad \in \mathcal{C}_{\Gamma_{i+1}}\\ t_{\lceil i\rceil} dq_k^{[-s]}/q_k^{[-s]} &\mapsto t_{\lceil i\rceil} dq_k^{[-s]}/q_k^{[-s]} && \bmod t_{i+1}^{\ell_{i+1}}, \quad \in \mathcal{C}_{\Gamma_{i+1}} \end{aligned} \tag{69}$$

for all $s = 0,\dots,i$ and all $j$ and $k$. Again, this map is designed so that the kernel is $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}|_U$. A local computation of transition functions shows that these maps glue together. □

*The proof of Theorem 9.2.* is completed by the two preceding lemmas. □

**Proposition 9.5.** *The Chern character of the twisted logarithmic cotangent bundle $\mathcal{E}_B\otimes\mathcal{L}_B^{-1}$ can be expressed in terms of the twisted logarithmic cotangent bundles of the top levels of non-horizontal divisors as*

$$\mathrm{ch}(\mathcal{E}_B\otimes\mathcal{L}_B^{-1}) \;=\; Ne^\xi - 1 \;-\; \sum_{\Gamma\in\mathrm{LG}_1(\mathrm{B})} \mathrm{i}_{\Gamma*}\left(\mathrm{ch}(\mathcal{E}_\Gamma^\top)\cdot \mathrm{ch}(\mathcal{L}_\Gamma^\top)^{-1}\cdot \frac{(1-e^{-\ell_\Gamma \mathrm{c}_1(\mathcal{N}_\Gamma)})}{\mathrm{c}_1(\mathcal{N}_\Gamma)}\right).$$

*Proof.* First, by Corollary 9.1 we have $\mathrm{ch}(\mathcal{K}) = Ne^\xi - 1$. Second, from the sequence (67) we get

$$\mathrm{ch}(\mathcal{E}_B\otimes\mathcal{L}_B^{-1}) \;=\; \mathrm{ch}(\mathcal{K}) - \mathrm{ch}(\mathcal{C}). \tag{70}$$

From the additivity of the Chern character we get $\mathrm{ch}(\mathcal{C}_\Gamma) = \oplus_{\Gamma\in\mathrm{LG}_1(\mathrm{B})}\mathrm{ch}(\mathcal{C}_\Gamma)$. We now aim to apply Lemma 9.3 and the Grothendieck-Riemann-Roch Theorem (10) to the map $f = \mathrm{i}_\Gamma$, a smooth embedding. The contribution of the Todd classes simplifies, since the normal bundle exact sequence

$$0 \to \mathcal{T}_{D_\Gamma} \to \mathrm{i}_\Gamma^*\mathcal{T}_{\overline{B}} \to \mathcal{N}_\Gamma \to 0$$

implies $\mathrm{td}(\mathcal{T}_{D_\Gamma}) \cdot \mathrm{td}(\mathcal{N}_\Gamma) = \mathrm{td}(\mathfrak{i}_\Gamma^* \mathcal{T}_{\overline{B}}) = \mathfrak{i}_\Gamma^* \,\mathrm{td}(\mathcal{T}_{\overline{B}})$. If $\mathcal{F}_\Gamma$ is a vector bundle on $D_\Gamma$, we can thus simplify (10) and get

$$\begin{aligned}\mathrm{ch}(\mathfrak{i}_{\Gamma,*}\mathcal{F}_\Gamma) \;&=\; \mathfrak{i}_{\Gamma,*}(\mathrm{ch}(\mathcal{F}_\Gamma)\cdot \mathrm{td}(\mathcal{T}_{D_\Gamma}))\cdot \mathrm{td}(\mathcal{T}_{\overline{B}})^{-1} \;=\; \mathfrak{i}_{\Gamma,*}(\mathrm{ch}(\mathcal{C})\cdot \mathrm{td}(\mathcal{T}_{D_\Gamma})\cdot \mathfrak{i}_\Gamma^*\,\mathrm{td}(\mathcal{T}_{\overline{B}})^{-1})\\ &=\; \mathfrak{i}_{\Gamma,*}(\mathrm{ch}(\mathcal{F}_\Gamma)\cdot \mathrm{td}(\mathcal{N}_\Gamma)^{-1})\,.\end{aligned}$$

Using the previous remark and Lemma 9.3 we get

$$\begin{aligned}\mathrm{ch}(\mathcal{C}_\Gamma) \;&=\; (\mathfrak{i}_\Gamma)_*\Big(\sum_{j=0}^{\ell_\Gamma-1} \mathrm{ch}(\mathcal{E}_\Gamma^\top)\cdot \mathrm{ch}(\mathcal{L}_\Gamma^\top)^{-1}\cdot \mathrm{ch}(\mathcal{N}_\Gamma)^{-j}\,\mathrm{td}\,([\mathcal{N}_\Gamma])^{-1}\Big)\,.\\ &=\; \sum_{j=0}^{\ell_\Gamma-1} \mathfrak{i}_{\Gamma*}\left(\mathrm{ch}(\mathcal{E}_\Gamma^\top)\cdot \mathrm{ch}(\mathcal{L}_\Gamma^\top)^{-1}\cdot \frac{e^{-j\,\mathrm{c}_1(\mathcal{N}_\Gamma)}(1-e^{-\,\mathrm{c}_1(\mathcal{N}_\Gamma)})}{\mathrm{c}_1(\mathcal{N}_\Gamma)}\right).\end{aligned}$$

Canceling terms in the telescoping sum and substituting back the previous expression in (70) gives the proposition. □

From this proposition we get some concrete expansions.

*Proof of Theorem 1.1.* Since the first Chern character is the same as the first Chern class, by extracting the first degree parts from the expression given in Proposition 9.5 we compute the left hand side to be

$$\mathrm{ch}_1(\mathcal{E}_B \otimes \mathcal{L}_B^{-1}) \;=\; \mathrm{c}_1(\mathcal{E}_B) + (N-1)\sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \ell_\Gamma [D_\Gamma]$$

and the right hand side to be

$$N\xi - \sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \ell_\Gamma \mathfrak{i}_{\Gamma,*}((N_\Gamma^\top - 1)[1_{D_\Gamma}]) \;=\; N\xi - \sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \ell_\Gamma (N_\Gamma^\top - 1)[D_\Gamma]\,.$$

By comparing the two expressions, we get the claim. □

In order to translate Proposition 9.5 into a formula that can be recursively evaluated, we compare the bundle $\mathcal{E}_\Gamma^\top$ to the analogous object

$$\mathcal{E}_{B_\Gamma^\top} \;=\; \Omega^1_{B_\Gamma^\top}(\log D_{B_\Gamma^\top})$$

on the top level of the divisor $D_\Gamma$ for $\Gamma \in \mathrm{LG}_1(\mathrm{B})$, where $D_{B_\Gamma^\top}$ is the total boundary of the generalized stratum $B_\Gamma^\top$, including the horizontal divisor.

**Lemma 9.6.** *We have*

$$p_\Gamma^{\top,*}\,\mathcal{E}_{B_\Gamma^\top} \;=\; c_\Gamma^*\,\mathcal{E}_\Gamma^\top\,. \tag{71}$$

*Proof.* The statement can be checked on the local generators. Indeed recall that the generators of $\mathcal{E}_\Gamma^\top$ as introduced before Lemma 9.3 are $dc_j^{[0]}$ at a generic point of $D_\Gamma$, and $dc_j^{[-i]}$ and $dt_i/t_i$ for $i = 0, \dots, k-1$. Note that even though the map $c_\Gamma$ is branched at the preimage of $\{t_i = 0\}$, say given by $\{\tilde{t}_i = 0\}$, the pullback of the standard generators $dt_i/t_i$ of the log cotangent bundle are proportional to the standard generators $d\tilde{t}_i/\tilde{t}_i$. We can apply the same argument for the finite degree map $p^\top \times p^\perp$, and check that the pull-back of the local generators of $\mathcal{E}_{B_\Gamma^\top}$ coincide with the previous ones. □

For the inductive proof we introduce the following shorthand notation. Let

$$P_B \;=\; \mathrm{ch}(\mathcal{E}_B) \prod_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} e^{-\ell_\Gamma[D_\Gamma]} \quad \text{and} \quad P_{B_\Gamma^\top} \;=\; \mathrm{ch}(\mathcal{E}_{B_\Gamma^\top}) \prod_{\Delta\in \mathrm{LG}_1(B_\Gamma^\top)} e^{-\ell_\Delta[D_\Delta]}$$

be the Chern characters of the logarithmic cotangent bundles twisted by a boundary contribution and let

$$P_\Gamma^\top \;=\; \mathrm{ch}(\mathcal{E}_\Gamma^\top)\cdot \mathrm{ch}(\mathcal{L}_\Gamma^\top)^{-1} \;=\; \mathrm{ch}(\mathcal{E}_\Gamma^\top) \prod_{\Gamma \overset{[0]}{\rightsquigarrow} \widehat{\Delta}} e^{-\ell_{\Delta,1}[D_\Delta]}\,. \tag{72}$$

In these terms, Proposition 9.5 reads

$$P_B \;=\; (Ne^\xi - 1) \;-\; \sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \mathfrak{i}_{\Gamma *}\left(\ell_\Gamma P_\Gamma^\top\, \mathrm{td}(\mathcal{N}_\Gamma^{\otimes \ell_\Gamma})^{-1}\right)\,. \tag{73}$$

We set $\delta_{L+1}(\Gamma) = \{\cdot\}$, the only graph with one level corresponding to the open stratum $B$, for $\Gamma \in \mathrm{LG}_L(B)$, to make boundary terms well-defined in the sequel. In particular $N^\top_{\delta_{L+1}}(\Gamma) = N$.

**Proposition 9.7.** *The twisted Chern character $P_B$ is given by*

$$P_B \;=\; \sum_{L=0}^{N-1} \sum_{\Gamma\in \mathrm{LG}_L(B)} \left(N^T_{\delta_1(\Gamma)} e^{\xi_B} - 1\right) \mathfrak{i}_{\Gamma *}\left(\prod_{i=1}^{L} -\ell_{\Gamma,i}\, \mathrm{td}\left(\mathcal{N}^{\otimes \ell_{\Gamma,i}}_{\Gamma/\delta_i^{\complement}(\Gamma)}\right)^{-1}\right)\,. \tag{74}$$

*Proof.* We prove the formula by induction. For one-dimensional strata ($N = 2$) the formula is (73), since $P_\Gamma^\top$ is trivial then. We claim that by induction hypothesis

$$P_\Gamma^\top \;=\; \sum_{L=0}^{N-2} \sum_{\substack{\widehat{\Delta}\in \mathrm{LG}_{L+1}(B)\\ \delta_{L+1}(\widehat{\Delta})=\Gamma}} \left(N^T_{\delta_1(\Gamma)} e^{\xi_B|_{D_\Gamma}} - 1\right) \mathfrak{j}_{\widehat{\Delta},\Gamma *}\left(\prod_{i=1}^{L} -\ell_{\widehat{\Delta},i}\, \mathrm{td}\left(\mathcal{N}^{\otimes \ell_{\widehat{\Delta},i}}_{\widehat{\Delta}/\delta_i^{\complement}(\widehat{\Delta})}\right)^{-1}\right) \tag{75}$$

holds in $\mathrm{CH}^*(D_\Gamma)$. We insert this formula into (73). Note that for the degeneration of arbitrary codimension appearing in 75 we have

$$j^*_{\widehat{\Delta},\Gamma} c_1(\mathcal{N}_\Gamma^{\otimes \ell_\Gamma}) \;=\; c_1\left(\mathcal{N}^{\otimes \ell_{\widehat{\Delta},L+1}}_{\widehat{\Delta}/\delta_{L+1}^{\complement}(\widehat{\Delta})}\right) \tag{76}$$

by splitting the degeneration into codimension one degenerations and applying successively Corollary 7.7 in the case $\delta_{L+1}(\widehat{\Delta}) = \Gamma$. An application of the push-pull formula now gives the expression in the proposition.

To prove the claim, note that the induction hypothesis directly implies that

$$P_{B_\Gamma^\top} \;=\; \sum_{L=0}^{N-2} \sum_{\Delta\in \mathrm{LG}_L(B_\Gamma^\top)} \left(N^T_{\delta_1(\Delta)} e^{\xi_{B_\Gamma^\top}} - 1\right) \mathfrak{i}_{\Delta *}\left(\prod_{i=1}^{L} -\ell_{\Delta,i}\, \mathrm{td}\left(\mathcal{N}^{\otimes \ell_{\Delta,i}}_{\Delta/\delta_i^{\complement}(\Delta)}\right)^{-1}\right) \tag{77}$$

in $\mathrm{CH}^*(B_\Gamma^\top)$. We now pull back this equation and our claimed equation to $D_\Gamma^s$ and compare. Agreement in $D_\Gamma^s$ implies the claim, since we are working with rational Chow groups throughout. The agreement follows from the comparison of the normal bundles in the argument of the Todd classes, which in turn is a consequence of the comparison results Proposition 4.9 and (59). □

**Corollary 9.8.** *The Chern character of the logarithmic cotangent bundle is*

$$\mathrm{ch}(\mathcal{E}_B) \;=\; \sum_{L=0}^{N-1} \sum_{\Gamma\in \mathrm{LG}_L(B)} \left(N^T_{\delta_1(\Gamma)} e^{\xi_B} - 1\right) \mathfrak{i}_{\Gamma *}\left(e^{\mathcal{L}_\Gamma} \prod_{i=1}^{L} -\ell_{\Gamma,i}\, \mathrm{td}\left(\mathcal{N}^{\otimes -\ell_{\Gamma,i})}_{\Gamma/\delta_i^{\complement}(\Gamma)}\right)^{-1}\right)$$

*where* $\mathcal{L}_\Gamma = \sum_{i=0}^{-L} \mathcal{L}_\Gamma^{[i]}$.

The subsequent simplifications of this formula are based on the following observation. Suppose that $\Gamma \mapsto a_\Gamma$ is an assignment of a rational number to every level graph $\Gamma \in \mathrm{LG}_L(B)$ for every $L$ with the property that if $L > 1$ then

$$a_\Gamma \;=\; \prod_{i=1}^{L} a_{\delta_i(\Gamma)} \tag{78}$$

is the product of those numbers over all undegenerations to two-level graphs. We use the abbreviation $a_{\Gamma,i} = a_{\delta_i(\Gamma)}$.

**Lemma 9.9.** *For a collection of* $a_\Gamma$ *satisfying* (78) *the identity*

$$\exp\Bigg(\sum_{\Gamma\in \mathrm{LG}_1(\overline{B})} a_\Gamma [D_\Gamma]\Bigg) \;=\; 1 + \sum_{L=1}^{N-1} \sum_{\Gamma\in\mathrm{LG}_L(\overline{B})} a_\Gamma\, \mathfrak{i}_{\Gamma,*}\Big(\prod_{i=1}^{L}\mathrm{td}\big(\mathcal{N}_{\Delta/\delta_i^{\complement}(\Delta)}^{\otimes -a_{\Gamma,i}}\big)\big)^{-1}\Big)$$

*holds in* $\mathrm{CH}^*(\overline{B})$.

*Proof.* The proof shows that this equality holds in fact if we restrict to any subset $S \subset \mathrm{LG}_1(\overline{B})$ on the left hand side and if we restrict on the right hand side to the sum of those $\Gamma \in \mathrm{LG}_L(\overline{B})$ such that all their two-level undegenerations belong to $S$. The proof now proceeds by induction over $|S|$.

For $|S| = 1$ this is the identity $\exp(a_\Gamma[D_\Gamma]) = 1 + a_\Gamma \mathfrak{i}_{\Gamma,*}(\mathrm{td}(\mathcal{N}_\Gamma^{\otimes -a_\Gamma})^{-1})$ that follows from the adjunction formula $\mathfrak{i}_\Gamma^*\mathfrak{i}_{\Gamma,*}\alpha = c_1(\mathcal{N}_\Gamma)\cdot\alpha$ and the relation between the generating series of the exponential and the Todd class.

For $|S| > 2$ this follows from the uniqueness of the intersection orders shown in Proposition 5.1 and induction. We give details for $|S| = 2$, leaving it to the reader to set up the notation for the general case. Let $\Gamma_k \in \mathrm{LG}_1(\overline{B})$ for $k = 1, 2$ and abbreviate $D_k = D_{\Gamma_k}$, $\mathcal{N}_k = c_1(\mathcal{N}_{\Gamma_k})$, $\mathfrak{i}_k = \mathfrak{i}_{\Gamma_k}$ and $\mathfrak{j}_k = \mathfrak{j}_{\Delta,\Gamma_k}$ for any graph $\Delta$ with $\delta_k(\Delta) = \Gamma_k$ for $k = 1, 2$. We denote by $[1, 2]$ this set of 3-level graphs. Then by (76)

$$\begin{aligned}
&\sum_{\Delta\in[1,2]} \mathfrak{i}_{\Delta,*}\Big(c_1\big(\mathcal{N}_{\Delta/\delta_1^{\complement}(\Delta)}\big)^{x-1} c_1\big(\mathcal{N}_{\Delta/\delta_2^{\complement}(\Delta)}\big)^{y-1}\Big) \;=\; \sum_{\Delta\in[1,2]} \mathfrak{i}_{\Delta,*}\big(\mathfrak{j}_1^*\mathcal{N}_1^{x-1}\mathfrak{j}_2^*\mathcal{N}_2^{y-1}\big) \\
&= \sum_{\Delta\in[1,2]} \mathfrak{i}_{1,*}\big(\mathfrak{i}_1^*([D_1])^{x-1}\mathfrak{j}_{1,*}\mathfrak{j}_2^*\mathcal{N}_2^{y-1}\big) \;=\; [D_1]^{x-1}\cdot \sum_{\Delta\in[1,2]} \mathfrak{i}_{2,*}\mathfrak{j}_{2,*}\mathfrak{j}_2^*\mathcal{N}_2^{y-1} \\
&= [D_1]^x \cdot \mathfrak{i}_{2,*}\mathcal{N}_2^{y-1} \;=\; [D_1]^x\cdot[D_2]^y\,.
\end{aligned}$$

Taking the generating series over this expression proves the claim. □

*Proof of Theorem 1.2.* In order to deduce this theorem from Corollary 9.8, we introduce shorthand notations for the products of inverse Todd classes, namely for any $\Gamma \in \mathrm{LG}_L(\overline{B})$ we let

$$X_{\Gamma,i} \;=\; \mathrm{td}\left(\mathcal{N}_{\Gamma/\delta_i^{\complement}(\Gamma)}^{\otimes -\ell_{\Gamma,i}}\right)^{-1} \quad \text{and} \quad X_\Gamma \;=\; \prod_{i=1}^{L} X_{\Gamma,i}, \tag{79}$$

and

$$X_{\Delta\setminus\Gamma} \;=\; \prod_{i\in I^{\complement}} \mathrm{td}\left(\mathcal{N}_{\Gamma/\delta_i^{\complement}(\Gamma)}^{\otimes -\ell_{\Gamma,i}}\right)^{-1}$$

if $\Gamma = \delta_I(\Delta)$ is the undegeneration keeping only the level passages in $I$ of $\Delta$. Now the argument of Lemma 9.9 with $\ell_\Gamma$ playing the role of $a_\Gamma$ and with both sides restricted to degenerations of a fixed $\Gamma \in \mathrm{LG}_L(\overline{B})$ gives

$$\exp(\mathcal{L}_\Gamma) \;=\; \exp\left(\sum_{\Gamma\in\mathrm{LG}^{\Gamma}_{L+1}(\overline{B})} \ell_\Gamma[D_\Gamma]\right) \;=\; 1 \,+\, \sum_{L'=L+1}^{N-1}\; \sum_{\Delta\in\mathrm{LG}^{\Gamma}_{L'}(\overline{B})} \ell_\Delta\, \mathsf{j}_{\Delta,\Gamma,*}(X_{\Delta\setminus\Gamma})\,,$$

where $\mathrm{LG}^{\Gamma}_{L'}(\overline{B})$ are the graphs with $L'$ levels below zero that are degenerations of $\Gamma$. We inject this formula into the right hand side of Corollary 9.8. Since

$$\mathsf{i}_{\Gamma,*}\big(\mathsf{j}_{\Delta,\Gamma,*}(X_{\Delta\setminus\Gamma})\cdot X_\Gamma\big) \;=\; \mathsf{i}_{\Delta,*}(X_\Delta)$$

by the projection formula and equation (76), we obtain

$$\mathrm{ch}(\mathcal{E}_B) \;=\; \sum_{L=0}^{N-1}(-1)^L \sum_{\Gamma\in\mathrm{LG}_L(B)} \left(N^T_{\delta_1(\Gamma)}e^{\xi_B}-1\right) \sum_{L'=L}^{N-1}\; \sum_{\Delta\in\mathrm{LG}^{\Gamma}_{L'}(\overline{B})} \ell_\Delta\, \mathsf{i}_{\Delta,*}(X_\Delta)\,.$$

It remains to sort this expression as sum over $\ell_\Delta \mathsf{i}_{\Delta,*}(X_\Delta)$. Since each $\Delta\in\mathrm{LG}_{L'}(B)$ appears in the expression of each $\Gamma$ with $\delta_I(\Delta)=\Gamma$, its coefficient in the final expression of $\mathrm{ch}(\mathcal{E}_B)$ is (defining $\min(\{\emptyset\}) = L'+1$ )

$$\begin{aligned}\sum_{I\subseteq\{1,\cdots,L'\}} (-1)^{|I|}\cdot\left(N^T_{\delta_{\min(I)}(\Delta)}e^{\xi_B}-1\right) \;&=\; e^{\xi_B}\cdot \sum_{I\subseteq\{1,\cdots,L'\}} (-1)^{|I|} N_{\delta_{\min(I)}(\Delta)} \\ &=\; e^{\xi_B}\cdot\left(N - N^T_{\delta_{L'}(\Gamma)}\right)\,,\end{aligned}$$

where the disappearance of $(-1)^{|I|+1}$ in the first equality and the cancellation in the second equality stem from canceling the contributions of pairs under the involution $I\mapsto I\cup\{L'\}$ if $L'\notin I$ and $I\mapsto I\setminus\{L'\}$, if $L'\in I$. $\square$

In preparation for the next theorem we switch to the language of profiles introduced in Section 5 and recall that the notation depends on the choice of the numbering of $\mathrm{LG}_1(\mathrm{B}) = \{\Gamma_1,\ldots,\Gamma_M\}$. We claim that Theorem 1.2 can equivalently be restated as

$$\mathrm{ch}(\mathcal{E}_B) \;=\; e^{\xi_B}\cdot \sum_{L=0}^{N-1}\; \sum_{[j_1,\ldots,j_L]\in\mathscr{P}_L} \big(N - N^T_{j_L}\big) \prod_{i=1}^{L}\left(e^{\ell_{j_i}[D_{j_i}]}-1\right)\,. \tag{80}$$

To see the equivalence, it suffices to expand the product (80) and to use Proposition 5.1 about the uniqueness of the order of letters in a profile. Note that we cannot replace $\mathscr{P}_L$ by $\mathrm{LG}_L(B)$ in (80), as this would give wrong multiplicities.

We abbreviate the difference of dimensions $r_{\Gamma,i} = N - N^\top_{\delta_i(\Gamma)}$ and write $r_\Gamma = \prod_{i=1}^L r_{\Gamma,i}$. It is useful to remember that $r_{\Gamma,i} = \sum_{j=i+1}^L N^{[-j]} = \sum_{j=i+1}^L (d^{[-j]}+1)$ is the sum of the unprojectivized dimensions of the lower levels. If we work with profiles and the elements of $\mathrm{LG}_1(\mathrm{B})$ are numbered, we write $r_j = r_{\Gamma_j}$ and $\ell_i = \ell_{\Gamma_i}$. We can now state an additive and a multiplicative decomposition of the Chern polynomial.

**Theorem 9.10.** *The Chern polynomial of the logarithmic cotangent bundle is*
(81)

$$
\begin{aligned}
c(\mathcal{E}_B) \;&=\; \prod_{L=0}^{N-1} \prod_{[j_1,\dots,j_L]\in\mathscr{P}_L} \prod_{I\subseteq\{1,\dots,L\}} \big(1+\xi+\sum_{i\in I}\ell_{j_i}[D_{j_i}]\big)^{(-1)^{|I^{\complement}|}\cdot r_{j_L}} \\
&= \sum_{L=0}^{N-1} \sum_{\Gamma\in \mathrm{LG}_L(B)} \ell_\Gamma \mathrm{i}_{\Gamma,*}\Bigg(\sum_{\mathbf{k}} \binom{N-\sum_{i=1}^L k_i}{k_0}\xi^{k_0}\cdot\prod_{i=1}^{L}\binom{r_{\Gamma,i}-\sum_{j>i}^L k_j}{k_i}(\ell_i\nu_{\Gamma,i})^{k_i-1}\Bigg),
\end{aligned}
$$

*where* $\mathbf{k}=(k_0,k_1,\dots,k_L)$ *is a tuple with* $k_0\geq 0$ *and* $k_i\geq 1$ *for* $i=1,\dots,L$ *and where* $\nu_{\Gamma,i}=c_1(\mathcal{N}_{\Gamma/\delta_i^{\complement}(\Gamma)})$*. For* $L=0$ *the exponent* $r_{j_L}$ *is to be interpreted as* $N$*.*

*Proof.* We first deduce the first line from (80). We compute the degree-$d$-part of its interior product to be

$$
\Big[e^{\xi_B}\prod_{i=1}^{L}\Big(e^{\ell_{j_i}[D_{j_i}]}-1\Big)\Big]_d \;=\; \frac{1}{(d-1)!\cdot d}\sum_{I\subseteq\{1,\dots,L\}}(-1)^{|I^{\complement}|}\big(\xi+\sum_{i\in I}[D_{j_i}]\big)^d .
$$

On the other hand, recall from [ACG11, p. 586] that the Chern polynomial is given in terms of the graded pieces of the Chern character by

$$
c(\mathcal{E}_B) \;=\; \exp\Big(\sum_{d\geq 1}(-1)^{d-1}(d-1)!\,\mathrm{ch}_d(\mathcal{E}_B)\Big).
$$

Using the generating series of the logarithmic function, we then obtain the first line of the statement by combining the previous two expressions.

In order to pass to the second line, we first show that the first line formally fits with Lemma 9.11. We want to replace the two exterior products over all $L$ and profiles $\mathscr{P}_L$ by all subsets of the integer interval $[[1,\dots,M]]$ without altering the value of the product. For this purpose we claim that for each element of $\mathscr{P}_L$ the interior product

$$
P \;=\; \prod_{I\subseteq\{1,\dots,L\}}\big(1+\xi+\sum_{i\in I}\ell_{j_i}[D_{j_i}]\big)^{(-1)^{|I^{\complement}|}\cdot r_{j_L}}
$$

considered as an element in the polynomial ring is in $1+D_1\cdots D_L\cdot\mathbb{Q}[\xi,D_1,\dots,D_L]$. This claim implies that the additional products give zero in the Chow ring and considering the profiles as subsets of $[[1,\dots,M]]$ rather than as ordered tuples is no loss of information thanks to Proposition 5.1. To justify the claim we may assume $r_{j_L}=1$, since the claim persists when raising to an integral power. For $L=1$ the claim is obvious and for the inductive step one replaces $\xi$ successively by $\xi+\ell_{j_k}D_k$ to see that $P-1$ is divisible by $D_i$ for all $i\neq k$.

Now we are in the situation to apply the image of the formula of Lemma 9.11 in the Chow ring. To match the second line of the lemma and the theorem we define for tuple $\mathbf{k}=(k_0,k_1,\dots,k_M)$ as in the lemma the integer $L$ to be the number of entries $k_i$ that are positive. Consider a summand $\mathbf{k}=(k_0,k_1,\dots,k_M)$ in the second line of the line, and say that $i_1,\dots,i_L$ are those indices where the entries $k_{i_j}$ are positive. Then the contribution of this summand to the second line of (82) equals the contributions of the (possibly empty) set of level graphs in $D_{i_1}\cap\dots\cap D_{i_L}$ to the second line of (81). $\square$

**Lemma 9.11.** *In the polynomial ring* $\mathbb{Q}[\xi, D_1, \dots, D_M]$ *the identity*

$$\begin{aligned}(82)\qquad &\prod_{[j_1,\dots,j_L]\subseteq\{1,\dots,M\}} \prod_{I\subseteq\{1,\dots,L\}} \big(1+\xi+\sum_{i\in I}\ell_{j_i}D_{j_i}\big)^{(-1)^{|I^\complement|}\cdot N_{j_L}} \\ &= \sum_{\mathbf{k}} \binom{N-\sum_{i=1}^M k_i}{k_0}\xi^{k_0}\cdot\prod_{i=1}^{M}\binom{\sum_{j\geq i}N^{[-j]}-\sum_{j>i}^M k_j}{k_i}(\ell_i D_i)^{k_i}\end{aligned}$$

*holds, where* $\mathbf{k} = (k_0, k_1, \dots, k_M)$ *is a tuple of non-negative integers, and where* $N_s := \sum_{j=s+1}^M N^{[-j]}$ *and* $N = N_\emptyset = \sum_{j=0}^M N^{[-j]}$.

*Proof.* We proceed by induction, $M = 0$ is the binomial expansion. The effect of the passage from from $M-1$ to $M$ is given on the left hand side by replacing $N^{[-(M-1)]}$ with $N^{[-(M-1)]} + N^{[-M]}$ in all those factors where $j_L < M$, and by multiplying by the factors where $j_L = M$, i.e. by multiplication with

$$\begin{aligned}&\prod_{\substack{[j_1,\dots,j_{L-1}]\subseteq\{1,\dots,M-1\}\\ I\subseteq\{1,\dots,L-1\}}} \big(1+\xi+D_M+\sum_{i\in I}\ell_{j_i}D_{j_i}\big)^{(-1)^{|I^\complement|}\cdot N^{[M]}} \\ =\ &\sum_{k_M\geq 0}\binom{r_M}{k_M}D^{k_M}\cdot\prod_{\substack{[j_1,\dots,j_{L-1}]\subseteq\{1,\dots,M-1\}\\ I\subseteq\{1,\dots,L-1\}}}\big(1+\xi+\sum_{i\in I}\ell_{j_i}D_{j_i}\big)^{(-1)^{|I^\complement|}\cdot(N^{[M]}-k_M)}\end{aligned}$$

Applying the induction hypothesis with $N^{[-(M-1)]}$ replaced by $N^{[-(M-1)]}+N^{[-M]}-k_M$ gives the claim. ☐

The following step concludes the proof of all main theorems.

**Lemma 9.12.** *Suppose that* $\alpha_\Gamma \in \mathrm{CH}_0(D_\Gamma)$ *is a top degree class and that* $c_\Gamma^*\alpha_\Gamma = \prod_{i=0}^{-L(\Gamma)} p_\Gamma^{[i],*}\alpha_i$ *for some* $\alpha_i$. *Then*

$$\int_{D_\Gamma}\alpha_\Gamma \;=\; \frac{K_\Gamma}{|\operatorname{Aut}(\Gamma)|\ell_\Gamma}\prod_{i=0}^{-L(\Gamma)}\int_{B_\Gamma^{[i]}}\alpha_i\,.$$

*Proof.* We have

$$\int_{D_\Gamma}\alpha_\Gamma \;=\; \frac{1}{\deg(c_\Gamma)}\int_{D_\Gamma^s}c_\Gamma^*(\alpha_\Gamma) \;=\; \frac{\deg(p_\Gamma)}{\deg(c_\Gamma)}\prod_{i=0}^{-L(\Gamma)}\int_{B_\Gamma^{[i]}}\alpha_i.$$

and the claim follows from by Lemma 4.5. ☐

*Proof of Theorem 1.3.* By Proposition 2.1, it is enough to compute the top Chern class $c_d(\mathcal{E}_B)$, where $d = \dim(B) = N-1$. We investigate for each $L$ and each $\Gamma \in \mathrm{LG}_L(B)$ the contribution of the second line of (81) in Theorem 9.10 to $c_d(\mathcal{E}_B)$. It suffices then to show that the expression inside the $\mathrm{i}_{\Gamma,*}$ is equal to $N_\Gamma^\top\prod_{i=0}^{L-1}(\xi_\Gamma^{[i]})^{d_\Gamma^{[i]}}$. Note that by Proposition 7.5 the first chern class of the normal bundle $\mathcal{N}_{\Gamma/\delta_i^\complement(\Gamma)}$ is supported on the levels $-i+1$ and $-i$ of $\Gamma$. Considering the bottom level we deduce that if the summand $\mathbf{k}$ contributes non-trivially to the top Chern class $c_d$, then we must have $k_L \geq d^{[L]}+1$ so that the $\nu_{\Gamma,L}$-power is large enough for its binomial expansion to contain a top $\xi$-power for the bottom level. On the other hand, for the binomial coefficient in front of it to be non-zero we need $r_{\Gamma,L} \geq k_L$, which is equivalent to $k_L \leq d^{[L]}+1$. So in fact $k_L = d^{[L]}+1$, the binomial coefficient is one, and we have to select from the expansion of $\nu_{\Gamma_L}^{d^{[L]}}$ the term that does not contribute to level $-i-1$. Since the top entry of the binomial coefficient is $r_{\Gamma,i} - \sum_{j>i}^{L} k_j = 1 + N_i + \sum_{j>i}^{L}(d^{[j]} + 1 - k_j)$ we can inductively repeat this argument for all levels and deduce $k_j = d^{[j]} + 1$ for all $j \geq 1$ and $k_0 = d^{[0]}$. The only non-trivial factor is now $N_\Gamma^\top$ that stems from the first binomial coefficient in the second line of (81). The final shape of the statement follows then directly from Proposition 4.9 and Lemma 9.12, after noticing that the $\ell_\Gamma$ coefficients cancel. □

## 10. Examples: Geometry and values

In this section we explain how to evaluate top degree classes, we provide examples illustrating the geometry at infinity of the compactification $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and examples of our formulas for the normal bundles, the Chern polynomials and the Euler characteristic.

10.1. **Evaluation of top $\xi$-powers.** First of all we explain how to evaluate the expression in Theoreom 1.3, see [CMZ20] for many algorithmic details. We only need to explain how to evaluate $\int_{\overline{B}} \xi^d$, i.e. top powers of $\xi$ on generalized strata.

Suppose that $B = \mathbb{P}\Omega\mathcal{M}_{g,1}(2g-2)$ is a stratum parametrizing connected surfaces with a single zero. Then the generating series of top $\xi$-powers is given by a simple power series inversion that arises in the computation of Masur-Veech volumes, see [Sau18] and [CMSZ19, Theorem 3.1], and Table 2 for some values.

Suppose that $B = \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is a stratum parametrizing connected surfaces of holomorphic type, i.e., with all $m_i \geq 0$ and with $n \geq 2$. Then $\int_B \xi^d = 0$ by [Sau18, Proposition 3.3].

It remains to explain how to evaluate $\int_{\overline{B}} \xi^d$ top powers of $\xi$ on meromorphic generalized strata. First of all, we write $\xi$ with the help of Proposition 8.2 as a $\psi$-class and boundary strata. The product of such objects, which are standard generators as in 6 of the tautological ring, can be rewritten as a sum of standard additive generators via the algorithm explained in the proof of Theorem 1.5, more specifically in the part in which we show that $R^\bullet_{fg}(\overline{B}) = R^\bullet(\overline{B})$ is a ring. Now that we have rewritten $\xi^d$ in terms of standard additive generators, by Lemma 9.12, it only remains to explain how to evaluate a top-dimensional standard generator, i.e. the top power of a $\psi$-class, on a generalized stratum $\overline{B}$. Since $\psi$-classes are pulled back from $\overline{\mathcal{M}}_{g,n}$, we can use a push-pull argument and express

$$\int_{\overline{B}} \psi_i^d = \int_{\overline{\mathcal{M}}_{g,n}} \pi_*([\overline{B}])\psi_i^d$$

where $\pi : \overline{B} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism and where we used as always the abuse of notation $\psi_i = \pi^*(\psi_i)$.

If we can express the class $\pi_*([\overline{B}])$ in terms of the standard generators of $\overline{\mathcal{M}}_{g,n}$, we can use the sage package `admcycles` in order to obtain a number.

If $\overline{B}$ is a stratum parameterizing meromorphic differentials on connected surfaces without residue conditions, the class $\pi_*([\overline{B}])$ was computed in [Sau19] and [BHPSS20], and the algorithmic task can be performed again by the sage package `admcycles`, which implements the algorithm based on the formula in [Sch18] and [BHPSS20].

If the stratum $\overline{B}$ is more general parametrizing differentials on disconnected surfaces and with residue conditions, we first of all use repetitively Proposition 8.3 to write the class of $\overline{B}$ into the associated stratum without residue conditions in terms of additive generators of the stratum with not conditions. We then reduced to the computation of the class $\pi_*([\overline{B}])$ in the case that $\overline{B}$ has no more residue conditions, but is potentially disconnected. If $\overline{B}$ is disconnected then actually $\pi_*([\overline{B}])$ is zero. In fact, since we can scale the differentials on the components independently, the fiber dimension to a product of $\overline{\mathcal{M}}_{g_i,n_i}$ is positive and by definition of push-forward we get the zero class.

| $\mu$ | $(0)$ | $(2)$ | $(4)$ | $(6)$ | $(0,0,-2)$ | $(2,-2)$ |
|---|---|---|---|---|---|---|
| $\int_{\overline{B}} \xi^{\dim(B)}$ | $\frac{1}{24}$ | $-\frac{1}{640}$ | $\frac{-305}{580608}$, | $-\frac{87983}{199065600}$ | $1$ | $-\frac{1}{8}$ |
| $\mu$ | $(1,1,-2)$ | $(4,-2)$ | $(3,1,-2)$ | $(2,1,-3)$ | $(5,-3)$ | $(8,-2,-2,-2)$ |
| $\int_{\overline{B}} \xi^{\dim(B)}$ | $0$ | $-\frac{-23}{1152}$, | $0$ | $\frac{5}{8}$ | $-\frac{21}{20}$ | $-\frac{4527}{32}$ |

TABLE 2. Integrals of top $\xi$-powers for some connected strata

For the subsequent examples we present some cases where we can directly evaluate the top $\xi$-power for meromorphic strata in genus 0 and genus 1.

**Proposition 10.1.** *The integrals of the top $\xi$-power are given ($a_i, k \geq 0$)*

$$
\begin{aligned}
&\textit{for} \quad B = \mathbb{P}\Omega_{0,n+1}(-2-\sum_{i=1}^{n} a_i, a_1, ..., a_n) && \textit{by} \quad \int_B \xi_B^{n-2} = (-1-\sum_{i=1}^{n} a_i)^{n-2}\,, \\
&\textit{for} \quad B = \mathbb{P}\Omega_{1,2}(-k,k) && \textit{by} \quad \int_B \xi_B = -\frac{(k-1)(k^2-1)}{24}\,, \\
&\textit{for} \quad B = \mathbb{P}\Omega_{1,3}(-k-1,1,k) && \textit{by} \quad \int_B \xi_B^2 = \frac{(k^4-1)}{24}\,.
\end{aligned}
$$

*Proof.* The first statement follows easily from Proposition 8.2, which in this case implies $\xi = (-1-\sum_{i=1}^{n} a_i)\psi_1$. Indeed there cannot be any divisors which have the pole on lower level. Hence

$$\int_B \xi^{n-2} = (-1-\sum_{i=1}^{n} a_i)^{n-2} \int_{\mathcal{M}_{0,n+1}} \psi_1^{n-2} = (-1-\sum_{i=1}^{n} a_i)^{n-2}.$$

The second statement follows immediately as the previous one, since again there cannot be divisors where the pole is on lower level. Hence

$$\int_B \xi \;=\; -(k-1)\int_{\mathcal{M}_{1,2}} \pi_*([B])\psi_1 = -\frac{(k-1)(k^2-1)}{24}\,,$$

where we used [CC14, Proposition 3.1] for the computation of $\pi_*([B])$.

For the proof of the last statement, notice that there can only be one non-horizontal divisor $D_3$ which has the pole on lower level (see Section 10.3 for the full boundary description). Using Proposition 8.2 we find then $\xi = -k\psi_1 - D_3$, which

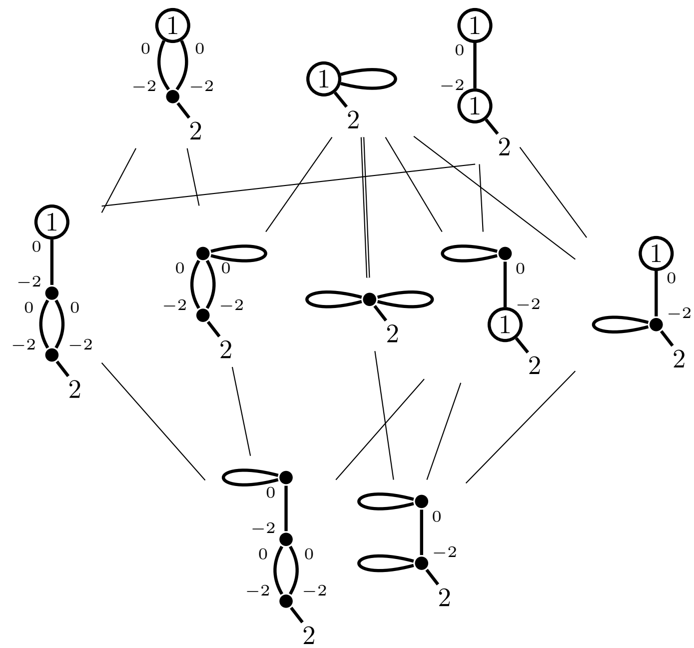


FIGURE 4. Level graphs appearing in the boundary of $\Omega\mathcal{M}_{2,1}(2)$. Graphs corresponding to components of the same dimension are in the same row (divisors in the first row, points in the bottom row). The lines connecting the graphs symbolize degeneration.

yields

$$\int_{\overline{B}} \xi^2 \;=\; -\int_{\overline{B}} \xi(D_3 + k\psi_1) \;=\; \Big(-1/24 + \int_{\overline{B}} k\psi_1(D_3 + k\psi_1)\Big)$$
$$= \Big(-1/24 + k^2 \int_{\mathcal{M}_{1,3}} \pi_*([B]) \cdot \psi_1^2\Big) = (k^4 - 1)/24$$

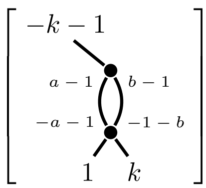
again by the computation in [CC14, Proposition 3.1] of the class of the class $\pi_*(B)$ of the stratum in $\mathcal{M}_{1,3}$. $\square$

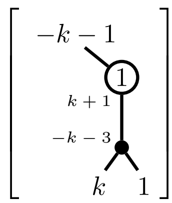
10.2. **The minimal stratum** $\mathbb{P}\Omega\mathcal{M}_{2,1}(2)$**.** This stratum is small enough so that we can show all the level graphs, including those with horizontal nodes and their adjacency in Figure 4. The picture shows the dual graphs of stable curves in the boundary of this stratum, the top level is on top of each graph. The number in the vertex denotes the genus, a black dot corresponds to genus zero. The numbers associated to the legs are the orders of zero. In this stratum, all interior edges have enhancement $\kappa_e = 1$, so the discussion of prong-matchings is void here. There are only three graphs without horizontal nodes, in fact $|\operatorname{LG}_1(B)| = 2$ and $|\operatorname{LG}_2(B)| = 1$, where $B = \mathbb{P}\Omega\mathcal{M}_{2,1}(2)$ as usual. Taking into also account the entire stratum and the stack structure of the banana graphs, and using the values of top $\xi$-powers from Section 10.1, we get

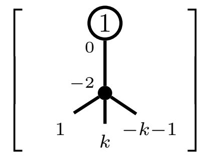
$$(-1)^3 \cdot \chi(B) \;=\; 4 \cdot \frac{-1}{640} + 0 + 2 \cdot \frac{1}{24} \cdot \frac{-1}{8} + 2 \cdot \frac{1}{2} \cdot \frac{1}{24} \cdot 1 \cdot 1 \;=\; \frac{1}{40}$$

as in the table in the introduction, in accordance with the fact that this stratum is a 6-fold unramified cover of $\mathcal{M}_2$ and $\chi(\mathcal{M}_2) = -\frac{1}{240}$.

10.3. **The stratum $\mathbb{P}\Omega\mathcal{M}_{1,3}(-k-1, 1, k)$.** This example illustrates the quotient stack structure at the boundary of the smooth compactification that result from prong-matchings, i.e., from points with $\mathrm{Tw}_\Gamma \neq \mathrm{Tw}_\Gamma^s$. We have chosen a genus-one stratum with a simple zero, since the projection to $\mathcal{M}_{1,2}$ provides an alternative way to compute all invariants in this case. We label the points $z_1$ (pole), $z_2$ (simple zero) and $z_3$. The boundary divisors here are $D_\mathrm{h}$ and five more types of divisors, namely there are the divisors

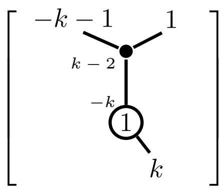
$D_{1,a} =$ 

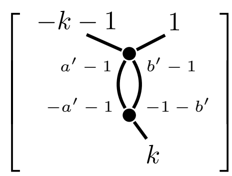
$D_2 =$ 

, $D_3 =$ 

,

where $a, b \geq 1$ and $a + b = k + 1$. Here $D_{1,a} = D_{1,k+1-a}$ and if $k$ is odd, the middle divisor $D_{1,(k+1)/2}$ has an $\mathbb{Z}/2$-involution. Moreover, there are the divisors

$D_4 =$ 

, $D_{5,a'} =$ 


where $a' \in \{1, \ldots, k-1\}$ and $b' = k - a'$. Again, $D_{5,a'} = D_{5,k-a'}$ with an involution on $D_{5,k/2}$ if $k$ is even. The local exponents are

$$\ell_{1,a} = \mathrm{lcm}(a, k+1-a), \quad \ell_2 = k+2, \quad \ell_3 = 1, \quad \ell_4 = k-1, \quad \ell_{5,a'} = \mathrm{lcm}(a', k-a')$$

and the dimensions of the top level components are

$$N_1^\top = 1, \quad N_2^\top = 2, \quad N_3^\top = 2, \quad N_4^\top = 1, \quad N_5^\top = 2\,.$$

We abbreviate $D_1 = \frac{1}{2}\sum_{a=1}^{k} D_{1,a}$ and $D_5 = \frac{1}{2}\sum_{a'=1}^{k-1} D_{5,a'}$.

**The local geometry of the boundary divisors.** We give a summary of the boundary points and intersection behaviour of the boundary divisors listed above. We start with the boundary divisors that map to the interior of $\mathcal{M}_{1,2}$ under the map to $\mathcal{M}_{1,1}$ forgetting the second point. These are represented by the thin lines in Figure 5, while thick lines are mapped to the point at infinity of $\mathcal{M}_{1,1}$. The divisor $D_3$ is simply a copy of the modular curve, intersecting once $D_\mathrm{h}$.

The divisor $D_2$ minus its intersection with other boundary divisors is the union of the modular curves $X_1(d) = \mathbb{H}/\Gamma_1(d)$ for all divisors $d > 1$ of $k+1$. The only intersections with other boundary divisors are $\lceil (k+1)/2 \rceil - 1$ intersection points with $D_\mathrm{h}$ and $\gcd(a,b)$-points with $D_{1,a}$.

The divisor $D_4$ minus its intersection with other boundary divisors is the union of the modular curves $X_1(d) = \mathbb{H}/\Gamma_1(d)$ for all divisors $d > 1$ of $k$. The only intersections with other boundary divisors are $\lceil (k)/2 \rceil - 1$ intersection points with $D_\mathrm{h}$ and $\gcd(a',b')$-points with $D_{5,a'}$.

The curves $D_{1,a}$ and $D_{5,a'}$ form the exceptional divisor when realizing the level compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{1,3}(-k-1, 1, k)$ as a blowup of $\overline{\mathcal{M}}_{1,2}$ in the node of the

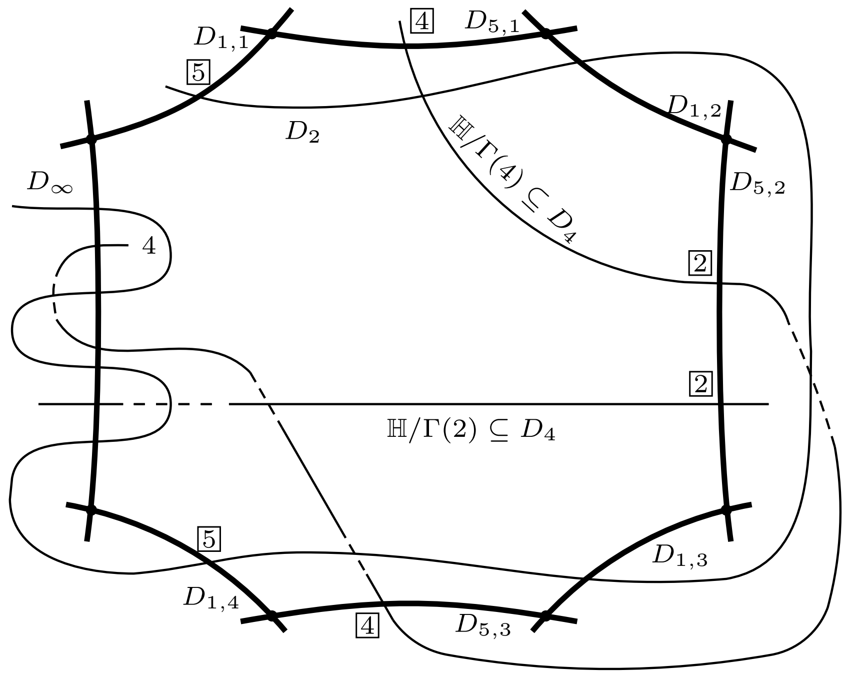



FIGURE 5. The intersection behavior of the boundary in the stratum $\mathbb{P}\Omega\mathcal{M}_{1,3}(-4,1,5)$. The figure has to be considered as quotient by the elliptic involution that interchanges $D_{1,1}$ with $D_{1,4}$ and $D_{5,1}$ with $D_{5,3}$ etc.

forgetful map to $\overline{\mathcal{M}}_{1,1}$. Without prong-matchings the curve $D_{1,a}$ were just an $\overline{M}_{0,4}$ (with a stack structure of an involution if $a = (k+1)/2$). The three boundary points correspond to the intersection with $D_{5,a-1}$ and $D_{5,a}$ (respectively with $D_{\mathrm{h}}$ and $D_{5,a}$ if $a = 1$), and with $D_2$. By the formulas in Section 3.4, at the generic point of $D_{1,a}$ (and also near the intersection with $D_2$) there are $\gcd(a,b)$ prong-matching equivalence classes. At the intersections with $D_5$ there is just one prong-matching equivalence class. This implies that each $D_{1,a}$ is a $\gcd(a,b)$-fold cover of $\overline{\mathcal{M}}_{0,4}$, totally ramified over the two points of intersection with $D_5$.

Similarly, the divisor $D_5$ is a $\gcd(a',b')$-fold cover of $\overline{M}_{0,4}$, totally ramified over the two points that correspond to the intersections with $D_1$. We compute the normal bundles of the divisor using the special geometry of this example, independently of Theorem 7.1.

**Proposition 10.2.** *The self-intersection number of $D_{1,a}$ is*

$$D_{1,a}^2 \;=\; -\delta_a^{k+1} \cdot k g_{1,a}/\ell_{1,a} \quad \textit{where} \quad g_{1,a} \;=\; \gcd(a,b)$$

*and where $\delta_a^{k+1} = 1/2$ if $a = (k+1)/2$ and $\delta_a^{k+1} = 1$ otherwise. The self-intersection number of $D'_{5,a}$ is*

$$D_{5,a'}^2 \;=\; -\delta_{a'}^{k} \cdot (k+1)\, g_{5,a'}/\ell_{5,a'} \quad \textit{where} \quad g_{5,a'} = \gcd(a',b')\,.$$

*Proof.* We consider the fibration $\pi : \mathbb{P}\Xi\overline{\mathcal{M}}_{1,3}(-k-1,1,k) \to \overline{\mathcal{M}}_{1,1}$ obtained from forgetting the last two marked points and take a smooth chart of the quotient stack near the image of the curves $D_{1,a}$ and $D_{5,a'}$. From the intersection discussion above we deduce that the fiber over $\infty$ in $\overline{\mathcal{M}}_{1,1}$ consists of a ring of rational curves intersecting in the order

$$D_{\mathrm{h}} - D_{1,1} - D_{5,1} - D_{1,2} - D_{5,2} - \cdots - D_{1,k-1} - D_{5,k} - D_{1,k+1} - D_{\mathrm{h}}\,,$$

see again Figure 5. We claim that the multiplicity of $D_{1,a}$ in the fiber $F = \pi^{-1}(\infty)$ is equal to $(k+1)/\gcd(a,k+1)$. This can be deduced from the fact that $\pi|_{D_2}$ is a cover of degree $(k+1)^2-1$ and from the order of the cusp stabilizers (see [DS05, Section 3.8], in particular the explanation around Figure 3.2) since $D_2$ and $D_{1,a}$ intersect transversally in $\mathbb{P}\Xi\overline{\mathcal{M}}_{1,3}(-k-1,1,k)$. Similarly, the multiplicity of $D_{5,a'}$ in this fiber is $k/\gcd(a',k)$. Using the orbifold degree of the intersection points given in (23) and $D_{1,a}\cdot F = 0$ we find with $a' = a-1$ and $b' = b-1$ that

$$\begin{aligned} D_{1,a}^2 \;&=\; -\frac{\gcd(a,b)}{k+1}\cdot\frac{abk}{\ell_{1,a}}\cdot\Big(\frac{a'}{\ell_{5,a'}\gcd(a',k-a')}+\frac{b'}{\ell_{5,b'}\gcd(b',k-b')}\Big)\\ &=\; -\frac{\gcd(a,b)}{k+1}\cdot\frac{abk}{\ell_{1,a}}\cdot\frac{k+1}{ab} \;=\; -k\cdot g_{1,a}/\ell_{1,a}\,. \end{aligned}$$

The proof of $D_{5,a'}$ is similar. □

Proposition 10.2 agrees with Theorem 1.6. Indeed, since the dimension of the top (resp. bottom) level stratum of $D_{1,a}$ (resp. $D_{5,a'}$) is zero dimensional, we compute

$$\begin{aligned} D_{1,a}^2 \;&=\; \mathrm{c}_1(\mathcal{N}_{D_{1,a}}) \;=\; \frac{1}{\ell_{1,a}}\left(-\xi^{\top}_{D_{1,a}} - \mathcal{L}^{\top}_{D_{1,a}} + \xi^{\perp}_{D_{1,a}}\right)\\ &=\; \frac{K_{1,a}}{\ell_{1,a}^2\operatorname{Aut}(D_{1,a})}\xi_{B^{\perp}_{1,a}} = \frac{g_{1,a}}{\ell_{1,a}\operatorname{Aut}(D_{1,a})}\cdot(-k) \end{aligned}$$

where in the last two equalities we used Lemma 9.12 about evaluating top classes and the computation of top powers of $\xi$ which can be done analogously as in the first case of Proposition 10.1. Similarly we also get

$$\begin{aligned} D_{5,a'}^2 \;&=\; \mathrm{c}_1(\mathcal{N}_{D_{5,a'}}) \;=\; \frac{1}{\ell_{5,a'}}\left(-\xi^{\top}_{D_{5,a'}} + \xi^{\perp}_{D_{5,a'}} - \mathcal{L}^{\top}_{D_{5,a'}}\right)\\ &=\; -\frac{K_{5,a'}}{\operatorname{Aut}(D_{5,a'})\ell_{5,a'}^2}\xi_{B^{\top}_{5,a'}} - \frac{1}{\ell_{5,a'}}\mathcal{L}^{\top}_{D_{5,a'}}\\ &=\; \frac{g_{5,a'}}{\operatorname{Aut}(D_{5,a'})\ell_{5,a'}}\cdot(-k) - \frac{1}{\ell_{5,a'}}\left([D_{5,a'}]\cdot([D_4]+[D_{1,a'}]+[D_{1,a'+1}])\right)\\ &=\; \frac{g_{5,a'}}{\operatorname{Aut}(D_{5,a'})\ell_{5,a'}}\cdot(-k-1). \end{aligned}$$

**The Euler characteristic.** We give two ways proof of the following fact.

**Proposition 10.3.** *The moduli space $\mathbb{P}\Omega\mathcal{M}_{1,3}(-k-1,1,k)$ has Euler characteristic equal to $k(k+1)/6$.*

*Proof.* The first proof uses the description of $B = \mathbb{P}\Omega\mathcal{M}_{1,3}(-k-1,1,k)$ as the complement of $D_2$ and $D_4$ in $\mathcal{M}_{1,2}$. By the above description of $D_2$ and $D_4$ we need to compute

$$\sum_{\substack{d|k \\ d\neq k}} \chi(X_1(k/d)) \;=\; \chi(\mathcal{M}_{1,1}) \sum_{\substack{d|k \\ d\neq k}} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(k/d)] \;=\; -\frac{k^2-1}{12}\,,$$

which holds, since the rightmost sum counts the number of non-zero $k$-torsion points in an elliptic curve. Together with $\chi(\mathcal{M}_{1,2}) = -1/12$ implies the claim.

The second proof evaluates Theorem 1.3, given in the surface case concretely by

$$\chi(B) \;=\; 3\xi_B^2 + \sum_{\Gamma\in \mathrm{LG}_1(\mathrm{B})} \frac{K_\Gamma \cdot N_\Gamma^\top}{|\operatorname{Aut}(\Gamma)|} \left( \int_{B_\Gamma^\top} \xi_{B_\Gamma^\top} + \int_{B_\Gamma^\perp} \xi_{B_\Gamma^\perp} \right) + \sum_{\Delta\in \mathrm{LG}_2(B)} \ell_{\delta_0(\Delta)} \ell_{\delta_1(\Delta)} [D_\Delta]\,.$$

Using the third statement of Proposition 10.1 we find

$$3\int_{\overline{B}} \xi^2 \;=\; 3(k^4-1)/24.$$

For the divisors $D_{1,a}$ and $D_4$ the contribution from $\xi_{B_\Gamma^\top}$ is zero and that of $\xi_{B_\Gamma^\perp}$ is non-zero, while for $D_2$, $D_3$ and $D_{5,a'}$ the converse holds. We evaluate in detail the contribution of that last divisor type. Its top levels are $D_{5,a'}^\top = \mathbb{P}\Omega_{0,4}(1, a'-1, b'-1, -k-1)$. Using again Proposition 10.1 we get

$$\sum_{\substack{\Gamma=D_{5,a} \\ a=1,\dots k/2}} \frac{K_\Gamma \cdot N_\Gamma^\top}{|\operatorname{Aut}(\Gamma)|} \int_{B_\Gamma^\top} \xi_{B_\Gamma^\top} \;=\; \frac{1}{2}\sum_{a'=1}^{k-1} 2a'(k-a')\cdot(-k) \;=\; \frac{-k^2(k^2-1)}{6}$$

Similar computations using again Proposition 10.1 yield

$$\sum_{\substack{\Gamma=D_{1,a} \\ a=1,\dots(k+1)/2}} \frac{K_\Gamma N_\Gamma^\top}{|\operatorname{Aut}(\Gamma)|} \int \xi_{B_\Gamma^\top} \;=\; -k^2\frac{k^2+3k+2}{12}\,, \quad 2\ell_{D_3}\int \xi_{D_3}^\top \;=\; \frac{1}{12}$$

$$2\ell_{D_2}\int \xi_{D_2}^\top \;=\; -2k(k+2)\frac{(k+1)^2-1}{24}\,, \quad \ell_{D_4}\int \xi_{D_4}^\top \;=\; -(k-1)^2\frac{k^2-1}{24}$$

Using the evaluation result of Lemma 9.12 we finally get

$$\sum_{\Delta\in \mathrm{LG}_2(B)} \ell_{\delta_0(\Delta)} \ell_{\delta_1(\Delta)} [D_\Delta] \;=\; k(k+1)\frac{k^2+k+1}{4}\,.$$

Adding these contributions gives the claim. □

10.4. **Hyperelliptic components.** We recall from [KZ03] that strata of holomorphic Abelian differentials have up to three connected components, distinguished by the parity of the spin structure and possibly hyperelliptic components. The strata $\Omega\mathcal{M}_{g,1}(2g-2)$ and $\Omega\mathcal{M}_{g,2}(g-1,g-1)$ have hyperelliptic components. Their Euler characteristics are easy to compute.

**Proposition 10.4.** *The Euler characteristic of the hyperelliptic components are*

$$\chi(\mathbb{P}\Omega\mathcal{M}_{g,1}(2g-2)^{\mathrm{hyp}}) \;=\; \frac{-1}{4g(2g+1)} \quad \textit{and}$$

$$\chi(\mathbb{P}\Omega\mathcal{M}_{g,2}(g-1,g-1)^{\mathrm{hyp}}) \;=\; \frac{1}{(2g+1)(2g+2)}\,.$$

*Proof.* In the first case the surfaces are double covers of surfaces the stratum $\mathcal{Q}_0(-1^{2g+1}, 2g-3)$ of quadratic differentials with unnumbered points, which is isomorphic to $\mathcal{M}_{0,2g+2}/S_{2g+1}$. The claim follows from $\chi(\mathcal{M}_{0,n+3}) = (-1)^n \cdot n!$, taking into account the global hyperelliptic involution on the stratum.

Double covers of surfaces the stratum $\mathcal{Q}_0(-1^{2g+2}, 2g-2)$ of quadratic differentials with unnumbered points produce the Abelian differentials second case, and this stratum is isomorphic to $\mathcal{M}_{0,2g+3}/S_{2g+2}$. The extra factors 2 from labelling the zeros of order $g-1$ and $1/2$ from the global hyperelliptic involution cancel each other. $\square$

10.5. **Meromorphic strata and cross-checks.** In this section we provide in Table 3 some Euler characteristics for meromorphic strata. We abbreviate $\chi(\mu) = \chi(\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu))$. Moreover we provide several cross-checks for our values. First

| $\mu$ | $(4,-2)$ | $(3,1,-2)$ | $(2,2,-2)$ | $(2,1,1,-2)$ | $(1^4,-2)$ |
|---|---|---|---|---|---|
| $\chi(B)$ | $-\frac{19}{24}$ | $\frac{28}{15}$ | $\frac{17}{10}$ | $-6$ | $26$ |
| $\mu$ | $(4,-1,-1)$ | $(3,1,-1,-1)$ | $(2,2,-1,-1)$ | $(2,1^2,-1,-1)$ | $(1^4,-1,-1)$ |
| $\chi(B)$ | $-\frac{8}{5}$ | $-4$ | $-4$ | $14$ | $-63$ |

TABLE 3. Euler characteristics of some meromorphic strata

note that the union of the strata of types $(4), (3,1), (2,2), (2,1,1)$ and $(1^4)$ glue together to the projectivized Hodge bundle over $\overline{\mathcal{M}}_3$, if all of them are taken with unmarked zeros. In fact, we read off from Table 1 that

$$\chi(4) + \chi(3,1) + \frac{1}{2}\chi(2,2) + \frac{1}{2}\chi(2,1,1) + \frac{1}{4!}\chi(1^4) \;=\; \frac{3}{1008} \;=\; \chi(\mathbb{P}^2)\cdot\chi(\mathcal{M}_3)\,.$$

The value $\chi(4) = -55/504$ can also be retrieved from Proposition 10.4 and the computations of Bergvall [Ber19, Table 4], that gives the cohomology of the stratum with odd spin structure $\Omega\mathcal{M}_{g,1}(4)^{\text{odd}}$ with $\mathbb{Z}/2$-level structure. Computing the alternating sum weighted by dimension gives $-141120$. Since $|\operatorname{Sp}(6,\mathbb{Z})| = 1451520$ this checks with

$$\frac{-55}{504} \;=\; \chi(4) \;=\; \chi(4^{\text{hyp}}) + \chi(4^{\text{odd}}) \;=\; \frac{-1}{84} + \frac{-141120}{1451520}\,.$$

(A few other strata in $g=3$ might be cross-checked with table in [Ber19], but one has to take into account that Bergvall glosses over the existence of hyperelliptic curves in non-hyperelliptic strata.)

Another cross-check is the Hodge bundle twisted by twice the universal section over $\mathcal{M}_{2,1}$. It decomposes into the unordered strata $(4,-2), (3,1,-2), (2,2,-2), (2,1,1,-2), (1^4,-2), (2,0), (1,1,0), (2)$ and the ordered stratum $(1,1)$, since the simple zero at the unique marked point is distinguished. Note that $\chi(2,0) = 3\chi(2)$ and $\chi(1,1,0) = 3\chi(1,1)$. We can now add up the contributions listed in Table 1 and Table 3 and find that the sum equals $\frac{1}{40} = \chi(\mathbb{P}^2)\cdot\chi(\mathcal{M}_{2,1})$. A similar cross-check can be made for the Hodge bundle over $\mathcal{M}_{2,2}$ twisted by every section once, using the second row of Table 3.

## References


[ACG11] E. Arbarello, M. Cornalba, and P. Griffiths. *Geometry of algebraic curves. Volume II*. Vol. 268. Grundlehren der Mathematischen Wissenschaften. With a contribution by J. Harris. Heidelberg: Springer, 2011, pp. xxx+963.

[BCGGM1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. "Compactification of strata of Abelian differentials". In: *Duke Math. J.* 167.12 (2018), pp. 2347–2416.

[BCGGM2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. "Strata of $k$-differentials". In: *Algebr. Geom.* 6.2 (2019), pp. 196–233.

[BCGGM3] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. *The moduli space of multi-scale differentials*. Preprint. 2019. arXiv: `1910.13492`.

[Ber19] O. Bergvall. "Equivariant cohomology of moduli spaces of genus three curves with level two structure". In: *Geom. Dedicata* 202 (2019), pp. 165–191.

[BHPSS20] Y. Bae, D. Holmes, R. Pandharipande, J. Schmitt, and R Schwarz. *Pixtons formula and Abel-Jacobi theory on the Picard stack.* (2020). arXiv: `2004.08676`.

[Boi15] C. Boissy. "Connected components of the strata of the moduli space of meromorphic differentials." In: *Comment. Math. Helv.* 90.2 (2015), pp. 255–286.

[BT82] R. Bott and L. Tu. *Differential forms in algebraic topology*. Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. xiv+331.

[CC14] D. Chen and I. Coskun. "Extremal effective divisors on $\overline{M}_{1,n}$". In: *Math. Ann.* 359.3-4 (2014), pp. 891–908.

[Che19] D. Chen. "Tautological ring of strata of differentials". In: *Manuscripta Math.* 158.3-4 (2019), pp. 345–351.

[CMSZ19] D. Chen, M. Möller, A. Sauvaget, and D. Zagier. *Masur-Veech volumes and intersection theory on moduli spaces of abelian differentials.* (2019). arXiv: `1901.01785`. to appear in Invent. Math.

[CMZ19] M. Costantini, M. Möller, and J. Zachhuber. *The area is a good metric.* (2019). arXiv: `1910.14151`.

[CMZ20] M. Costantini, M. Möller, and J. Zachhuber. *diffstrata – a Sage package for calculations in the tautological ring of the moduli space of Abelian differentials.* (2020). arXiv: `2020.xxxx`.

[Del70] P. Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970, pp. iii+133.

[DS05] F. Diamond and J. Shurman. *A first course in modular forms*. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436.

[DSZ20] V. Delecroix, J. Schmitt, and J. van Zelm. *admcycles – a Sage package for calculations in the tautological ring of the moduli space of stable curves*. (2020). arXiv: `2002.01709`.

[EV86] H. Esnault and E. Viehweg. "Logarithmic de Rham complexes and vanishing theorems". In: *Invent. Math.* 86.1 (1986), pp. 161–194.

[EV92] H. Esnault and E. Viehweg. *Lectures on vanishing theorems*. Vol. 20. DMV Seminar. Birkhäuser Verlag, Basel, 1992, pp. vi+164.

[Fio17] A. Fiori. "A Riemann-Hurwitz theorem for the algebraic Euler characteristic". In: *Canad. Math. Bull.* 60.3 (2017), pp. 490–509.

[FP18] G. Farkas and R. Pandharipande. "The moduli space of twisted canonical divisors". In: *J. Inst. Math. Jussieu* 17.3 (2018), pp. 615–672.

[Ful98] W. Fulton. *Intersection theory*. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 1998, pp. xiv+470.

[GP03] T. Graber and R. Pandharipande. "Constructions of nontautological classes on moduli spaces of curves". In: *Michigan Math. J.* 51.1 (2003), pp. 93–109.

[HM79] J. Hubbard and H. Masur. "Quadratic differentials and foliations". In: *Acta Math.* 142.3-4 (1979), pp. 221–274.

[HS19] D. Holmes and J. Schmitt. *Infinitesimal structure of the pluricanonical double ramification locus*. (2019). arXiv: `1909.11981`.

[HZ86] J. Harer and D. Zagier. "The Euler characteristic of the moduli space of curves". In: *Invent. Math.* 85.3 (1986), pp. 457–485.

[KZ03] M. Kontsevich and A. Zorich. "Connected components of the moduli spaces of Abelian differentials with prescribed singularities". In: *Invent. Math.* 153.3 (2003), pp. 631–678.

[Mas82] H. Masur. "Interval exchange transformations and measured foliations". In: *Ann. of Math. (2)* 115.1 (1982), pp. 169–200.

[Mum83] D. Mumford. "Towards an enumerative geometry of the moduli space of curves". In: *Arithmetic and geometry, Vol. II*. Vol. 36. Progr. Math. Birkhäuser Boston, Boston, MA, 1983, pp. 271–328.

[MUW17] M. Möller, M. Ulirsch, and A. Werner. "Realizability of tropical canonical divisors". In: (2017). arXiv: `arXiv:1710.0640`. to appear in: J. Eur. Math. Soc.

[SageMath] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.0)*. `https://www.sagemath.org`. 2020.

[Sau18] A. Sauvaget. "Volumes and Siegel–Veech constants of $\mathcal{H}(2g-2)$ and Hodge integrals". In: *Geom. Funct. Anal.* 28.6 (2018), pp. 1756–1779.

[Sau19] A. Sauvaget. "Cohomology classes of strata of differentials". In: *Geom. Topol.* 23.3 (2019), pp. 1085–1171.

[Sch18] J. Schmitt. "Dimension theory of the moduli space of twisted $k$-differentials". In: *Doc. Math.* 23 (2018), pp. 871–894.

[SZ18] J. Schmitt and J. van Zelm. *Intersections of loci of admissible covers with tautological classes*. (2018). arXiv: `1808.05817`.

[Vee82] W. Veech. "Gauss measures for transformations on the space of interval exchange maps". In: *Ann. of Math. (2)* 115.1 (1982), pp. 201–242.

[Vee86] W. Veech. "The Teichmüller geodesic flow." In: *Ann. Math. (2)* 124 (1986), pp. 441–530.

*E-mail address*: costanti@math.uni-bonn.de

Institut für Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

*E-mail address*: zachhuber@math.uni-frankfurt.de

Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany

*E-mail address*: moeller@math.uni-frankfurt.de